Optimal Control
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Lecture - 37
Performance Indices and Linear Quadratic Regulator Problem

So, last class we have taken one numerical example to illustrate the solution of optimal control problem using the calculus of variation.

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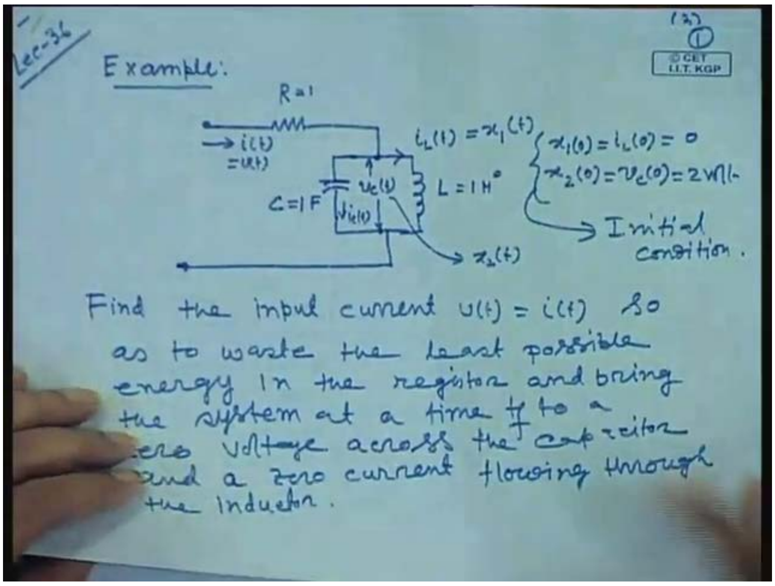

You see this is the simple circuit is there our initial condition of this circuit is given like this way voltage across the capacitor is given 2 volt and current flowing through the inductor is 0 ampere. So, I have to find out the control law u of t, means current flowing through the main circuit in such a way so that at time t is equal to t f.

The current voltage across the capacitor will be 0 and current flowing through the inductor will be 0. Find out the corresponding control law u of t in other words i of t, such that the loss in the circuit is minimum, that worst to the least possible energy power loss in the circuit is minimum as per the. So, this thing we have converted this description of this circuit it is converted into a dynamic equation.

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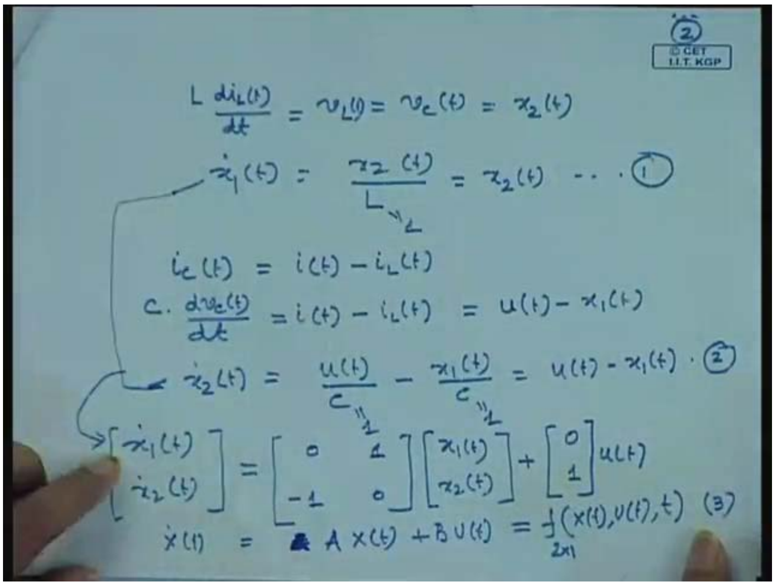

And x 1 dot is equal to x 2 and x 2 dot is equal to x 1 minus x 1 plus u of t. This is the state space description and it is a general structure is x dot is equal to f function of x t u t d t, agree, which is in this particular case, it has become into a state space form x dot is equal to A x of t B U of t. So, we know our dynamic equation of this one, our problem is to minimize this performing index.

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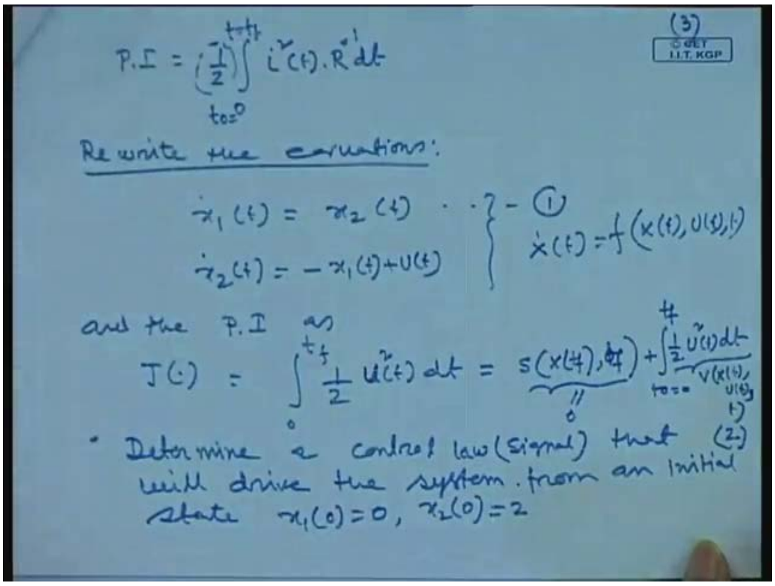

So, this we are solving this problem by using it is a constant optimization problem, we have converted into a, what is called unconstraint.

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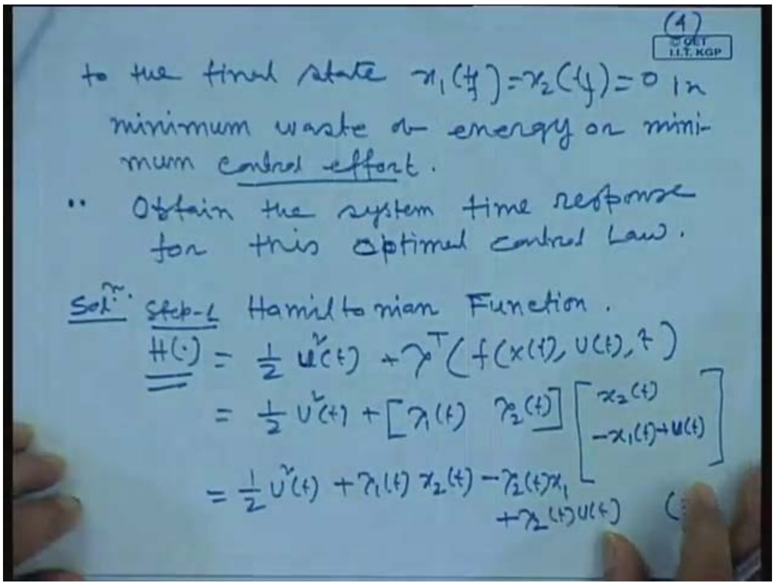

Optimization problem and using the, what is called instead of lagrangian function we are using.

