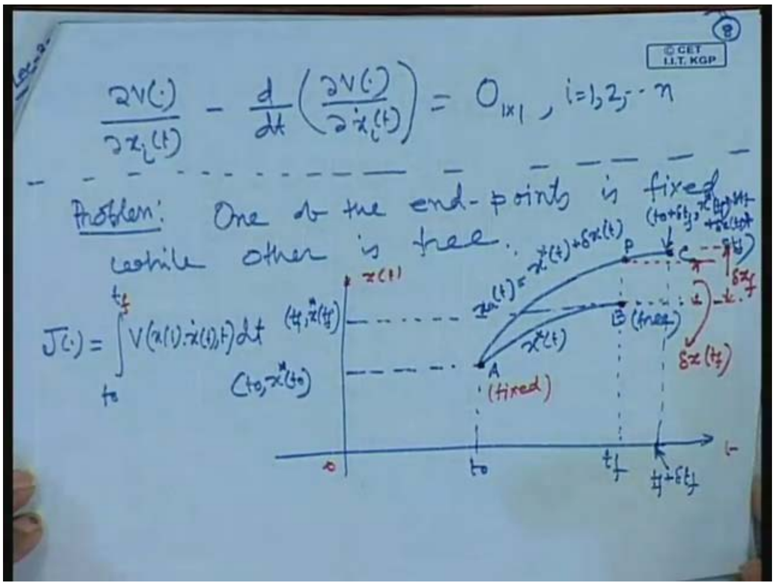
Optimal control

Prof. Dr. Goshaidas Ray
Department of electrical engineering
Indian institute of technology, Kharagpur

Lecture - 32
Dynamic Optimization Problem: Basic Concepts and Necessary and Sufficient Condition (contd.)

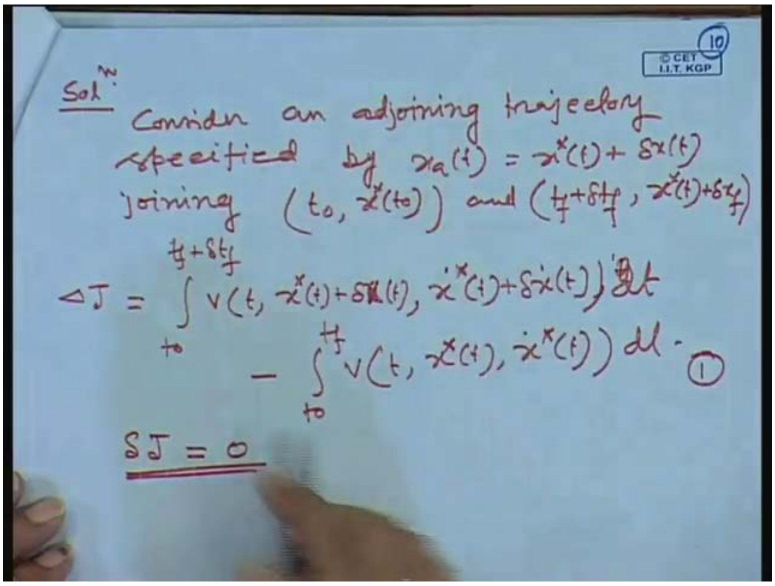
(Refer Slide Time: 00:21)



So, last class you have, we have considered the one problem. That we have a one point is fixed and other point is free, again the point A is fixed and point B is free. So, what we have considered? Our problem is that this is the functional, which is t 0 to t f. This functional we have to optimize either minimized or maximized, what should be the choice of the trajectory x t so that this functional is a maximized or minimized. So, let us consider that the x star is the optimal trajectory and its point A is fixed, B is free. It may be free in time and may be fixed in the n points of x.

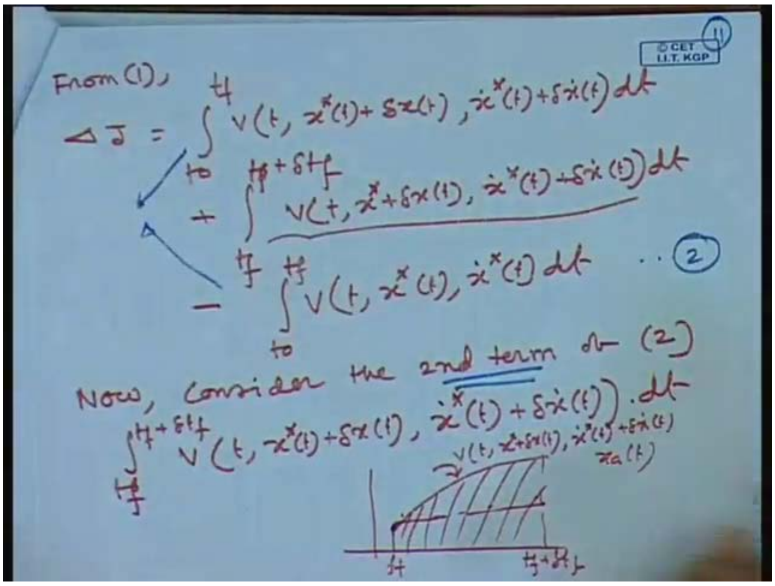
Now, measuring of this trajectory, we have considered another trajectory that x star delta of x t. That is x a and this trajectory is you have assumed a, that is DC. This trajectory and what every the notation we have used it, the same notation what we have used it earlier considered here. What is you have to do first? You have to find out the what is called our first variation of the functional value.

