Optimal control

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Lecture - 31
Dynamic optimization problem: Basic Concepts and Necessary and Sufficient Condition (contd.)

So, last class we are discussing about the how to optimize a functional; functional is nothing but a function of a function.

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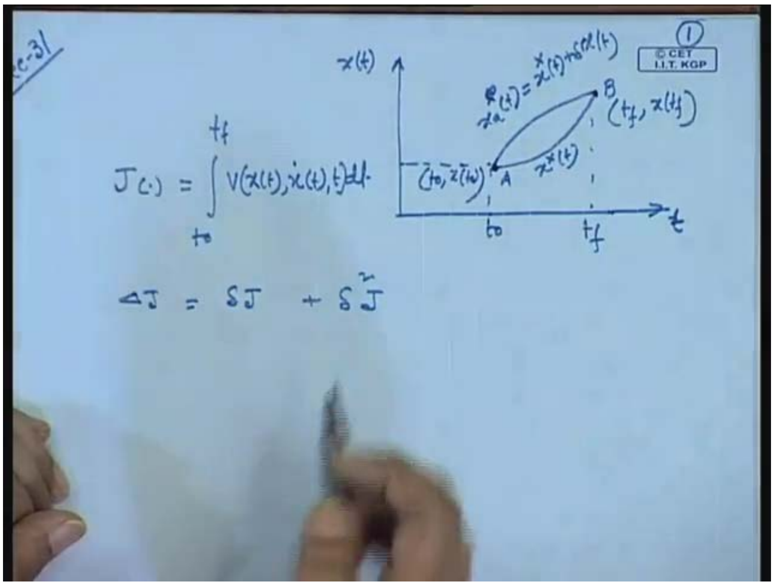

So, if you recollect this one that our problem is that A is point and B is another point, then our problem is to minimize j is a functional which is expressed mathematically t 0 to t f, again then v x of t which is a function of x dot of t and t dt. So, this point A is a point B is a point both the points are fixed, and this is the t and this is x of t. So, A and B both are fixed that time at time t equal to 0 the point A and its coordinates you can say this one t 0 x of t 0 this one, and this coordinate is at time t equal to t f, this coordinate is t f x of t f. Our problem is to find a trajectory between the 2 fixed point A and B in such a way, so that the functional is optimized either maximized or minimized that is our problem.

So, let us call this is the optimal trajectory x star of t, and near about this optimal trajectory. There is another trajectory is there which is neighboring of this trajectory, that is called that is x star, so x a is equal to x star of t plus delta x of t this one. So, our problem is I told you find that the optimal trajectory x such that this functional is minimized. So, for that one if you recollect we consider first that what is the incremental, what is called functional that means del J, we found out and del J is a two parts after doing the expansion then their first part of what is called is the necessary conditions for the functional to be minimized or maximized and second variation in functional is the first variation of functional. We denote it by del J and second variation of the functional is del square J and higher terms of expansion is neglected that. So, our first condition this delta of J is the increment of the functional increment of the change in functional values and delta J is first variation in functional value and delta square J is second variation in functional value.

So, our first condition is delta J must be equal to 0 that is the necessary condition in order to get this one we got it in step three if you recollect yesterday that what we got it, we got that del J is equal to t 0 to t f delta v x dot whole transpose. This value evaluate this is the evaluate along the trajectory and multiplied by delta x delta x is the change in coordinates of x when it is moving along the neighboring trajectory of optimal trajectory x star. Then second part is del v del v del x dot t whole transpose and put this value evaluate the value x equal to x star x dot equal to x star dot multiplied by delta x.

So, this second term of this one yesterday we have shown how to simplify this one, the second term I have written it here. See, instead of delta x t by definition d of d t delta x t into d t as it is so nothing but if you consider this is v delta t delta t you can consider something like this consult delta v d del d v and this whole quantity is known to you which is constant. So, u d v integration from 0 to t 0 t f I can write is nothing but a u v and its limit 0 t 0 to t f minus v del v del u this is the del u and I multiplied d t and d t this one, so this things at this term which can express into sum of two terms. Now, look at this one because both the points are fixed a point is fixed along the optimal trajectory or along the part of trajectory which is very close to the optimal trajectory both the ends A and B are fixed.

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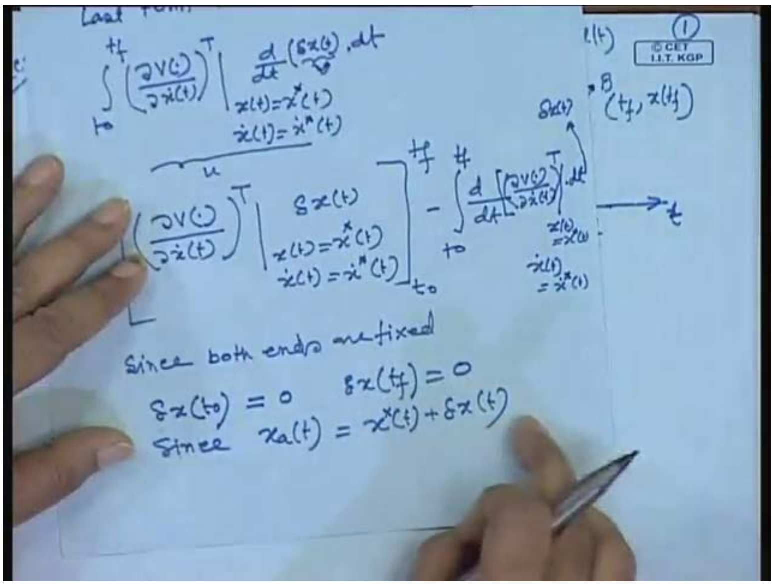

So, this delta x t if you see there since both the ends are fixed then I write delta x at t is equal to 0 is nothing but a 0 and delta x t is equal to t f is equal to 0. Since, if you see our delta a of x t is nothing but a x star of t plus delta x of t and t is equal to t 0 x of t 0 and x of x star of t both are both are at the same point. So, delta x t delta x t must be 0, so that is why this is their similarly, at t is equal to t f here at t is equal to delta x t t is equal to t f delta x t is 0 because x star of t and x a star of x a is equal to f both are same value delta a is 0 if you put this condition here the first will not come into the picture only this term. So, this the second term of this first variation of functional can be written as this term when both the ends are fixed.