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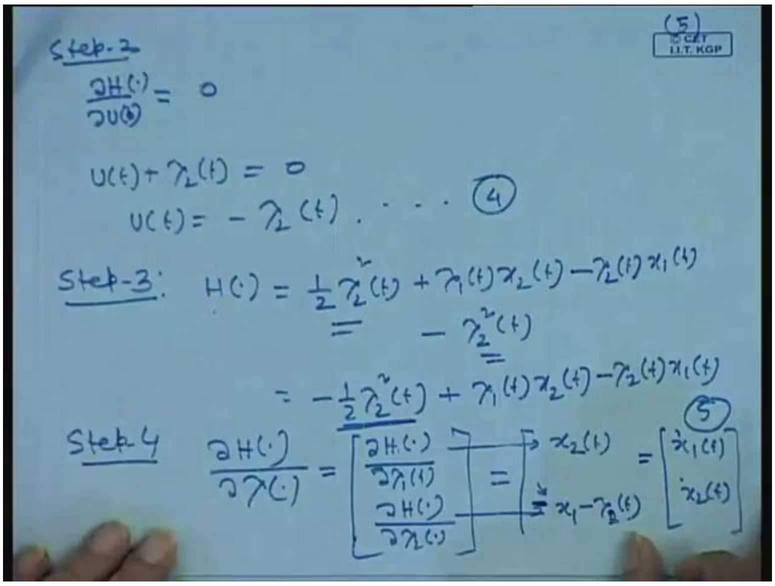

The what is the Hamiltonian function, that necessary condition for the Hamiltonian function is once you form the Hamiltonian function, that necessary condition for the Hamiltonian function is first is del H del U is equal to 0. Then you have a del H del lambda is equal to x dot.

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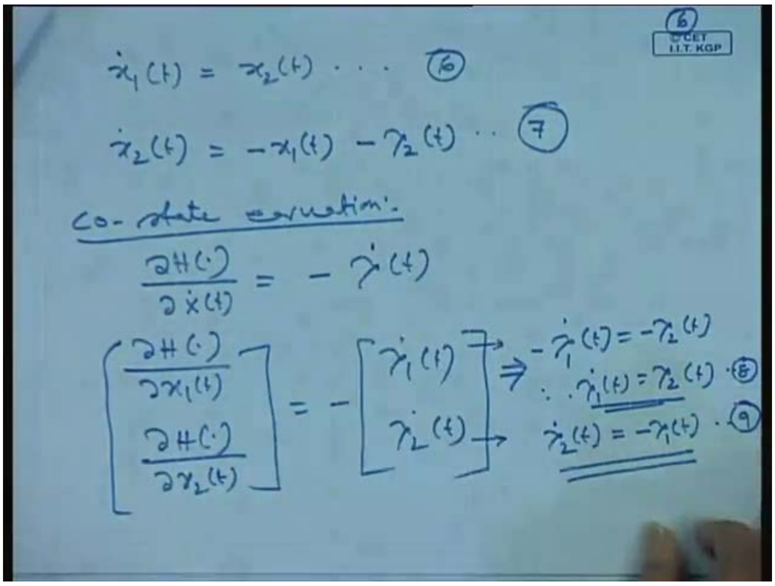

Then next is del H del x dot is equal to minus lambda dot, the three necessary equation, necessary condition we have to solve it in addition to the what is called the boundary conditions. And our boundary condition if you see we have shown you earlier, that our boundary condition that del H, that is what we have written.

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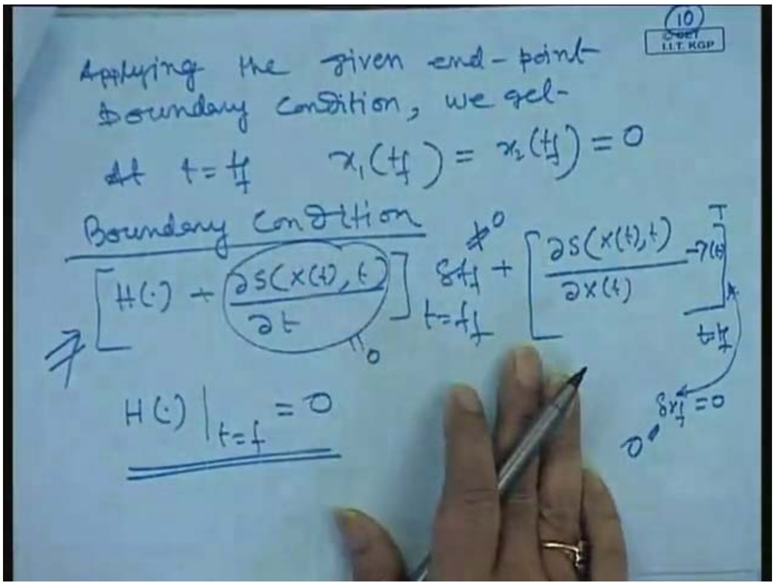

See this one, this is the boundary condition and we have to use this boundary condition corresponding to our problems. Since that what is called del t f, because we know at what time this that final state of the circuit. That means voltage across the capacitor and current flowing in the inductor will be 0, that t f is fixed so. And so we know the final state of this one is you know x voltage across the capacitor is O and current flowing through the inductor is 0. So, you know the final state of the final state of this systems, that means delta x f is equal to 0, but t f is not 0. We have find out the t f for which the system final state which leads to the origin mean 0 0 position.