(Refer Slide Time: 01:46)



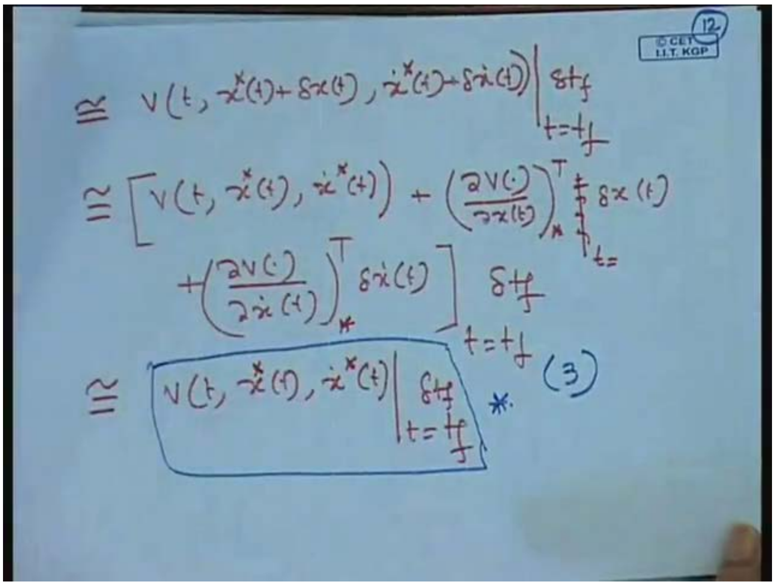
Del J, change in or incremental change in functional value this and this t 0 to t f plus delta t f because t f is free and x t f also free x value, final value of state also free. So, this we can split up into two parts t 0 to t f plus t f to delta t f. So, what we did it here?

(Refer Slide Time: 02:18)



We did exactly same, t 0 t f and delta t f to t f plus delta t f. Now, this part and this part we have already simplified in earlier, so there is no problem with these two parts and this two parts will give you the condition for Euler's Lagrange equation. Now, what about this one? So, this we will see it is nothing but A, let us call this is the functional B and it is the integration from t 0 to t f. Let us call t f is here, t f delta. So, how we are approximated this integration term?

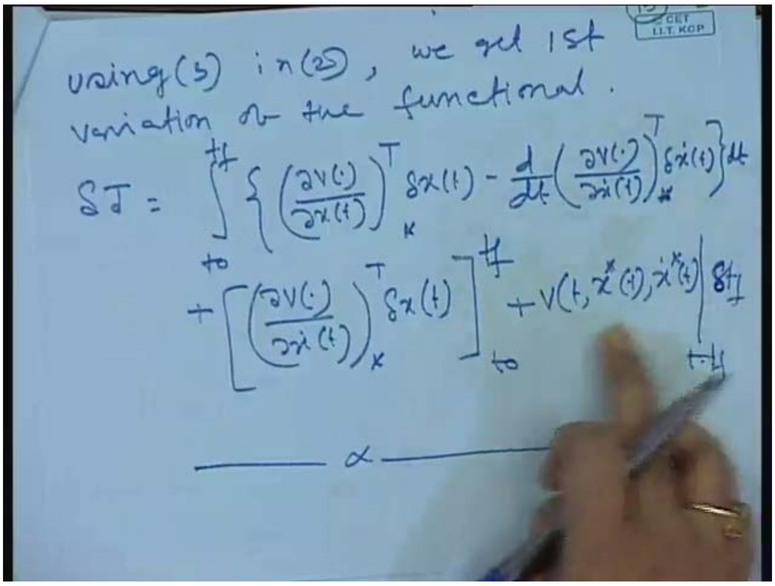
(Refer Slide Time: 03:04)



Ultimately I have explained earlier ultimately, we have approximate debt indication part by this one, that means it indicates whatever the slope at this point is t f V of x t curve. That we have this is the view of extra curve, take the slope of this one again. Then multiplied by delta t f at t is equal to t f, find out the functional value and multiplied by t f.

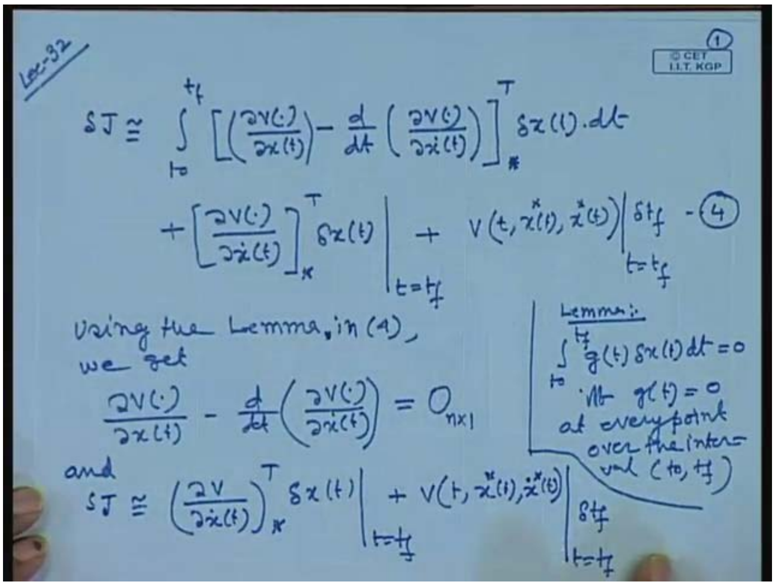
So, this is the approximate value of what is called this integration part when t f is very small, delta t f is very small. So, using this expression in equation, this expression and this approximation we did it by this one, using this expression here then we will get it finally, this expression. I told you if you consider, I told you earlier this and this we can simplify as we did earlier by this term, this and this term.