So, keeping that thing in mind delta J, now I can write it here if you if you use this expression in equation number three equation numbers three. Then I can write it you see this limit is t 0 to t f delta t transpose this when x is equal to find out this value x x t is equal to x star x dot t is equal to x dot star into delta x this is replaced by this 1 d of d t del v x dot of t transpose evaluate.

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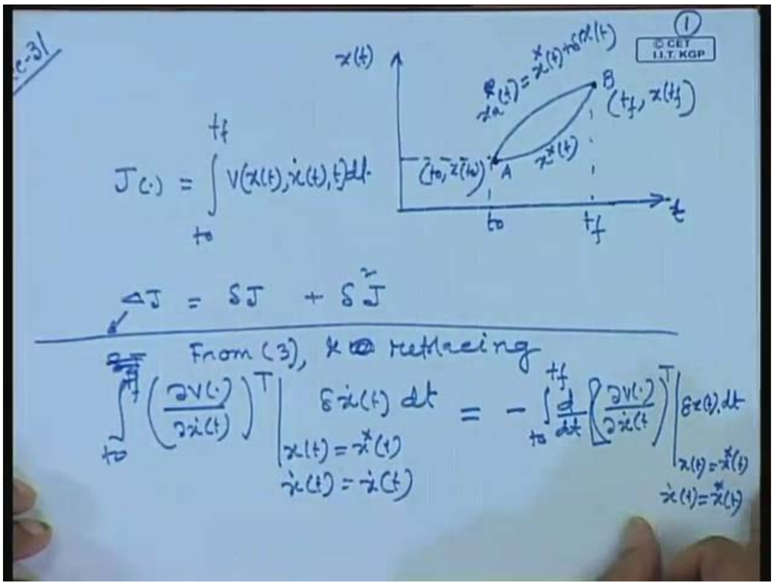

This values x t is equal to x star t x dot t is equal to x star dot into delta x I have considered. Let us call x is a scalar variable x dot is a scalar variable, so whether you multiplied because delta x I keep it here. So, if you do this one from equation three, I can write from equation three and using the expression and replacing that del v dot del x dot of t whole transpose x t is equal to x star of t x dot t is equal to x dot star of t delta x dot of t into d t that things whole thing, so this limit is t 0 to t f.

This I can replace by is equal to minus that means from equation replacing that quantity minus this quantity replacing is replacing by this one t 0 to t f d of d t del v dot del x dot of t whole transpose. Put the values, evaluate the values at x of t is equal to x star of t and x dot t is equal to x star dot of t this in this equation you just put it, then into this into what we can write it this into our del of x del of x into d t.

So, if you put in equation three replacing this by this then finally, we will get the expression like this way integration t 0 to t f t 0 to t f. This is the del v dot del x t whole transpose minus d of d t del v dot del x dot of t whole transpose that this curly bracket. Then put x t is equal to x star of x dot of t is equal x star dot x star dot of t into delta x t into d t. So, if you put this equation here and after simplification I will get it this let us call this equation is four, now this left hand side is the delta J.

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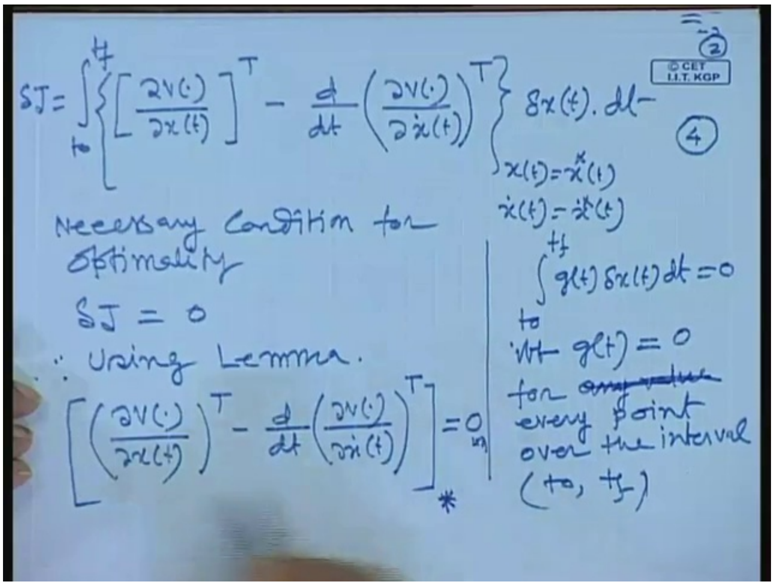

So, in order to become that functional to be optimized either extremal or minimal then what is called this del J must be 0. If you recollect lemma which we discussed yesterday that if t 0 to t f and g of t delta x t into d t is equal to 0 if and only if g t is a continuous function g t is equal to 0 for any value, for every not the for every point of every point over the interval over the interval t 0 to f. That means this quantity is 0 if and only if necessary and sufficient condition g t will be 0 for every point over the interval if it is so I will apply this lemma here.