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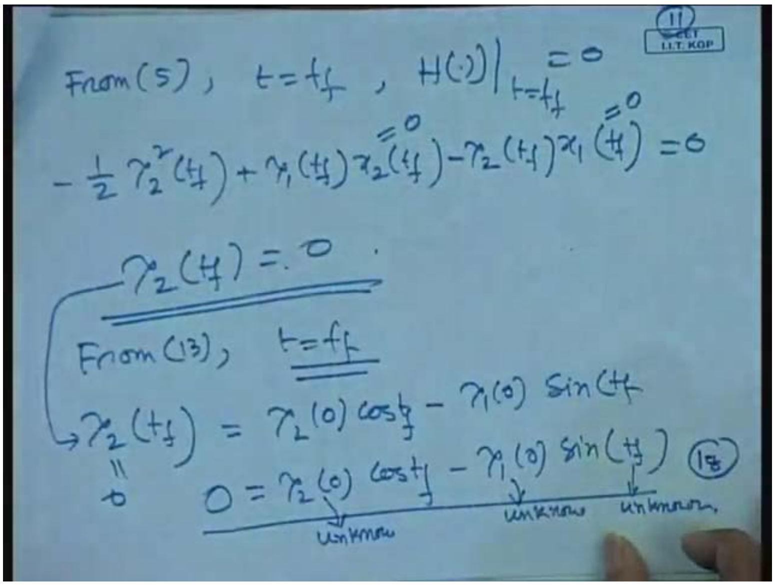

So, we have seen by using the three necessary conditions as well as boundary condition we have come to this point that what is called lambda 2 0 cos t f, lambda 1 0 sin tf. These are the three unknowns are there, but this equation 18 contains the three unknowns, lambda 1 of 0 lambda 2 of 0 and lambda sin t f which is not known. But our main aim if you see our problem is to find out the u control law by solving that equation.

So, u t you need the description of lambda 2 of t. Now, first lambda 2 of t if you see the expression for lambda 2 of t, we got it lambda 2 of t expression that this is the expression, lambda 2 of t expression. So, here unknowns are lambda 2 0, lambda 1 0, agree? So, this we must know this one, so we know this equation and another two equations, because three unknowns are there.

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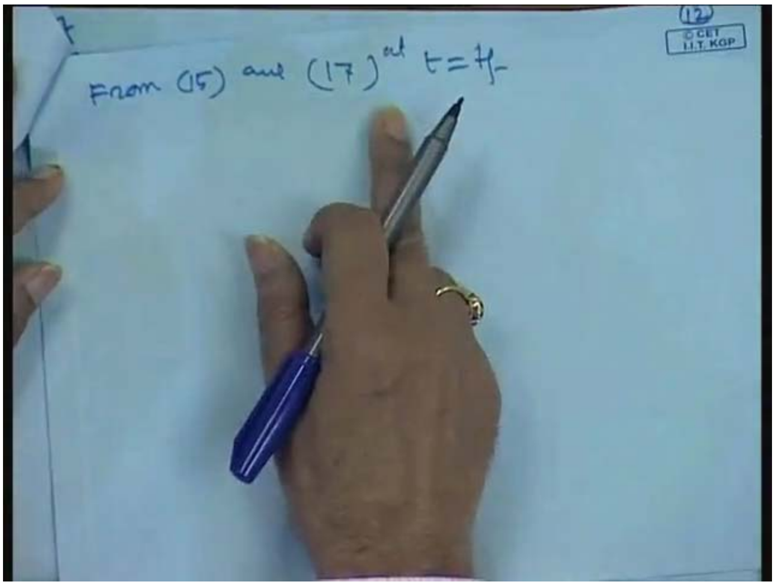

Another two equations, you get from the equation of the state equation trajectory, that x 1 t expression and x 2 t expression, the 15 corresponding to x 1 t and 17 corresponding to x 2 of t.

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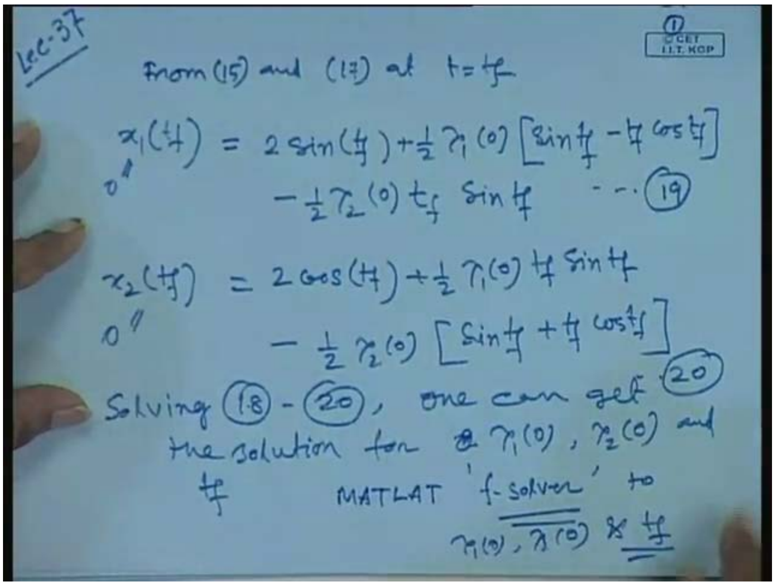

If you put the t is equal to t f, we will get the expression for that one what we are going to write it here, from 15 at t is equal to t f, from 15 and 17 at t is equal to t f we will get x 1 t f 15 is the expression for x 1 t expression. And we are putting t is equal to t f, in that expression, 15 expression. So, twice sin t f plus half lambda 1 0 sin t f minus t f cos t f minus half lambda 0 of t t f. Then sin t f, see the equation 15. Just now, I told you that equation 15 you just see, this is the equation 15. Last class we have seen this equation in that expression, we are putting t is equal to t f t, wherever t is there t is equal to t f, final time this one, so you got it.

Similarly, from equation 17 this one I will put t is equal to t f in the left hand side and right hand side both expression this. So, if you write it this one x 2 t f is equal to twice cos t f plus half lambda 1 0 t f sin t f minus half lambda 0 t sin t f plus t f cos t f, that there is the at t is equal to t f the x t f final. Because we know the final state of the x 1, x 2 is 0, that mean current flowing through the inductor is 0. We want to find out a control law u in such a way that resistor part in the circuit is minimum and not only that current flowing at a time t is equal to t f, that current will be will be 0 through the inductor and voltage across the inductor capacitor will be 0, so this things.

Now, we have a let us call this equation is 19 and this equation is 20. Now, see this one there are three equations, 18 to 20. Three equation three unknowns, but unfortunately these equations are the non-linear algebraic equation. So, you have to solve it by numerical techniques, you can solve it. So, we are writing solving 18 to 20, one can get the solution for lambda 1, sorry lambda 1 of 0, lambda 2 of 0 and t f.

Once you know that lambda 1 of 0 and lambda 2 of 0 then our problem is solved, because we can find out the our control law. If you see our control law u of t is nothing but A minus lambda 2 of t, lambda minus minus lambda 2 of t, u of t minus lambda. And lambda 2 of t is nothing but A lambda 2 of t. You see the expression for lambda 2 of t, here lambda 2 of t is lambda 2 of 0 sin t lambda 1 of 0 cos t sin t. So, this these are now known, so lambda 2 of t we know, in turn u of t we know.