(Refer Slide Time: 04:09)



So, last time we have simplified this one, that is middle term of this increment of functional value is a approximated this one. Now, see this one, this term t is equal to t 0 delta x t 0 is 0, because this is the fixed point, but t f and delta t f, t f delta t x, delta x t f is not 0. So, further we can what we can simplify this one, that we will discuss. Now, so if you see this one that that same expression.

(Refer Slide Time: 04:57)



I can write delta J equal to t zero to t f and delta del V del x of t, this one minus d of d t del V dot, del V dot, del x dot of this whole then whole they get transpose at along the optimal trajectory. This one and delta x t d t plus del V dot, del x dot whole transpose star means along the trajectory star means along the del x t. So, t is equal to t f because delta x t equal to is 0. That term we are not considering from the previous step and these two terms includes the first term of incremental transformation and third term of incremental transformations. That is this indicates the after simplification all this things this indicates that, I told you incremental transformation. This term as we did earlier is the simplification, after simplification we got it that 1 plus middle term of this one.

We have approximated the integration t f to t f plus delta t f, V d t is approximated by take the slop of the functional at t is equal to t f, then multiply it by delta t f. So, this is t x, t x star or whole this you write it bracket, then complete the bracket and then put star, whatever you like it. Then you evaluate this as t is equal to t f, find the value of t at t is equal to t f. Then multiply just like approximation by rectangular area, we got it this and it is just if you recollect this one. What we did it here that find out the value of the function at V function at t is equal to t f.

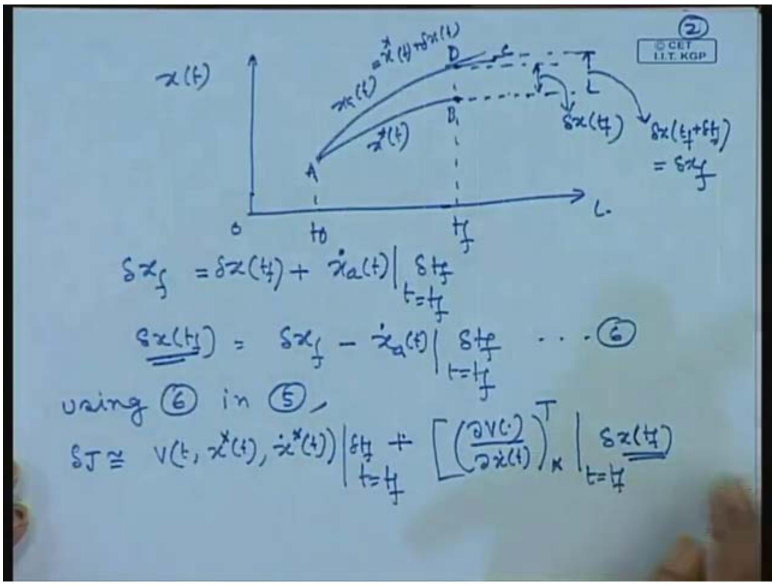
So, this is the ordinate and that ordinate multiplied by delta t f will give you the value of the function value of this integration t g, t f to t f plus delta t f. It is a approximation is valid when delta t f is small so that what we did it here. Then next is let us called this is equation number 4. We have considered up to equation number 3, last equation if you see we have done up to equation 3, this one. So, this is equation number 4.

Now, look at this expression of this one, that we have a use this lemma. Use this lemma as we discussed the important lemma is t 0 t f g t delta x t d t is equal to 0 if and only if g t is continuous. And it is 0 at every point, our the interval t 0 to t f that if and only g t is 0 at every point over the interval t 0 to t f.

So, we will use this lemma in this equation 4. So, this part will be 0, provided this is 0 because I have to necessary condition for the function to be optimized is delta J must be equal to 0, first variation of this functional must be 0. This is the necessary condition. So, using the lemma in 4, we get del V del x t minus d of d t del V dot, del x dot of t whole is equal to you can find out. You can put all these things is star or because I have to find out the optimal trajectory of this one or you solve this one, whatever the solution of x you will get it, that will give you the extract.

Either you can put whole thing star or omit this one, and this dimension is n cross 1. If x is the vector whose dimension is n cross 1, that n variables are there x 1, x 2, x, then the dimension is this one. So, in addition to this, this must be 0, in order to make this is direct. So, our in order to make this is del J is 0. So, if this is 0 in the expression 4, bounce down to that one that nearly equal to del V del x dot of t whole transpose delta x delta because S J is scalar quantity. This is the column vector. Now, you have to take transpose of this one. When you write it this one and this you have to calculate t is equal to t f plus V dot V t x star of t x dot star of t whole t is equal to t is equal to t f into delta t f and that what we can simplify that one we can see here.

(Refer Slide Time: 11:50)



Now, look at this expression for that one what you can write it for this one refer to our original figure E, this is a this is B. This is t f, this is t 0 and this is our D and this is our C. So, as this and this value is if you see this value is our delta x t f and from here to here, from here to here if you recollect the this is our nothing but A delta x. This is at C t f plus delta t f and that we donate it by delta x f.