So, you see whole thing I can write g t which is a continuous function at t is equal to x t is equal to x star x dot is x dot star, because I am doing the Taylor series expansion along the point x star along the optimal trajectory path. So, this is our g t this is the x t this is not equal to 0, so this quantity will be 0 provided this is equal to 0. So, using the necessary condition for optimality del J must be equal to 0 and by using lemma we can say this del J will be 0 provided this equal to 0 this implies using that lemma using lemma that is what we have discussed.

Now, our lemma is there g is a continuous function over the interval t 0 to t f and this g t if you have multiplied by non zero change in delta x into t over the interval t 0 to t f will be 0 if and only if g t is 0 for every point over the interval t 0 o t f that is this and using this lemma I can write that this 0 means this must be 0. So, our del v dot del x of t because this is a column vector transpose row vector multiplied by del delta x, if it is a x is a multi variable case, if it is a single transpose is not necessary.

So, this transpose d of d t del v dot del x dot of t whole transpose this you evaluate this values at now x is x star x dot is x dot star. I simply write star means x t is replaced by x dot star x t is replaced by x star of t and x dot is replaced by x dot star of t along the trajectory or of optimal trajectory path. So, this is equal to 0 if you write transpose that will be 1 cross n, if x is a vector, if you remove this transpose then it will be 1 cross n, if you remove this transpose.

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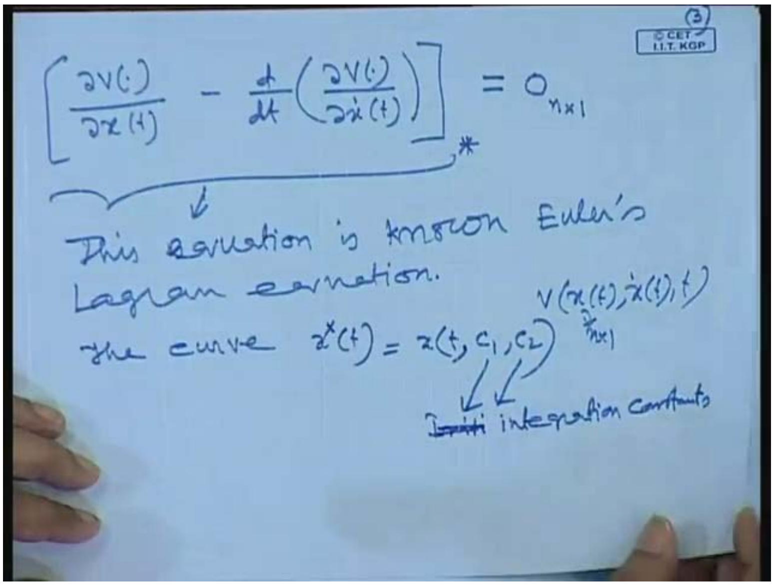

If you remove the suppose we remove the transpose this will be x of x t minus d of d t del v dot del x dot of t this evaluates this value at x is equal to x star x dot is equal to x dot star, if you evaluate this value is equal to 0 where the dimension is n cross n. So, this equation is known as, so this equation is known as Euler's Lagrange equation and it is the necessary condition or the functional to be optimized either maximum or minimum. But it is the necessary condition for this one a function to be optimized; next we have to check whether the function is minimum or maximum that sufficient condition we have to check. It is same as what we did it for what is called static optimization problems, so and if you see this one v is a function of x and x dot and t.

So, if you differentiate this 1 with respect to x dot which will be coming function of x x dot and so on and again you are differentiate with respect to t. So, this will be a second derivative of x will come, so this is nothing but a second order differential equation. Second order differential equation if x is a single variable case, if x is a vector then we will get for n 2 and differential equation from this because for each variable, we will get two differential equation. If we have a variables are there the function that integral part that v, v is express of this the integral part of this integral this and this x is a n cross 1 then how many variables are there, n variable for each variable we will get second order differential equation.

So, all together it will be a twice n differential equation in general these differential equations are non linear time varying equation, so if you solve this Euler's equation you will get the necessary condition for this one. So, the solution of this one and you can say it is a if x is a scalar then it is a second order differential equation, if x x is a vector of dimension n then it will be a twice n differential equation they are coupled each other. Now, this next is the solution the curve the curve x star of t a solution of Euler's equation x star of t is equal to x which is a function of t, c 1 and c 2 and c 1 c 2 are the initial, what is called initial integration constant, integration constant.

Suppose, x is a single case it is a second order differential equation, we have two integration constants and that integration constant can be found out from initial and final condition of the problem. Suppose, x is n variable case then we will get that how many constants 2 n constants we will get it that 2 n constants, 2 n constants we can find out by using initial and final value conditions. So, if you make it the remarks about what is called Euler's Lagrange equation and this is nothing but what is called necessary condition if functional is to what is called optimized, this is the necessary condition.

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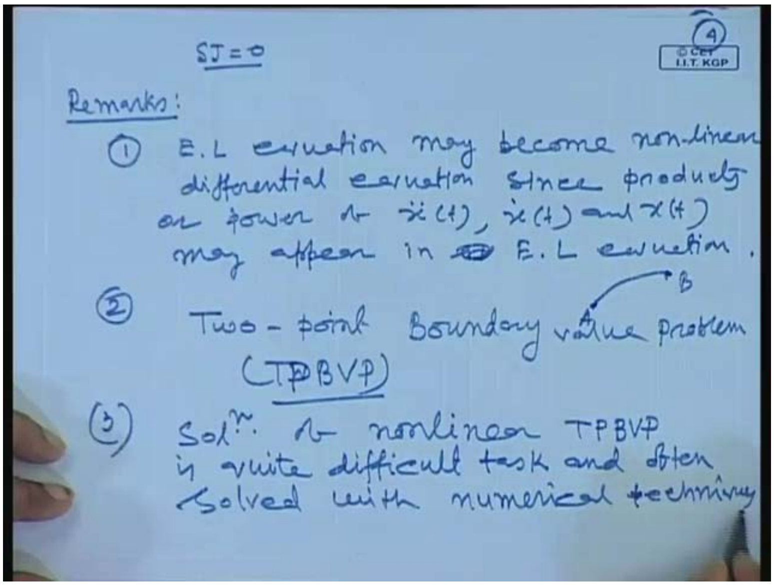