Once, you know u of t then we can get the response for optimal trajectory for x 1 of t, x 2 of t, see x 1 of t is what we got it. So, our x 1 of t this, so I know lambda 1 of 0, lambda 2 of 0. So, I know x 1 of t trajectory of that one. Similarly, x 2 of t putting these values I know the response of this one. So, these are the solution for this, we can get it so or one can use to solve equation 18 to 19, 18 to 20, 18, 19, 20 that is a set of non-linear algebraic equation. One can solve by using a software generally used in matlab software, what is called matlab toolbox. You can use it to solve this set of equation.

So, by using matlab toolbox, that is one solver is there, f solver, agree? You can use this one to get this three values, three unknown values, lambda 1 of 0, lambda 2 of 0 and t f you can solve this one. So, now what is our optimal control law? Just now we have seen our optimal control law is equal to optimal control law.

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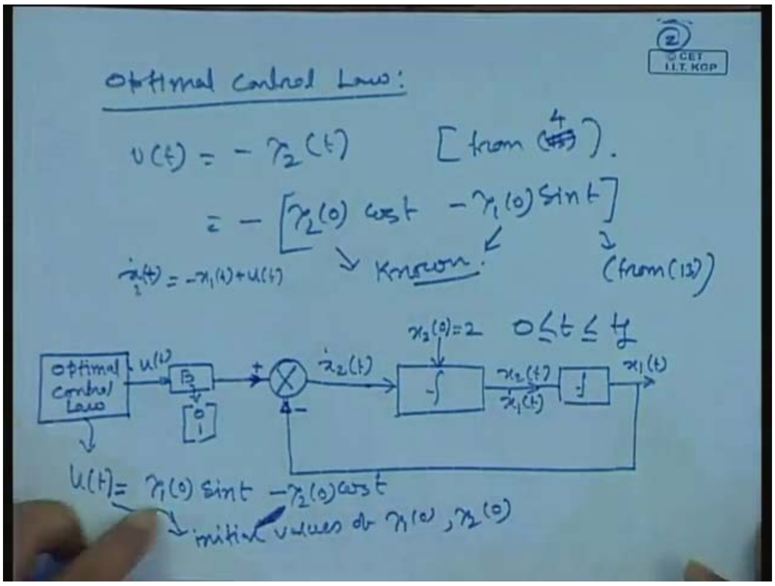

Optimal control law u of t is nothing but minus lambda 2 of t, see from equation 13. So, lambda 2 expression we know that lambda 2 0 cos t minus lambda 1 0 sin t, so that this is the lambda 2 expression, see this expression. So, first is this is equation number 4, this is equation number 4 and this is expression lambda 2 expression from 13, see the equation 13. This is the lambda 2 expression, so once you know this one, this is known, all are known by solving three set of equations.

So, and we have to find out from t 0 to t f, we know at what time, you know at what time t is equal to t f, this final state of the system. That means voltage across the capacitor will be 0 and voltage across not voltage current flowing through the inductor will be 0 at time t is equal to t f if you solve this three set of non-linear algebraic equations.

So, if you see the block diagram of this one, what we got it this we have a say this one our system x dot, x 2 dot of t. You have integrated the output of integrated is x 2 of t and this is initial condition of this x 2 of zero is equal to 2, then this this x 2 is nothing but a x 1 dot of t, x 1 dot of t is nothing but x 2 integrated to realize that one is x 1 of t, agree?

So, this output is coming here if you see the basic equation x 2 dot. If you see the our basic equation x 2 dot is equal to minus x 1 of t plus u of t.

So, minus x 1 of t, so there is another is B, this is our B and this is our u star and from where u star is, this is u of t and this is B. B is in your case 0 1 this and what is this? This is the our optimal control law and how do what is this expression, u of t is equal to just now you have seen it you have to use that one. So, what is this one lambda 1 of 0 sin t, I am writing from this one, you have to sin t minus lambda 2 0 cos t. So, you see this control law is openly control law. So, it entirely depends on the solution of lambda 1 0 and lambda 2 0, agree?

So, this is optimality so it depends on the initial value. So, u depends on initial value of lambda 1 0 and lambda 2 0. So, this is the open loop, it does not depend on the state information to control the systems. So, it is a open loop, what is called system and the control law is generated with this expression and we have seen how lambda 1 0 and lambd 2 0 are obtained from this solution of this one. See lambda 2 0, if you see this one then when we are solving this equation.

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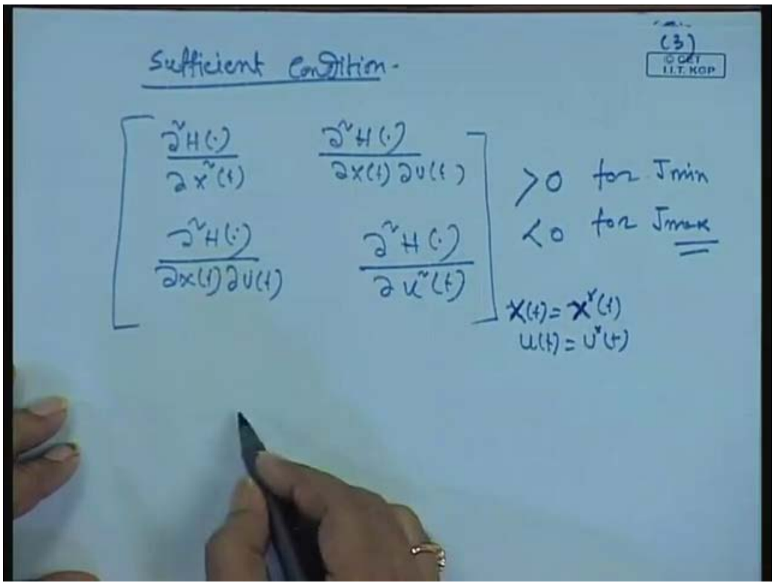

So, this initial condition lambda 2, how you are solving this equation? Lambda 2 expression is this one alpha 1 plus alpha 2, we know this lambda 2 of 0. So, whole control law depends on the initial condition of that alpha 1 0 and alpha 2 0. So, it is a open loop solution of this one and this is dependent on initial condition and in fact is it is not acceptable that control law.

So, what is this? Our sufficient condition and sufficient curve, whether the system that objective function is a minimized or maximized along this trajectory or whatever the control law we have generated u star, whether the corresponding performing index minimized or maximized that has to be checked with this sufficient condition. So, this is equal to x square of t, then del square H del X of t del U of t.