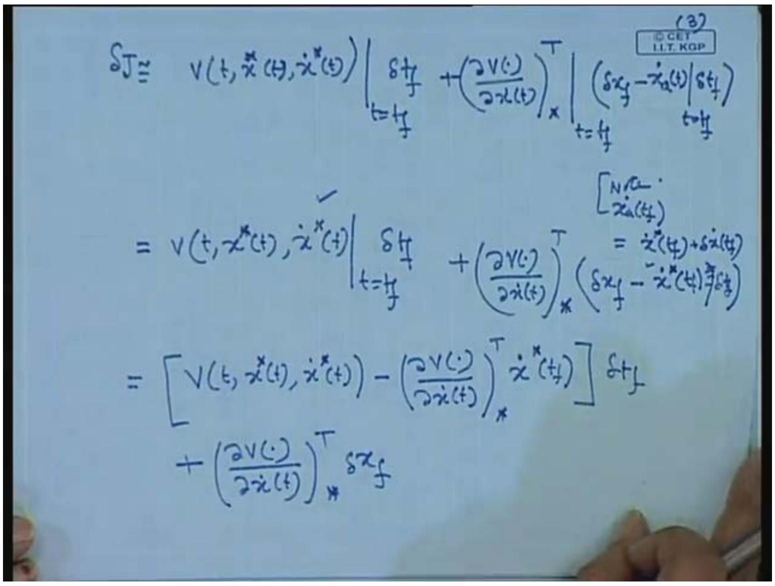
So, this is x of t 0. Now, see how we can write delta x of f value of the function along the x a of t. This is x star of t trajectory and this is the x of t is equal to x star of t plus delta x of t. So, this one is nothing but A, this one this length of this one, this length plus this length. So, I am writing it this delta this that delta x t f plus find out the slope at this point multiplied by delta t f, what is the incremental. So, x A dot of t find out this derivative at this point that at t is equal to t f into delta t f.

So, this expression I can write it now which in turn I can write it delta x t f, delta x t f this one is equal to delta x f minus x A dot t, t is equal to t f delta t f. So, this expression we will use in equation number 5. Let us call this is the equation number 5 for our case. This is the equation number 5. This is the equation number 5, so and this is the equation number 6, using 6 in 5, using 6 in 5 what you get it.

So, let us write it delta J nearly equal to this term as it is we will write it that V t x star of t x dot star of t that we can come t is equal to t f, second term of expression. Second term of equation 5 is this one plus delta t f is a delta t f plus first one I am writing. Now, del V dot, del x dot of t that transpose this as t is equal to t f t is equal to t f and we have A.

This delta x t f that this thing I have first put the value of this one, then this you can do it, put this value this is the at t is equal to t f. You put this value in the grading transpose, grading of V with x dot transpose. You put this value and then multiplied by x t is equal to t f, that is what is we did it then this one this expression in place of t f x t f, this I put this value here.

(Refer Slide Time: 16:48)



So, if you use this value here then we will get it delta J, we will get it delta J nearly equal to V t x dot x star of t. Then x dot star of t bracket close, t is equal to t f into delta t f that is first term, as it is second term of first part we will write as it is second term of first part. We will write as it is and second part we replace delta x t f is equal to delta x f minus x A dot of t, t is equal to t f. That means find out the slope of the trajectory of the optimal trajectory at find out the slope at t is equal to t f, then multiply it by delta t f.

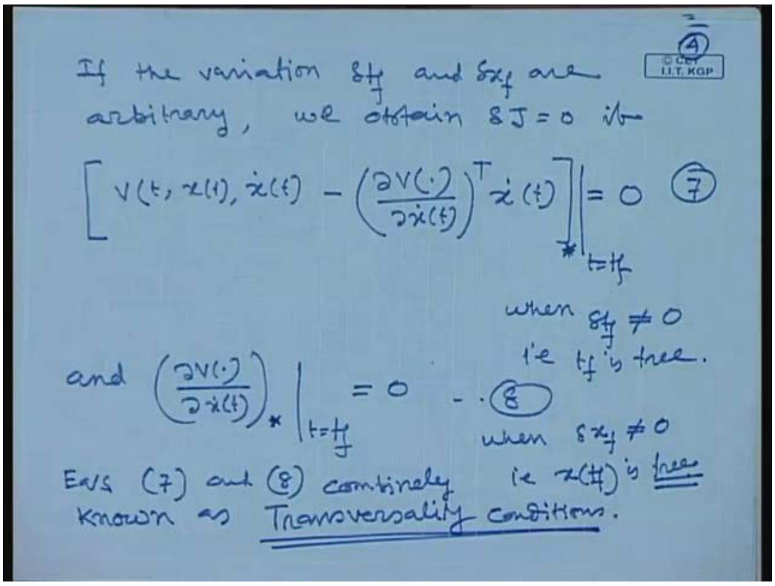
So, this is the second part, I have replaced x delta x bracket t f replaced by that one. Now, see this one what is note, just note what is x t f delta dot x t f is equal to x dot t f star plus delta x dot t f, this one. So, I put this value in this equation and if you put this value in this equation, you see delta x dot t f, this and delta x t f multiplied by x t f. So, this is the small quantity, again it is a small quantity. The second part is omitted, neglected.

So, what is leftover term is here after putting this expression t x star of t into x dot star of t, t is equal to t f delta t f plus V x dot of t whole transpose this. And this you see as it is that one delta x f and this term x A dot t x dot t is the trajectory, never root of the optimal trajectory. That one at this point you are finding out the slope multiplied by del. So, if you put this one, this will be replaced by this one multiplied by delta f. So, this term delta x dot t f multiplied by delta t f is neglected. So, it will be left with x dot star t f.