That means what is the necessary condition del J is equal to 0 implies that Euler's equation must be satisfied. This equation for that one you find out the solution of x t x of t and that is the optimal trajectory which will give you the minimum or maximum value of the functional. So, this now remarks on what we have seen Euler's equation is a non linear differential equation, why it is non linear because after getting the Euler's equation you will get this, this is the product of either x into x dot or x dot into x double dot or x square into x dot or x square into x double dot and so on.

So, it is a non linear differential equation first remark is the Euler's equation may become non linear differential equation since products or power of x double dot of t x dot of t and x of t may appear in there. There may appear in the Euler's Lagrange equation this is the first thing second thing that what is called knowing the initial and final value.

The two point initial and final point because x may be n dimensional, but one point is that, that is what we have considered knowing the initial point A and this is the final point B. There are two points are there in that sense it is called two point boundary balloon non linear differential equation two point boundary value problem, if t w t two point TPBVP two point the solution of such type problems non linear differential equation is not a easy task to do this one. So, gently you people can solve this one using what is called numerical technique, one can use numerical to find out this one next point.

So, you think solution of solution of non linear TPBVP is quite difficult analytically, quite difficult task analytically, but and often solved with numerical techniques. Third is if Euler's Lagrange equation is satisfies that it indicates that Euler's Lagrange equation is satisfied. It indicates that we have a extremal value of the functional either minimum or maximum if does not satisfies the Euler's equation agree it indicates that functional value J does not have any optimal value of this functional value if does not satisfies E L conditions.

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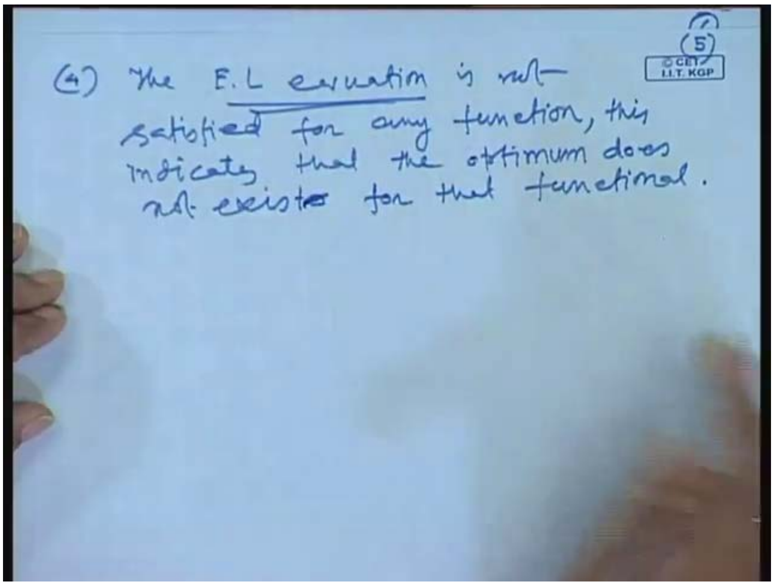

So, we can write the first remarks the E L equation, Euler's Lagrange equation is not satisfied for any function, this indicates that the optimal that the optimum does not exists, does not exist for that functional. So, we have to satisfy this 1 other wise this does not exist for the optimal value of the functional, so let us see in multi dimensional case. Now, that I told you if it is x is n variables are there. So, we will get 2 n differential equation 2 n coupled differential equation which we need to solve for finding out the solution of what is called optimal trajectory x which you will optimize the functional.

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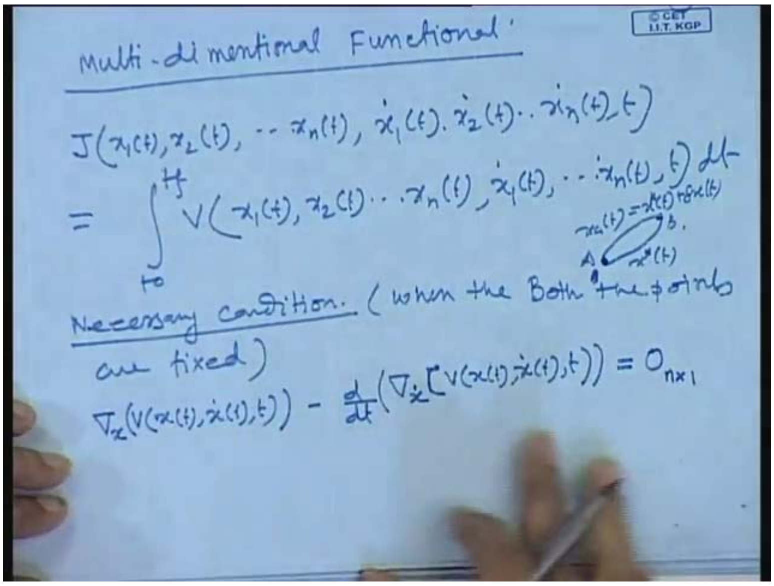

Now, mode multi dimensional functional, next extension of single variable case suppose is a function which is function of x 1 t x 2 t dot x n of t and is also function of derivatives x dot of t x 2 dot of t dot dot x n dot of t. This is a function of time which is equal to t 0 to t f v x 1 of t x 2 of t dot dot x n of t. Similarly, x is derivative dot dot x n of t dot into t and d t, so this is the multi, so what is the necessary condition. Let us call two fixed point both the points are in n in dimensional space that we have two points are that A and B and both the points are fixed.