Similarly, del square of H del X of t del U of t, then del square H this del U square of t, this you find out along the trajectories that whatever solution you got it X of t. Let us call that solution you got X star of t and U of t you got. Let us call U star of t, this is capital X U star of t and if it is greater than 0 for then you conclude. Implies that j minimum you got it, if it is less than 0 for j maximum that you have to check that things. So, you know how to solve this problems of that one.

Next we will come to the solution of what is called linear quadratic regulator problem with finite time. So, next is before that we will discuss the what is the performance indexes, performance indices.

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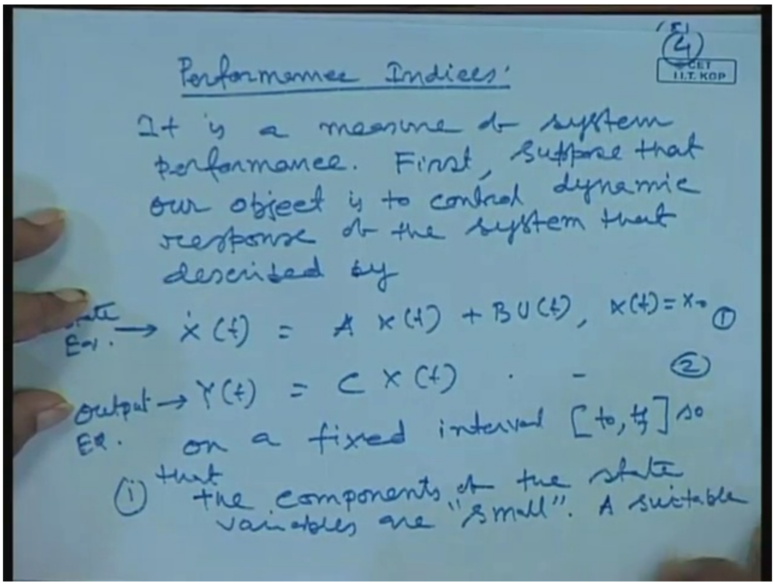

Performing index is a measure of system performance, first suppose our objective. So, it is measure of measure of system performance, first suppose that an objective function that our objective is to, if our objective is to control dynamic response of the system that described by X dot is equal to A X of t plus B U of t and initial condition is known or given. Let us call equation number 1 and our output equation U of t is equal to C X of t, this is the output equation. So, this is the equation what is called state equation and this is our output equation so that output equation on a fixed interval fixed interval t 0 to t f.

So, that one what is our objective? The component of the state variables, the component of the state variables X are small, our objective is to here the component of the components of the state variables are small so for that one. So, our problem is you design a control law U such that our component of the state variables are small for that one. What should be the choice of our performing index since component of the state variables are small? The performance index what we have selected in quadratic form, that means some of the error square must be small. So, a suitable performance index is selected.

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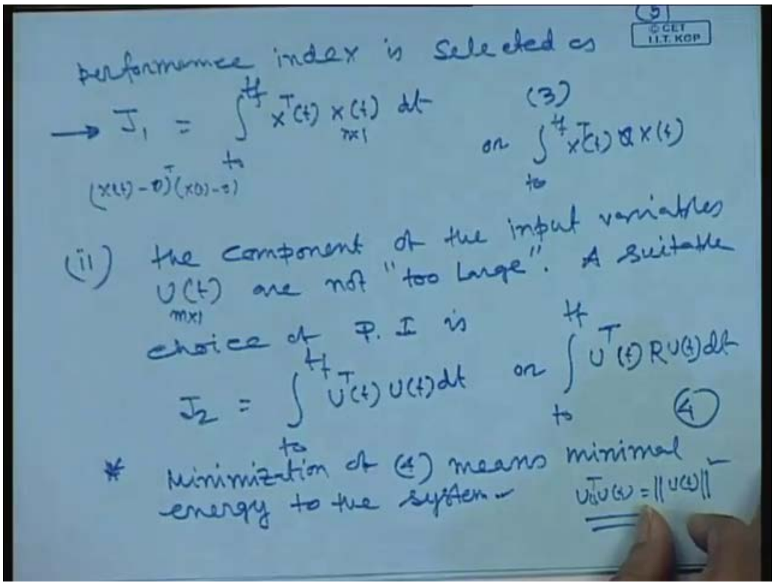

So, our performance index J 1 is equal to t 0 to tf, because our over the time interval t 0 to tf that our component of the state variable should be small, so that it can be represented by X transpose t into X of t dt should be small that this indicates over the time interval t 0 to tf. These are nothing but what is called echelon norm or distance of the vector, that is nothing but each component if it is x 1 x 2 dot dot xn. The dimension is x n is nothing but a x 1 square plus x 2 square plus x 3 square dot dot x n square this represents.

So, this should be a small so you have to find out the control law in such a way so that J is small, means each component of the state variable is small so that for that one what should be the our control law. So, if you want to design a control law there is no restriction on the input. So, input magnitude can be very high which is not acceptable in practical situation. So, this is one of our performance index you consider if our each component of the state variables, if you want to make it small that corresponding performing index is the that one. Now, this or one can write it this also one can write it by giving weightage on the state.

Suppose, our Q is a diagonal matrix, it indicates if you expand this one, if q is a diagonal matrix then it is nothing but a x 1 square multiplied by small q 1 and diagonal diagonal elements of q. If it is a small q 1, q 2, dot dot q n then if you expand this one you will get x 1 square into q 1 x 2 square into q 2, that means we are giving weightage on this performing index. This one that is also could be a one of the performing index when our objective to make the each component of this state variable is each component of the state variable to make small, this is the either one of this will be our choice, agree?

Second performing index if our motivation is to each component of the control variable, that means we want to make it what is called large, not too large. If you want to make it, each component of the input variable not to make too large then what should be the choice of our performing index, agree? So, the component of the input variables U of t and U of t dimension is m cross 1.

So, you have a m components are there, each component when to make it not to make very large and not too large, then our suitable choice of performance, then a suitable choice of performing index performance index is J 2 is equal to t 0 to tf over the interval. And 0 to t dt t transpose, this dt dt and this is nothing but a once again it is nothing but a i u is a control effort just like in our previous example.

It is a current, so it is nothing but a scalar case is nothing but a i square is nothing but a energy. So, U transpose, U indicates the energy or control effort. So, what is the energy is delivered this indicates physical interpretation is what is the energy delivered to the system to achieve our objectives. Our objective is now in this case to make the input variables are each input variable not too large, that is we have to make it or one can select this one t 0 to t f U transpose, giving some weightage on U. So, this u of t dt this is also may be our choice of performance index.