Now, we can what we can write it just see this one, delta and this is multiplied by this one, multiplied by delta t f. That is missed it here delta t f then bracket close because this one multiplied by delta t f. So, delta t f, delta t f I take common, then you will get V t x star of t x dot star of t bracket close minus del V del x dot of t, whole this star multiplied by x dot star t f. This because delta t f is common in both the expression, this expression and this and this expression delta t f plus left over term is del V x dot t, dividend of V with respect to x dot whole transpose star into star into delta x f this one.

So, this is the equation. Now, delta J will be 0 delta J if you impose, the first necessary condition is the assigned to 0 will give you the optimum value. The point what is called solution of this trajectory will give you the optimum value of the functional. So, in order to make 0 when delta t f is not 0, this must be 0 if t f delta t f is 0, means final time is fixed. On the other way delta x f arbitrary, that means x f that final point of x is arbitrary, then delta x f is not equal 0. So, in order to make that is delta 0, this term must be 0, but if it is fixed that is n point of or trajectory is fixed x f is fixed, but t f is free, then this one will be 0.

(Refer Slide Time: 22:50)



Now, let us see what is the condition can impose in order to make delta J first variation of the functional assigned to be 0, if the variation delta t f and delta x delta x f are arbitrary. We obtain delta J is equal to 0, if we assign that V t x of x, x dot of t minus del V del x dot of t. This one whole transpose x dot t and this value you see when delta t f is not arbitrary, then this cannot be 0, this must be 0. So, this equal to this equal to 0 at t is equal to t f.

So, here I have forget here, I have forgot to write you see this is t f at t is equal to t f. This is also you see x t is equal to t f, agreed? This whole term you see this is t f t is equal to t f, agree? This is also t is equal to t f. So, we can write it here t is equal to t f, this one this is missed here and next is this will be a t is equal to t f. So, if you put it this one here, you see t is equal to t f from there, here I missed it this one. So, this into t f, so whole thing I can write it that whole this and t is equal to t f.

So, this equal to 0 when delta t f is not equal to 0, in other words t f is free if x is free, arbitrary. Then this is not equal to this, you can assign it to 0 and another condition that delta V dot delta x dot of t star, t is equal to t f is equal to 0, agree? So, we have used the equation up to 6. Let us called this is 7, this equation 7 and this equation is 8 and when this is to that delta x t f, x f delta x f is not equal to 0, it means that x t f is free, agree?

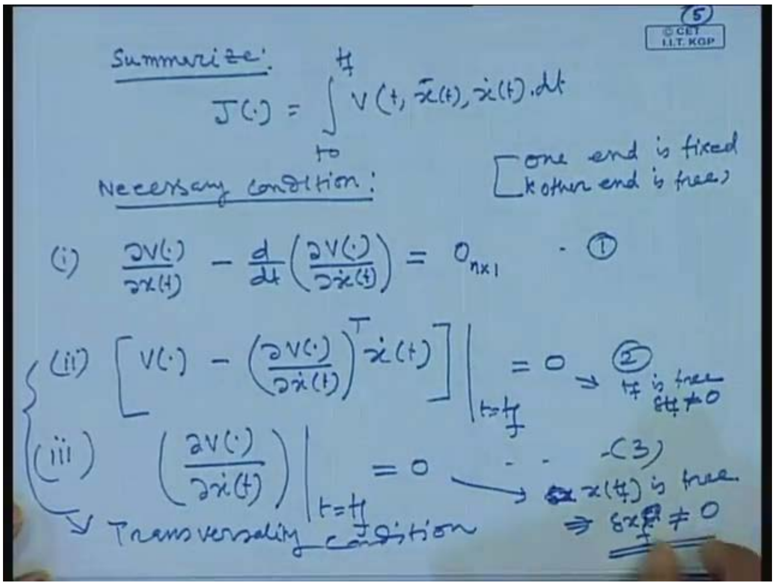
So, equations 7 and 8 combinedly know as that equation 7 and 8 combine known as or called as transversality condition, this is the important true condition. So, what is our condition if you have a problem is like this way if you have a see this one. If you have a J is a functional, you have to optimize this functional. Then what should the choice of our optimal trajectory x t so that this functional will be optimized?

Then we have considered one point is fixed, other point is free. When the time is free x t f is free, both are free then what we obtain first in order to the necessary condition for this one is delta. First variation of the functional must be 0 in order to assign that functional first variation of functional 0, first condition is our what is called Euler's Lagrange equation must be assigned to 0, that is Euler's Lagrange equation.

That is I have just what is that is del V del x minus d of d t, del V del x minus d of d t is d of d t bracket del V del x dot is equal to 0, this Euler's Lagrange equation must satisfied. In addition to this there are two conditions are there where both the pints, that means time and x t f are free. then this two conditions must be satisfied, that condition are called transversality conditions and this is the condition, that equation 5, agree? That this equation, this equation must satisfied.

So, one end is fixed, other end is free, in order to optimize the functional value again J is the functional we have to optimize. We have to satisfy first Euler's Lagrange equation, this equation and in addition to that there we have terminal condition, this 7 and 8. This transversal conditions are called n conditions are called that transversality condition. Simultaneously, we have to solve equation number, this equation and the end condition using n condition we have to solve. Then we will get a trajectory, optimal trajectory for this one.