That means optimal trajectory starts with a point and ends with B point that means it starts A point ends with B point. Neighborhood of this is the optimal trajectory for the functional to be optimized and neighborhood of this optimal trajectory there is another trajectory x a t which is equal to x star of t plus delta c t which is very small from along the trajectory x star. So, both the points are fixed, but we have a n dimensional variables this one.

So, what is the necessary condition necessary condition when both the points, this is a point are fixed is same as this one what we have write it if you see that one or this differentiation of scalar quantity with respect to A. If it is a scalar with respect to x then this differential form this to x is a scalar suppose x is a vector then differentiation with respect to vector, would it will give you the column vector is nothing but a gradient of v with respect to x. This is nothing but the gradient of v with respect to x dot and then because if you do this one column vector, then you multiplied by sorry differentiate with respect to time.

So, in short if I write it like this way is nothing but a gradient of x gradient of which function v v dot which v is function of x x dot of t and t you take the gradient of with respect to x this indicate with respect to x agree with respect to x minus d of d t. Take the gradient of x, sorry gradient of v v x of t x star of t, sorry x dot of t t. So, you get the take the gradient of v with respect to x dot that you that you multiply differentiate with respect to this.

Then, you will get it by left hand sums this is the Euler's equation I am writing same equation only using the symbol gradient here that equation will be n cross 1. That means each if you see this one del v of del x 1 del v of del x 1 del v of del x 2 del x 1 that expression del v of del x 1 minus d of d t del v del x 1 dot is equal to 0, whose dimension is square right hand side. That means this is a second order differential because am differentiating with respect to x 1 dot with respect to time is the second order differential equation.

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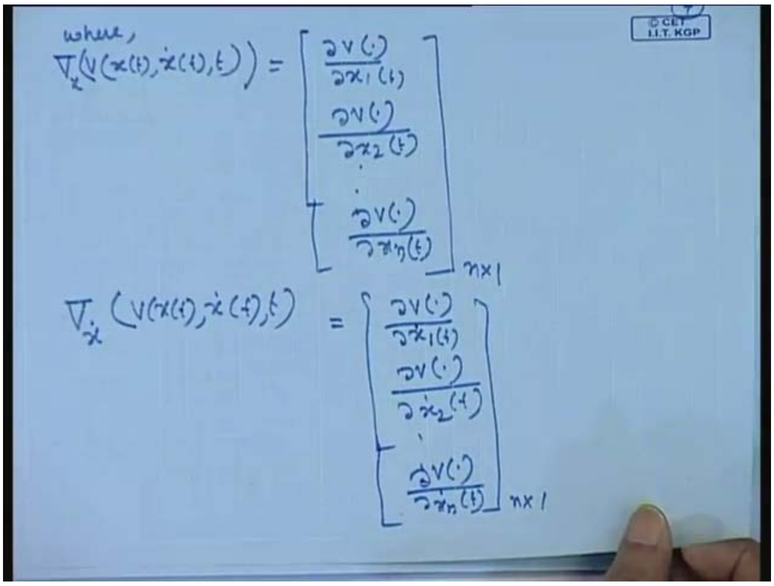

So, and if you see this the gradient of v with respect to where you can write where gradient of v with respect to v is functional of x x dot of t t. I am doing with respect to x is nothing but a del v dot del x 1 del v dot del x 2 dot del v dot del x n, what is the dimension of this 1 n cross 1 and each is a x 1, x 2, x 3 is dimensional. Then gradient of v with respect to x dot, what you will write it same thing del v del x 1 dot del v del x 2 dot and dot dot, last one will be del v dot del x n dot of t and this again dimension is there.

This quantity again if you differentiate, you will get it this is the function of x 1 dot you will get it and again you are differentiate x 2 double dot, so whole expression Euler's expression for each variable you will get second order differential equation. We have a n such differential equation we will get a twice n differential equation of that one.

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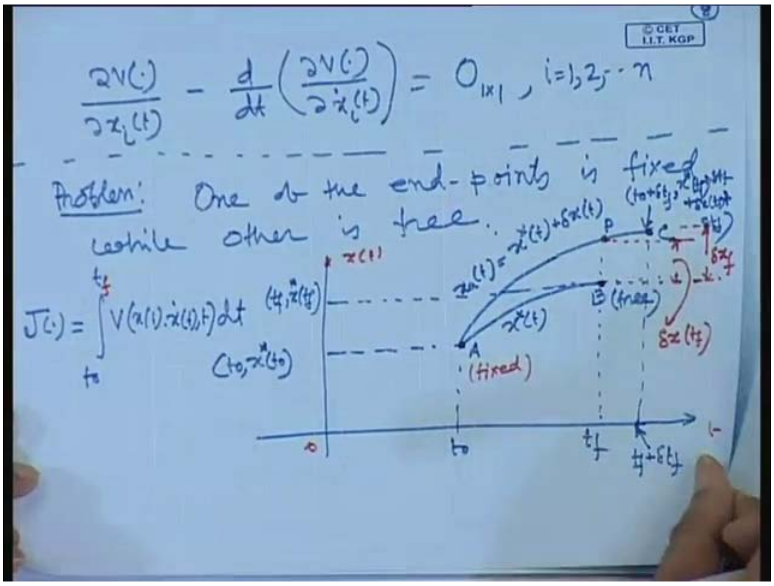

Ultimately, if you write it this one I can write it del v this expression more x i of t minus d of d t is equal to del v with respect to x dot of t this whole thing is equal to 1 cross 1 and i varies from 1, 2, n because we have a n variables are there. So, let us call now you instead of a specific problem both the ends of that trajectory is fixed then we can go for more general problem. So, our problem more general problem is like this one end is fixed other end is free, so one of the end points is fixed while other is free.