So, our problem is like this way, find out the control law U in such a way that this performance index is minimized subject to the constraint. The x dot is equal to x plus B U, but here we do not have any control on the state. State magnitude may be a very large all this things, agree? So, here we do not have any control on the state. Now, question is, so this is the, let us call this is 3, this is 4.

Now, we make a comment that what is a minimization? Minimization of 4 means the minimal energy to the system because this is the input energy is related to the input, generally related to the energy consumed in the circuit or in the system so that U transpose U, agree? tThis indicates nothing but a norm of U of t square, agree? So, it is nothing but a control effort or control energy involved in the system agree, control energy is supplied to the system this one this indicates.

So, if you see first case when the system is in equilibrium position, what we expect? We expect our state should be in equilibrium position. In linear system the our generally our equilibrium position is 0 0. So, if you give with the initial condition the state will deviate from equilibrium position and the corresponding deviation you see our x minus this is the our state x minus x 0 is the our equilibrium position. This transpose into x minus x t minus 0, this is the our equilibrium position. From there we have given a some, so this interpreted X transpose of x. So, this is to minimize this one, but you see when you are minimizing this one, when you are minimizing that quantity then we do not have a control on input.

When we are minimizing that second part then we do not have any control on the state. So, there is a what is called. So, remark it may be pointed out, we cannot simultaneously minimize the performance index indices one that is where I considered one that is J 1 of dot and 2 J 2 of dot, that means this is nothing but a t 0 to t f, x transpose of t x of t dt and this is nothing but a t 0 to tf U transpose t U of t dt. You cannot simultaneously minimize because why you cannot do it because the minimization of J of J 1 results large control effort or control signal control effort. Also you can control signal while minimization of

J 2 results small control signal, agree? So, this is the minimization of J 1 because we do not have any control.

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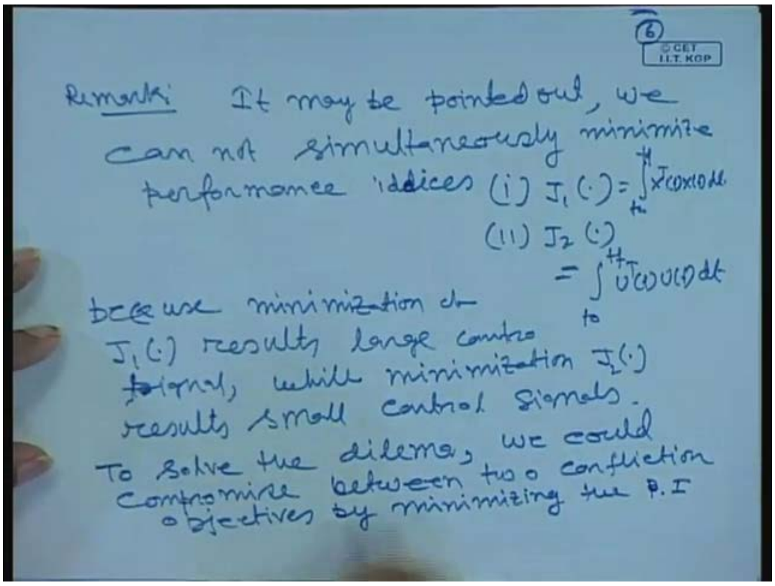

So, if you give it physically, if you give it pumping more energy to the system, that quantity will be minimized. Second part of this one is when we are minimizing that one agree, when we are minimizing that results the small control signal, minimization of this one means results small control signal. So, this two things simultaneously we cannot minimize because it is a contradict of one another. So, what you have to do, this one you have to make some to this dilemma. We have to make some compromise, agree? So, they compromises in we can make it by using the convex combination of this two performance index.

So, to solve this dilemma we could compromise between two conflicting objective functions objectives by minimizing the performance index, that is performing index minimizing the performing index that is a combination of that is a combination of.

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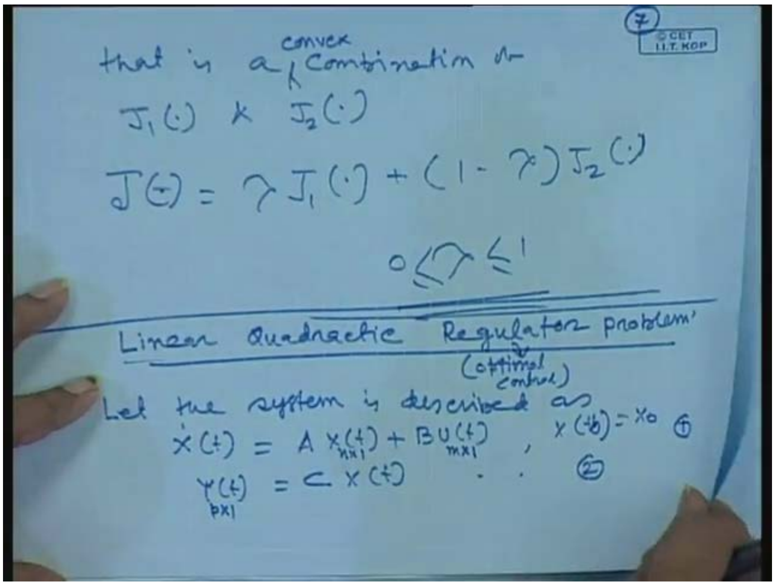

That is a convex combination of convex combination of J 1 and J 2. So, our resultant performance index is J dot is lambda into J 1 dot and plus 1 minus J 2 1 minus lambda is J 2 and lambda, that is from 1 to n. When lambda is equal to 1, then this is minimization of this one when lambda is equal to 0, then minimization of J 2. So, you have taken the combined convex combination of this two things. So, one can go, one can get good results good response of the system by making trial and error, giving more weightage here or less weightage here depending upon the situation, with lambda value we can get the tune the system response.

So, let us now see how to solve our control problem, but what do you mean by the linear quadratic regulator problems on finite time linear quadratic regulator problem and regulator in optimal control. So, linear quadratic regulator problem what we understand? So, describe the system equation like this, let the system is described as x dot is equal to A x of t plus B U of t, agree? So, that is this one and Y is equal to Y of t is equal to C x of t agree and this is the initial state x t of 0 is equal to x 0 either initial state is given to you. This is equation number 1, this is equation number 2 and this dimension is n cross 1, this input is m cross 1 and output is p cross 1, p outputs are there.