(Refer Slide Time: 29:52)



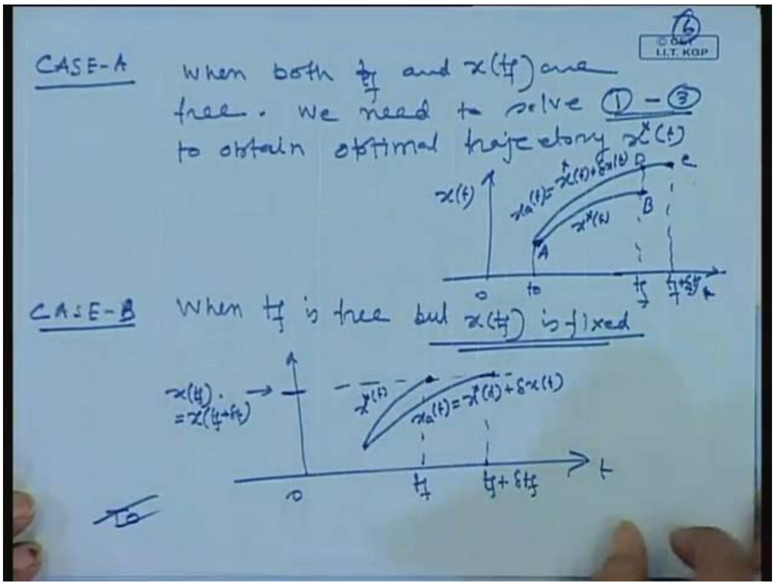
So, next is that what is called if you summarize this one, the results for this one that what we have mentioned this one we have a functional this. This t 0 to t f, V x of t x dot of t d t 1 end is necessary condition, we have assumed one end is fixed. Refer to our earlier figure, one end is fixed and other end is free. So, first condition of this one in order to make first variation of the functional that first condition is del V del x dot t minus d of d t.

Take the variant of this with respect to x 1 dot, this equal to 0 whose dimension in n cross 1. So, whole bracket star you can give because ultimately this equation is you need to solve it, not necessary to put it star. It indicates, the star indicates that if you solve this equation whatever trajectory you get it that is the optimal trajectory. That functional value may be maximum or minimum. At this stage necessary condition will give you the what trajectory will make the functional value is maximum or minimum. That means sufficient condition one has to check it, the next this is the one condition.

Next two conditions, the end condition we have to use for this problem is the called transversality condition, that V dot of x del V dot del x dot of t whole transpose, agree? x of x dot of t this agree? Put t is equal to t f is equal to 0 and this condition if you see, this one this condition are achieved from this condition. That means when t f is free, when t f is free this condition must be satisfying. So, whether you give it start or it does not, you have to solve this differential equation with this final condition. So, this is the one condition and the next condition is that del V dot del x dot, this t is equal to t f is equal to 0.

So, let us call equation number 1, equation number 2 and equation number 3. This is derived when t f is free, t f is free that means delta t f is not equal to 0. This is true when delta x f or you can say x t f is free, implies that delta x f, x t, x delta x f is not equal to 0. So, equation 1, 2, 3, equation 1 you have to solve in order to solve equation. The n condition of equation 2 and 3, 2 and 3 you have to use it that two things combined I told earlier transversality conditions, transversality condition, agree? This is called 2 and 3.

(Refer Slide Time: 34:40)



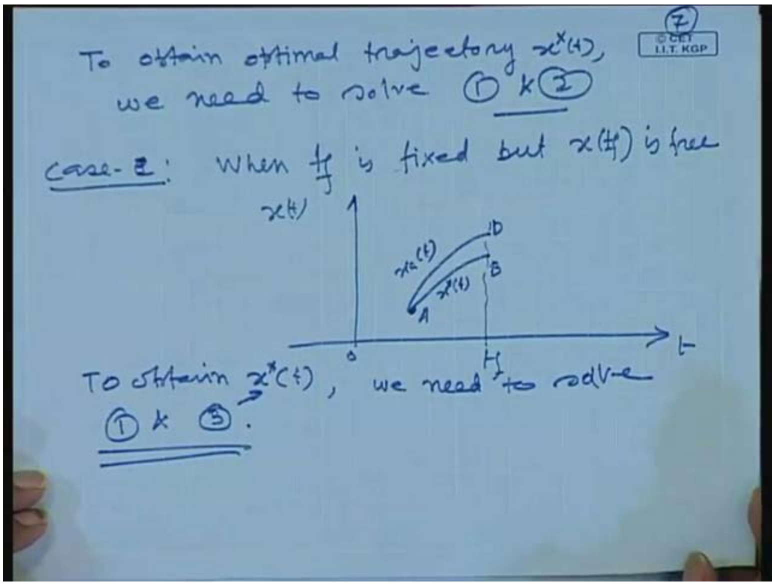
Now, in our case we take different cases, we know this one if you consider the case A, let us called consider case A. What is the case A? Case A is when both t f and x t f are free. We need to solve 1 to 3, this three equation you have to, need to solve. 1 we need to solve, 1 into 2 to obtain optimal, to obtain optimal trajectory. x star of t x is n dimensional case. Then this case t, then you know both the ends its free, that means corresponding figure if you see this is that one AB and this is D and this is C, this is t 0 t f.

This is t f plus delta t f, agree? And this point is you know this is x star, this trajectory and this is x a of t, which is equal to x star of t plus delta x of t. So, when both are free then you have to use equation number 1 and 2. Then some special cases can be derived from general case B and general case A when t f is free, but is t f is fixed. This is fixed, that means this corresponds to curve is like this way.