So, let us see the corresponding diagram for this one, again our job is this one this is the functional which is the function of x t, x dot of t and t, t 0 to t f, v x of t x dot of t, t d t. We have to find out the trajectory x t optimal trajectory such that this functional is optimized either maximized or minimized and what you making one end of this trajectory is fixed, other end of the trajectory is free. So, this is the one and this is the other point is B, then you have A, this is the t f this is the fixed point A with its intent t is equal to 0, this line is free t f plus delta t f.

This is our optimal trajectory, assume this is the optimal trajectory which will minimize this optimum what is functional and near about the optimal trajectory. There is another trajectory is here x a of t is equal to x star of plus delta x t this point is d now and we have A this point if you see this one, what is this corresponding coordinate of this one be B coordinate. What is the coordinate of A, immediately I can write it coordinate of x a x t 0 of t, what is coordinate of B is nothing but a f x of t f x star of t f or you can write x star of this 1 x t f this two coordinates. Now, so our point c this is the point c and coordinate of point c is what c t 0 plus t f and what is this one that value x t f is equal to t f x t f plus x t f plus delta x delta x t f.

So, you can write it x t f plus t f, what is this coordinate this is nothing but a x star x star t f plus x star t f plus delta x t f x star t f is what t f plus delta t f and x t is equal to what t f plus delta t f. So, you see t is what t is x t f plus delta t f, then this is t f plus delta t f that is what I am getting x star t f plus delta t f value and delta x x t f plus this is the coordinate of this one. Let us call this value is what is called this point is free, it is not fixed it can be anywhere here.

Now, from here to here what is this values at t is equal to t f, what is the value of that one is nothing but a delta x at t is equal to t f delta x t f. Then at this point at time t is equal to t f, what is this one this is nothing but a delta x f plus delta t f which I am denoting by a delta x f delta x f, this one and this is x t f and this value if you see. If you see this one is nothing but a delta x is equal to t f plus delta t f which I am writing in x delta x suffix f, so this is 0 this is time t and this is the value of x of t expression.

So, we are job is here one point is fixed this is the fixed point and B point is free, so what we have to find out you find out B point is free means x t f x, t f x star t f is free t f is free. So, our job is to find out a optimal trajectory x star t such that this functional is this is t f that functional value will be optimized. Then this is our problem, but B is free now near about the optimal trajectory, we have considered another neighborhood trajectory of x star is x a t x star of t delta x t.

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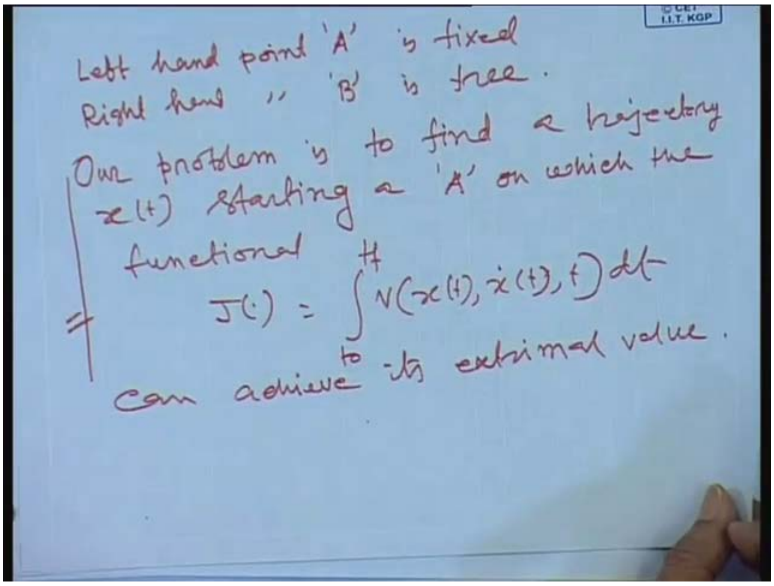

Now, you can describe this things like this way let the left hand point A is fixed right hand point B is free, our problem is just now I have mentioned it our problem is to find is to find out a trajectory. Trajectory x star of t starting at point A on which our problem is find out the find a trajectory starting at A on which the functional value j is equal to t 0 to t f v, x t x dot of t t d t can be can achieved its extremal value or extremal value can achieve its extremum or extremal value. So, that is our problem, this is our problem, so we will proceed in the same manner as we did for the two points of the trajectory is fixed I will just identically, we will proceed in the same manner.