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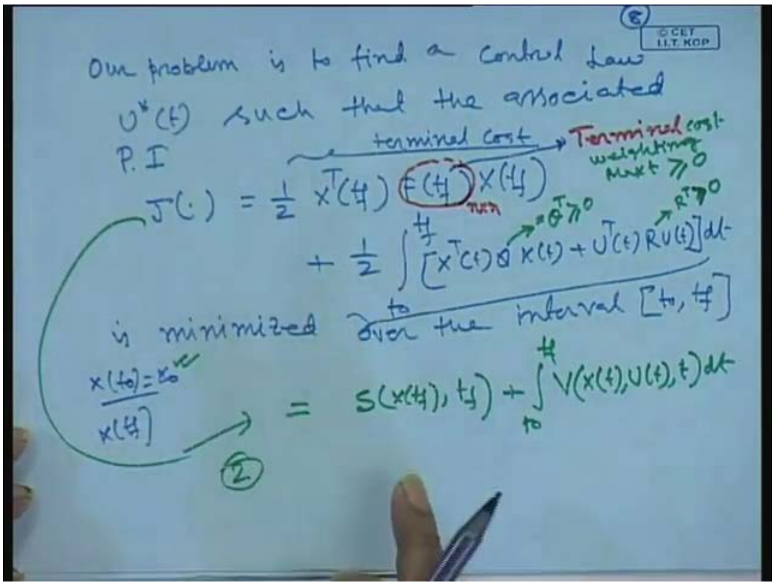

So, our problem is to find a control law, the statement of the problem. Our problem is to find a control law U star of t such that the associated performance index, this is a performance index J which is the function of U t X t. All this things half X transpose t f f of t f x of t f plus half t 0 to tf X transpose of this Q x of t plus u transpose R u of t dt, such that the associated performance is minimized over the interval t 0 to tf.

So, our problem is to find a control law U such that this a performance index is minimized and our state is moved. The initial state the control law will drive the initial state from X of t 0 to from initial state X t 0, X is equal to X t 0 to drive the final state X is equal to t f near to the what is called origin or near to the 0. So, that is the our problem.

So, once again I state the problem like this way that our problem is find out the control law U of t in such a way. So, this performance index is minimized agree, a performance index is minimized. In other words you can say you find the control law U that will drive the state from the initial state x 0 to the final state x t f which is close to the our origin or 0 for that you find out the control law U of t and what is the performance index is there.

Similar to this one is the terminal cost, similar to earlier discussion it is a terminal cos and this whole part is the integral cos functional, this is the integral cos functional. Now, this tf f of t, this is called the terminal weighting matrix. You can call this is the terminal f of t, this f of t is a matrix which dimension is n cross n is a terminal weighting matrix. Terminal cos weighting matrix and which is a positive semi definite matrix which a positive f is a positive semi definite and this is the symmetric, this is also symmetric. Q is the symmetric positive semi definite, that Q is equal to you can write Q transpose and which is a positive semi definite.

R is the weighting matrix with the control effort or control signal, R is the weighting matrix associated with the control signal and Q is the weighting matrix associated with the state vector and this R. Also symmetric matrix R is equal to R transpose and it is a only positive definite matrix, not positive semi definite is a positive definite matrix.

So, this indicates that this performing index, you find out the control law that will drive find out the optimal control law that will drive the state from initial state. This to a final state x t f which is near to the origin agree, by minimizing this performing index and each term of the performance index has a physical significance, because our original system is at rest. Whatever the position is there and we have given the system x t is equal to t 0, something so that state will deviate from the equilibrium position. And this indicate the sum of the deviations squares agree, with some weightage Q and this is the control effort that is U transpose U, if R is there U transpose is the control effort.

So, simultaneously we want to minimize both in such a way that you find, you minimize that U, you minimize that performance index J agree, such that that control effort U will drive the state from x t 0 to x t f which is near to the origin. So, this is called finite time regulator problems within a finite time the state should reach to the origin near origin.

So, this if you see it is equivalent to our original statement of the problem S. This is x t f comma t f plus whole t 0 to t f V X t U of t comma t, V of t. Now, only difference you see there is a half is present in this expression. This half is intentionally we kept it, we know if the objective function if you multiplied by a scalar quantity the optimal point does not change it, agree? But optimal value of the function value will change it.

So, I just multiplied by half the optimal value at which the objective function will be maximized or minimized, that value will not change optimal point at which the functional value is minimum or maximum, it will not change.

So, why I made it half you will see. When you will do the that find out the necessary condition of this objective function. There are two terms A 2 will come from the when you differentiate with respect to X or with respect to Y, all this things with respect to lambda we will do just like a your what is called necessary condition. What you got it the del H del U, the two term, 2 will come into the picture and this two term and half will cancel. If you do not keep it half, the 2 will be carried out throughout the derivations, that is only this. So, intentionally we make it half in the expressions.

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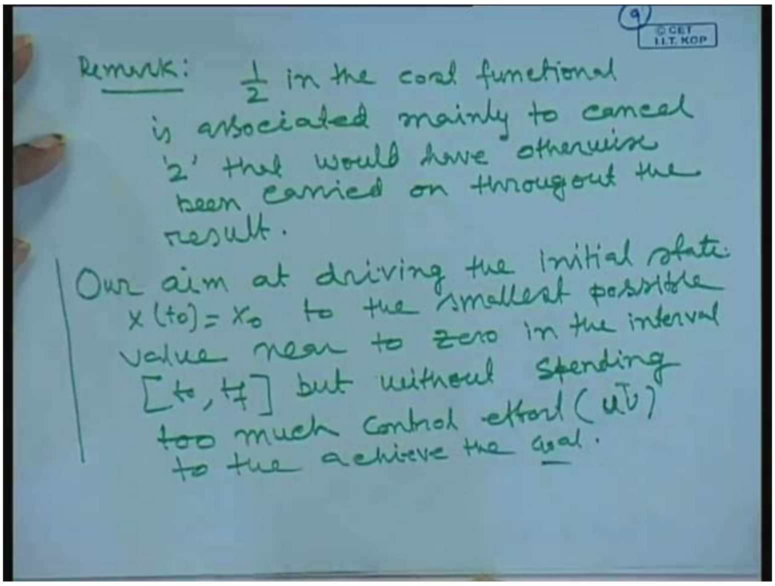

So, let us why we have introduced half, we just write it here. Remark the half in the cost function, functional is associated mainly to cancel 2 that would have otherwise been carried on throughout the that results or derivations. So, if you just see our main problem of this, the statement our problem, our aim this is a important thing. Our aim at driving the initial state, state x of t 0 is equal to x 0 to the smallest possible values, value near to 0 in the interval t 0 to t f, but without spending too much control effort. Too much control effort means U transpose U transpose U to achieve the goal.

So, this is our statement of the problem. If you see this one our problem is find out the control law U of t, such that our original state will with the application of this control law. Our original state will drive to a what is called at time t is equal to t f X t x t f which is near to the origin will drive, not only this, it will satisfy our original system dynamic equations agree, so this is our performance.