Suppose, this is you consider x star of t neighboring near very close to this optimal trajectory. There is a another trajectory x of t is equal to x star of t delta x of t. So, you see x t f, this is the x t f. So, x this is fixed, that means x t is equal to this is fixed, agree? That is fixed, but t f that is a t f plus delta t f, you just consider this situation like this way that our time t is equal to t f. This trajectory must reach x t f value, this at time t is equal to t f plus delta. If that very never root of this trajectory also should reach at this fixed value t f. What is the t f value?

It should reach to same below, that means x t f, this x t f plus delta t f, this must be both the value is same, this one. So, in this case you see this condition, that means x t f is fixed, x t f is delta t x t f is delta x t f is what, here you see that delta x t f is O. So, this condition will not come into the picture, only this condition. So, our we have to solve equation 1 and 2 to obtain optimal trajectory, to obtain optimal trajectory we need to solve 1 and 2, agree?

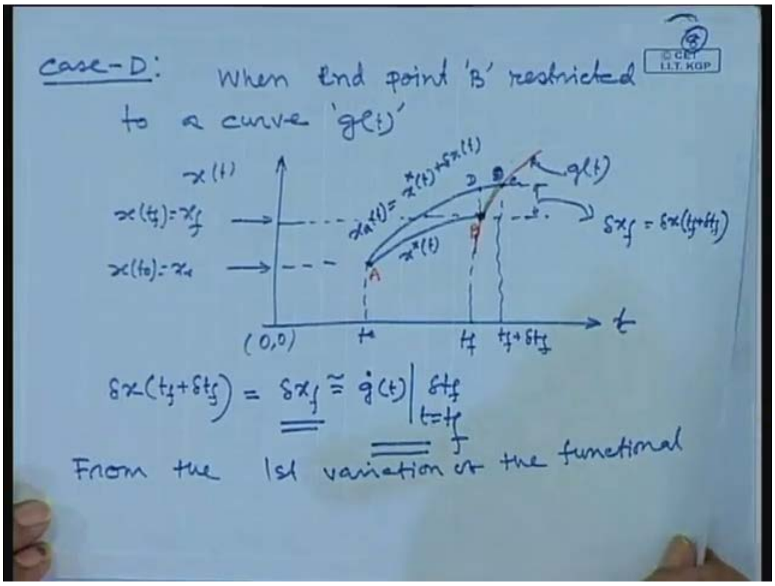
(Refer Slide Time: 39:06)



Now, case 3 or case C, you can say case C when t f is fixed, but x t f is free. That corresponding figure you can visualize like this way, t this is x of t and this is t f is fixed, t f and this is let us call 3, this is you can say ABD, agree? Here also if you want to give name of this one, this is AB and this is D, this is x star of t, this is a of t. So, in this situation you see t f is fixed, delta t f is 0. So, if you see this one our that equation delta j expression then delta t f is fixed, means t f is fixed, delta t f is 0.

So, this term will not come in to the picture in the end conditions. So, this is not equal to 0 because x t f is free, delta t f is not equal to 0, this condition is 0. So, we need to solve to obtain x star of t, optimal trajectory x star of t means optimal trajectory. In order to make the functional value is optimized, we need to solve, we need to solve equation 1 and 3. So, more general expression is one can write if you restrict the free end, that means point B restrict on a curve, then what will be the transversality condition corresponding to that point.

(Refer Slide Time: 42:03)

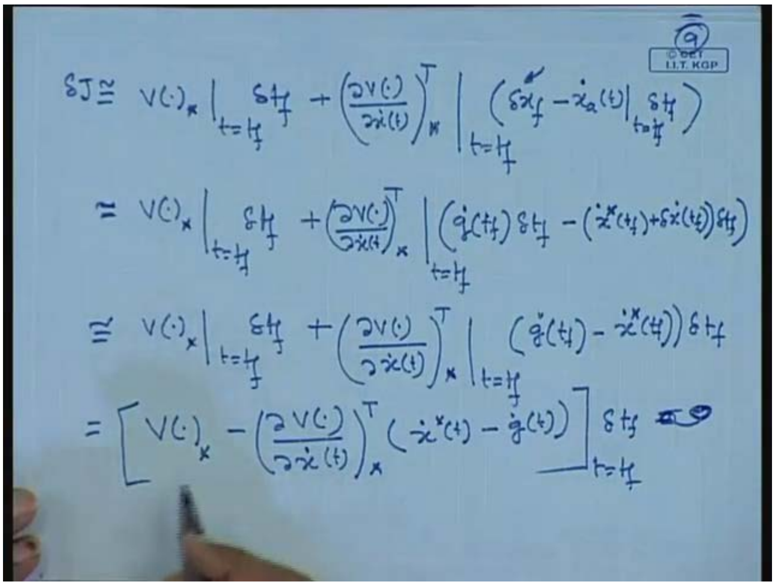


Case 5, that case D or 4, case D when the end point B restricted to a curve g of t, agree? Then what will happen, then corresponding diagram if you draw it here this is x, this is t, this is x of t, this is point this and our curve is g of t curve, the end this is the A point, this is your B point, B point is restricted on a curve.

So, our neighborhood of these optimal trajectories, there is another trajectory. We assume that one, so that is our D point, agree? This corresponding t is D point and this is corresponding to C point. So, this is x a t is equal to x star of t plus delta x of t. So, this is your what is called g of t, the end point of the trajectory restricted on the curve which is function of time. Then what will be the condition, but Euler's condition will remain same. Now, what we can write it for this one, see this is our, if you look at this one this is our t 0. This is t f and this is our t f plus delta t f and this corresponding point is x t 0 is equal to x 0 and this point corresponding to this one x t f is equal to x f, agree?