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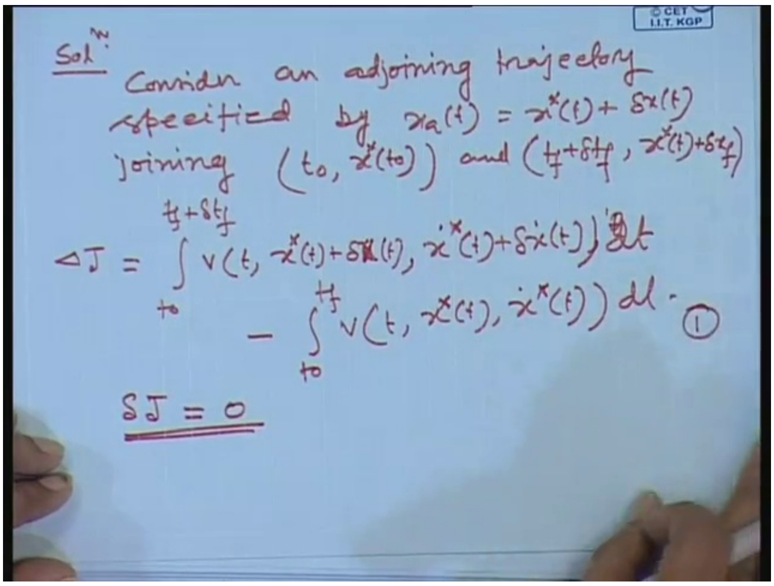

So, our solution if you see that is why I consider we have a adjacent adjoin we have just joining A and C. Consider an adjoining trajectory specified by x a t is equal to x star of t plus delta x of t joining which two point A point and C joining t 0 x of t 0 or x star of t 0 and t f plus delta t f and x star of t you can write it x star no problem plus delta x f what is delta x f delta x t is equal to t f plus delta t which I am writing delta x suffix f that is this one.

Now, what you will do this one, again find out the incremental value of the functional that means del J, del J incremental value of the function, but now our time is you see from t 0 to t f plus delta f time interval because t is free. So, it is t 0 to t f plus delta t f that is if you see that one is the direct at t f and our functional value is along the what is called adjacent trajectory that means x v of t x star of t plus delta x of t delta x of t comma x dot star of t plus delta x dot of t this into time already.

You have written d t minus t 0 to t f v t x star of t plus x dot star of t d t, this is the in incremental value of the functional and it has a two parts because if you consider the Taylor series expansion in the same manner. That will keep the first variation of the functional and second variation of the functional other variation of functional is neglected because delta x is very close to the optimal trajectory of this one, so its cube square is neglecting.

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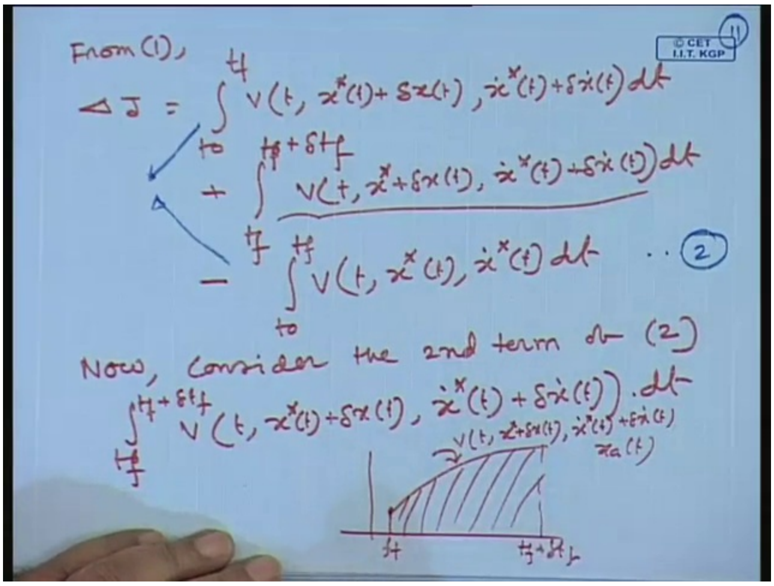

So, for necessary condition what is the necessary condition the first variation of the functional value will be 0, J equal to 0 this is our necessary condition. So, let this is equation number one from equation number one, what we can write it t into t 0 into t f and f to t f t f plus delta t f this integrant part.

So, from equation one delta J is equal to t 0 to t f v t x star of t plus delta x of t x dot star of t plus delta x dot of t this into d t plus t 0 t f to t f plus delta t f. So, this part this part I split up into two integral part this and this is augment is same x star of t delta x t plus x dot star of t plus x dot star of t d t minus whatever the along trajectory t f along the opposite trajectory this v t x star of t x star dot of t d t you.

If you recollect this one from this and this one we have computed this one earlier case where both the points are fixed this, we have done only when both the points only the difference is that. Here, that one end point is fixed, another end point is free this in this problem, so let us see let us all this is the equation number two. Now, consider mainly concentrate with this term the second term of equation two, consider the second term of two, so what is the second term of this two t 0 t f to t f plus delta t f v t x star of t plus delta x t plus x dot star of t plus x dot of t.

The whole d t, now look at this point to find out this integrant part of this one value along the neighborhood of the optimal trajectory this functional value. This is the scalar quantity find this scalar quantity value function along what is called neighborhood that optimal trajectory, what we have considered x a of t along that one. You find out the function value for each time up to t plus delta t f. So, what we have expected it is something like this let us call it is something like this it is t f I am just magnifying that 1 that is a t f plus delta t f what is this area under this curve during this period this one.

If it is a v t x tar delta t that mean along the x a and this is x star dot t plus x dot dot of t during region what is the area under this curve can find out because delta t f is very small though I am amplified to show you this one. This nothing but A area approximately I can write it area under this rectangular part, so this I can write it what is this point at t f, what is the value of function value of this function t is equal to t f this value function. That mean this multiplied this, so this rectangular area will be the same as whole shaded ground because t f is very small even though, so simplified this one.