So, now solution of this problem is this the same as what we have described earlier. Now, the state equation, we have now in state of what is called that dynamic equation is expressed in terms of state equations, agree? This is only difference is this one now that physical significance of this weighting matrix, this is called state weighting matrix Q is called the state weighting matrix.

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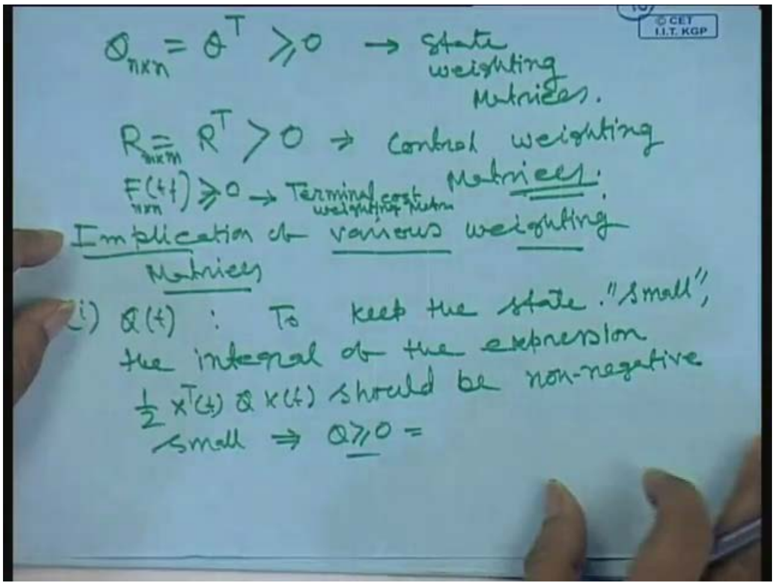

The Q is which dimension is n cross n, Q is a Q transpose and which is a positive semi definite matrix and it is called state weighting matrices. Then R is equal to R transpose, which is greater than 0, means positive definite, positive this is positive semi definite is equal control weighting matrices, agree?

So, if you want to keep the state small in the expression, if you see in the expression, if you want to keep the state small, then choice of Q must be positive definite matrix. So, our implication, you can say implication of various weighting matrices and what is this f of t t f is the, this is positive semi definite matrix. And this is the terminal weighting matrix terminal cos weighting matrix, terminal cos weighting matrix and this dimension is m cross m and this dimension is n cross n, because f is associated with the X, so it is n cross n.

So, next is implication of various weighting matrices. So, first implication Q of t state weighting matrix, this is the state weighting matrix agree, to keep the state small agree, because initially the system state at the equilibrium position. If you give to the system the state will deviate from the equilibrium position. So, if you want to keep the state small then the choice of then the integral of the expression half X transpose Q x of t. This should be non negative, non negative small agree, implies that Q must be greater than 0.

We want to keep this integral of this one small this should be a non negative small quantity, the Q must be a positive definite matrix.

So, how you can make it? See this expression, how can you make it, this Q this part is small by giving more control effort to the system. The U magnitude should be the U star U t of U t multiplied U t, transpose of U t should be large then it will drive the state to small value of this one. But when you give the control effort large then what will happen, this control effort is not permissible to act into the system, because it is a physical system, so you cannot make it too large of this one.

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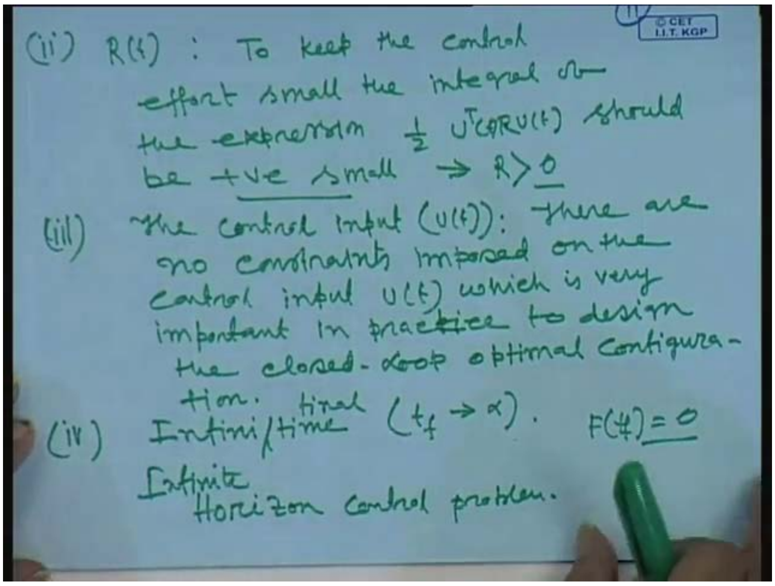

Second implication of that is our R with R of t, the control weighting matrix to keep the control effort small, the integral part integral of the expression half U transpose this R U of t should be positive small. Why positive small, this quantity that whatever the quantity is you are getting this quantity. If it is zero the U transpose u t if it is a 0 that there is we are not applying to any control effort to the system. So, it will not be able to control the state of this one, so this should be a positive definite, so this input positive implies that R should be greater than 0.

So, another important issue is most of the practical control problems, you will find there is the restriction on the control input. That means the control input as a minimum value and maximum value of this one. So, this is most of the practical problem, we have a restriction on the control input, but for the time being we will discuss back there is no restriction on the control input.

That control input U of t there are no constraint imposed on the control input U of t, in our statement of the problem we have not mentioned any control constraint in the statement of the problem, which is very uncommon in real practice. Or which is very important in practice to design the control systems, to design the closed loop optimal configurations. First we will design a linear optimal control linear regulator problem without constraints, then we will take that if there is a constraint in the control input, constraint in the input as well as in more practical problem constraint in the input as well as constraint in the state. So, that problem we will discuss later.

So, the last one is what is called infinite time, final time, that means t f tends to infinity when the final time t tends to infinity the terminal cos does not have any sense, because at time t is equal to infinity X of t f will approach to 0 or it will go to 0. So, in that situation the terminal weighting function has no, what is called sense at all. So, we will put that f f t f is equal to 0 when t tends to infinity. So, it is called this type of problem infinite final time, this type of problem is called what is called infinite horizon problems, infinite horizon control problem. And another two points are there, we will discuss next class of this one.

So, infinite time regulator problem t tends to infinity, you see the state will reach to the origin means near equilibrium position at that situation the terminal weighting function has no sense. So, we will assign that t f is equal to 0. So, that type of problem is called, what is called infinite horizon control problems. So, we will stop it here.