So, you can write it delta x t f plus delta t f, agree? That means this here to here is this one. I have denoted by delta x f which is nothing but a delta x t f plus delta t f. This is can write it is equal to delta x f which is which is same as you can say B to the g of t, take a tangent, make a tangent at the point of B. So, I can write it this nearly equal to g of t dot, find at t is equal to t f multiplied by delta t f, that is nearly equal to I can write it. So, x delta x f I just replaced by that one. Now, if you see this expression when you have derived that one, that is from the we can write it from the first variation of the functional. We rewrite this expression once again, that means from the first variation of the functional if you see that one this expression.

(Refer Slide Time: 46:47)



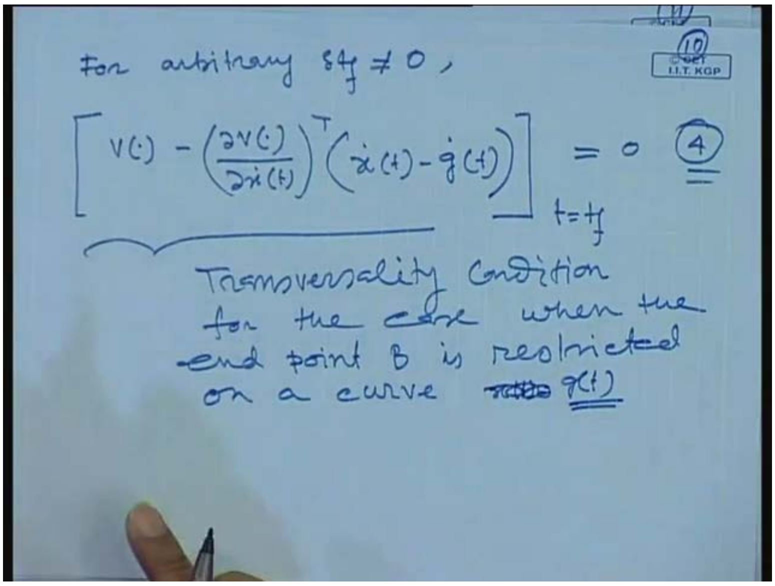
I am writing once again for you convenience, agree? So, delta J or I have given you the equation number, no I have not given. So, it is a del J is nearly equal to del V dot that star t is equal to t f delta t f, that is t is equal to t f delta t f plus del V dot del x dot t whole transpose. Then star t is equal to t f into del x f minus x a dot of t, t is equal to t f delta t f. Now, look this one, this value I will replace by what we made an approximation here, that is that we will replace it that one. If we replace that one then it nearly equal to V dot this star t is equal to t f delta t f plus v dot, this transpose x dot of t whole transpose star t is equal to t f and delta t f, what we will replace delta x x f.

We will replace by g dot t f, see g dot at t is equal to t f g dot t f into delta t f into delta t f and this expression if you know, this expression is nothing but x t a, x a t is nothing but a x t star plus delta x t f, agree? So, this is nothing but I write it that is x star t f plus delta x t f t is equal to t f, that one into delta t f bracket close. Now, this equal to delta x of this term is omitted, this both are small quantity of that one. So, this is can write it, now if you say that we dot of star t is equal to t f delta t f plus this term is omitted. So, it will be a del V dot function of or dividend of V with respect to x dot whole star, then find t is equal to t f, agree?

Then you can write it g dot t f minus x dot, this is dot, this is the dot because x a dot. So, dot star t f, this one into delta t f, agree? So, this can further I can simplify V dot is equal to del V dot del x dot of t whole transpose, agree? Whole transpose if you want to write it star, you can write it here star, you can write it here multiplied by x dot star of t minus g dot t whole bracket because we have t is equal to t f t is equal to t f main.

This value you call at t is equal to t f, this value you calculate at this value. You calculate at t is equal to t f, just I change the this I bring it first and this I bring it. So, minus sign has come here, so into t f is equal to 0 this one. So, in order to make this is 0, in order to make this is 0 then delta t f is see in this case. That is what you say delta t f is free, agree? So, this is not equal to 0.

(Refer Slide Time: 51:55)



So, this must be 0, so our condition for arbitrary delta t f is not equal to 0. Our condition the transversality condition case boils down to this way, V dot of this del V, the del x dot of t whole transpose x dot of t minus g do of t that this whole t is equal to t f. You can put star because it is boundary condition, you have to solve Euler's equation, equation number 1 as we have mentioned it here. That equation and this equation when your free end restrict on a curve, that equation number 1 and 4, we have to solve this is the equation number 4. So, this is also called transversality condition.

Transversality condition for the case when the point when the end point B is restricted on a curve x t is equal to g t, only curve g t. So, in short when the free end B is restricted on a curve, on a curve g t and the another end is fixed. Then we have to solve Euler's equation which is equation number 1 and equation number 4 that we have to solve it and now question after solving this one you will get the trajectory x of t, x star of t or x of t which is the optimal trajectory.

What is the, what you will ensure that this trajectory will give you the optimal value of the functional optimal, means whether it will give the maximum value of the or minimum value. J will be minimum or maximum that one, for that one, one has to check what is called the sufficient condition for this one. So, next class we will discuss the sufficient condition to establish the, what is the functional value is minimum or maximum. So, we will stop it here.