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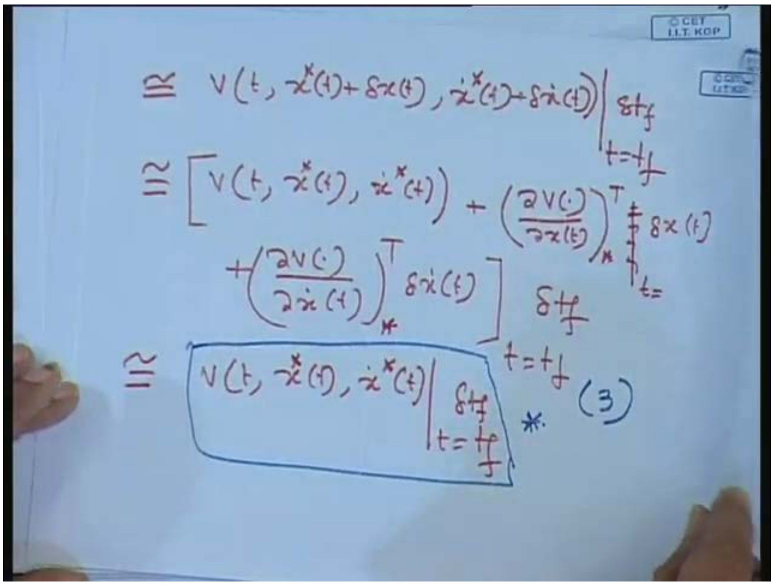

So, I can write it this nearly equal to I can say nearly equal to v t x star of t delta x t plus x star dot of t delta x dot of t find out the value of the function at t is equal to t f that is what we have studying at t is equal to t f. You find this value and this ordinate multiplied by x the delta small delta t f, what is called area under curve into delta t f. This is under approximately because t f is very small, so this I can write it nearly equal to if you do the Taylor series expansion of that 1 is it is nothing but a v t x star of t plus x dot star of t this plus v dot of this X of t whole transpose if it is a vector then this transpose.

If it is a scalar it is not necessary at all, so this if you are because this scalar at t is equal to t f t is equal to t f because I will write the Taylor series expansion. Then I will whole thing I will write t is equal to t f this into delta x del d of x t plus del v dot del dot of x t whole transpose. Then because along the trajectory of this, then this is will be x dot of t this whole thing nearly equal to I have written it that one and what is that value I am around the x star.

That is why x star I have written it this one that whole thing if you write from here to here evaluate at t is equal to t f multiplied by delta t f multiplied by t f. So, if I push delta t f here delta x t into delta t f both are very small quantity you can neglect that one and if you do this one delta x dot it is also very small delta t f is small this you can manage. So, this whole quantity after multiplying the delta x is delta t you can neglect it the whole quantity after multiplying delta x dot you can neglect. So, naturally this equal to del v x star of t x star dot of t f is equal to t f into delta t f that what we just mentioned.

From the graph this you see value of the function at t is equal to t f t f function multiplied by delta t will give you the approximate value of the area under this curve where you are integrating t f to t f to delta t f, but delta t f is very close to t f this. So, this is the important relationship we got it this one, so if you see this one this whole term this the like the second term of this one.

I can write just replace by this one, now what about this two things we have seen earlier, this two terms we can we can further because you do the Taylor series of that one. Then this and this will be cancelled then first term of this one will left over here. So, if you see this one for the that data is called this equation number three, we have given equation number two, here this one the whole thing, then equation number three.

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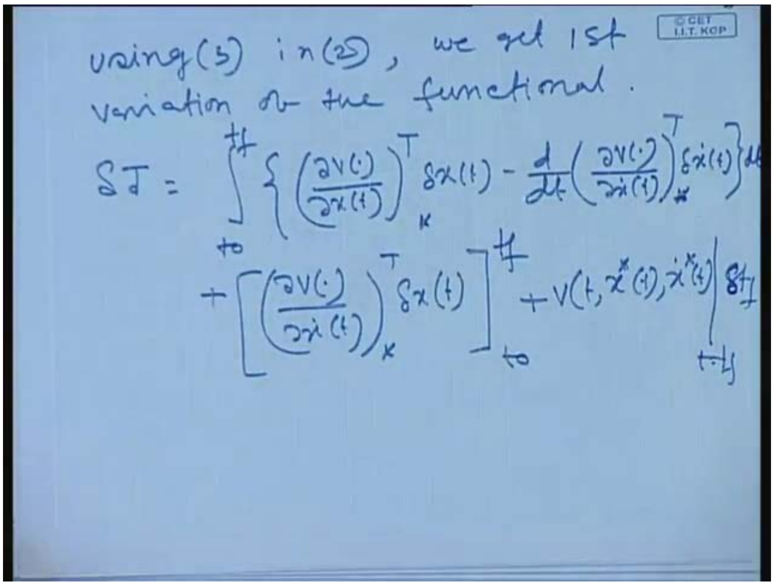

Using equation number three in two, we get first variation of the functional del J is equal to t 0 to t f, I am not repeating this one what I did it. Here, if you recollect this one this term you do it as before when both the points are fixed do the Taylor series expansion. Then first order term second order term written it and higher order terms are neglected then you will get this, this type of things del v dot x of t whole transpose x star del of x t minus d by d t that del v dot del x dot of t whole transpose star del x dot of t this.

Then, this thing is d t plus there is a initial and final condition is there del v dot del x dot of t whole thing transpose del x t when limit is from t 0 to t f please see our earlier diagram plus that term. The second term of this one we have shown is nothing but this term then that term is v t x star of t plus x dot star of t this t is equal to t f into delta t f into delta t f into delta t f. Now, see this one, a point is fixed delta x term will not be there at t is equal to 0, but only the t f is free, so this term with delta t f will be there as it is it will there.

This term will be there in out of this lower limit and upper limit lower limit will not be there because a point is fixed delta t f is fixed delta t x is 0. Then only the upper limit of t f will be there, so we will see next class how we will simplify. Ultimately, what is the necessary condition we will get to find the optimum value of the functional, so we will stop it here now.