Optimal Control
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Lecture - 30
Dynamic Optimization Problem Basic Concepts and Necessary and Sufficient Condition

This course optimal control course is divided into two parts. First part is the static optimization problem, which we have discussed extensible last few lectures. Next part is that the dynamic optimization problems. So, we briefly tell what we have covered in static optimization problems. First we have defined or just to a put an example we explained you what we mean by the optimization problems design. Then we have a stubbonized. Suppose, we have a function of GT function which is a multivariable function.

Then how to get the necessary and sufficient condition for that of GT function, which is an constant optimization problem. Let us say, then in order to find out the sufficient condition we need a some background of what we scroll for the indefinite matrix, negative definite matrix what did you similar, we need matrix. I made it difference simplifying matrix we have tested this matrix whether it is a positive definite matrix or negative definite matrix by using silver star matrix criteria.

Then we have solve this problem optimization problem which is un-constant optimization problem by solving some numerical methods. Newton, Ralph son method, then constant no than you are what is called conjugant variant method, variant method and other methods we have discussed extensible. After that we have seen that what is the difference between the un-constant optimization problem, and constant optimization problem.

Then we using the concept of lagrangian multiplier, we converted the constant optimization problem into un-constant optimization problem. One can solve this problem by what is called kikidi condition. We established the kikidi necessary and sufficient condition simultaneously. After that we have solve the problem of what is called that is linear programming problem by using the numerical method, that is called simplex method we have used it. Then after solving this simplex method, there optimization problem whether it is re constant or un constant optimization problem we can solve by linear programming method.

Then we have concept we have given the concept of what is called dwell and primary problems. One can solve the dwell problem from primary problem solution and vice versa. That we have seen it next we have given the concept of what is called convex set. Then in the what is called quadratic optimization problem, quadratically constant problems that radic optimization problems.

We have discussed it and the solution of this problem is one can solve by numerically or by using, once we obtained the necessary condition of this one, then you can solve by using what is called our LP method, linear programming methods. Then finally, we have discussed briefly the outline of what is multi-objective optimization problems again. In multi-objective optimization problems, we cannot get it optimal solution which this problem that we got it where to optimal solution. So, this is the first part the work course that is static optimization problems. Next part of the course is that dynamic optimization problems. Our constant everything objective function is that our constant is may be a dynamic in nature by equation.

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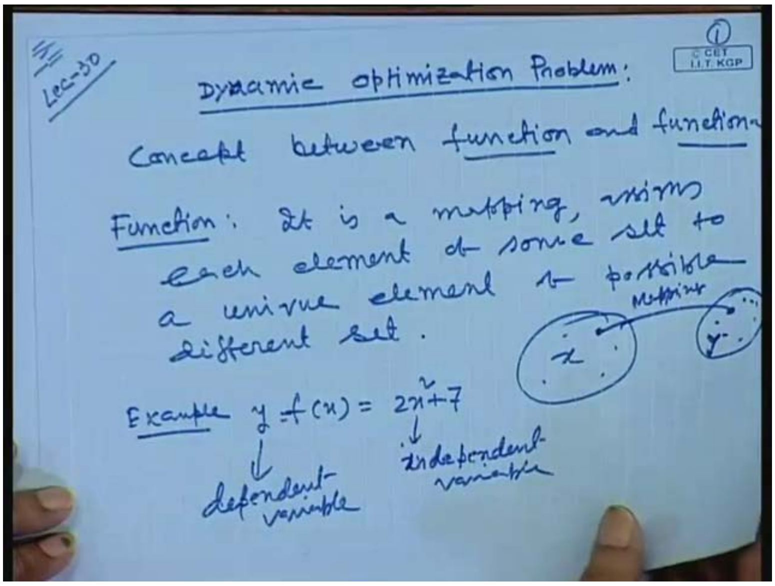

So, our next topic is our dynamic optimization problem. So, first we discuss the distinct concept of what is function, and what is functional. So, let us call concept between function and functional. This as we know the function is nothing but a mapping of one domain to another domain. For any element in some set when we will map this any element of this set to another set another personal set, it is called the mapping from one domain to another domain. It is nothing but a for an example, we can say function. It is a mapping assign each element of some set to a unique element of possible difference set.

Suppose this a set some set which elements are the x any point in this reminder x, that is map to a another set possible another set, let us call y again small y. This is the corresponding this let us called this point is map to this point. So, it is nothing but a mapping. In other words we can example, if you can know that if you have a function f of x y is equal to f of x, which is a function of two x square plus 7 again. Then this any value let us call in this axis map to other set value of y a. So, this x is a independent variable and y is a dependent variable. So, the value of y depends on the x. So, this is function is nothing but a form one each element of a set to a another set with a unique element.

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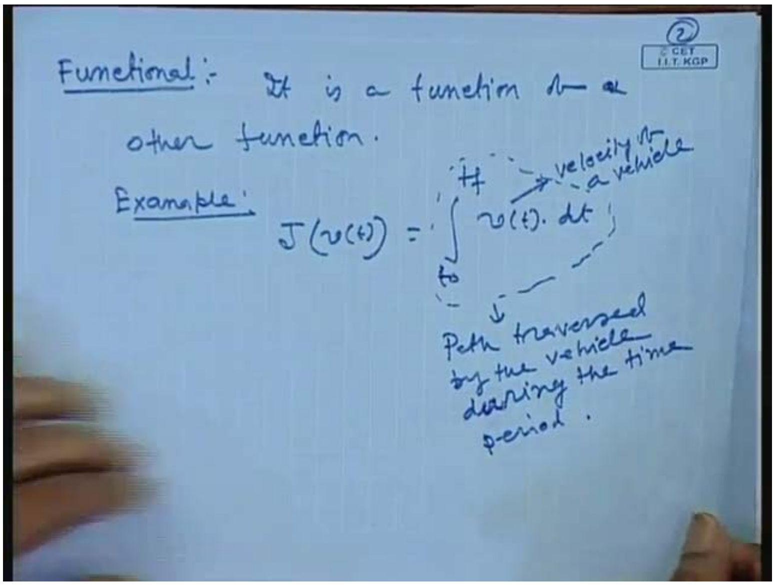

We can. So, ne next is functional. So, it is an important concept of calculus of variations again. So, this is functional is nothing but a function of a function. In other words you can say it is function of a other function. Let us take an one an example, functional is nothing but the function of the function again. Example, suppose this integration t 0 to t f and v of t d t, let us call v of t is the velocity of the vehicle. We want a started the vehicle at time t is equal t 0 and completed the its motion of a ties time t is equal to t f. Now, we want to this integration means this is the velocity of a vehicle.

This integration means how much path is traversed right the vehicle from the time t is equal to t 0 to t is equal to t f t is t 0 is the initial time and the t f is the final time. So, this indicates the path traversed by the vehicle during the period, during the time period. So, this is as is you see v is the function of time and J is the function of b. So, it is a function of a functional again. So, let us see that and if see in general in general you can write.

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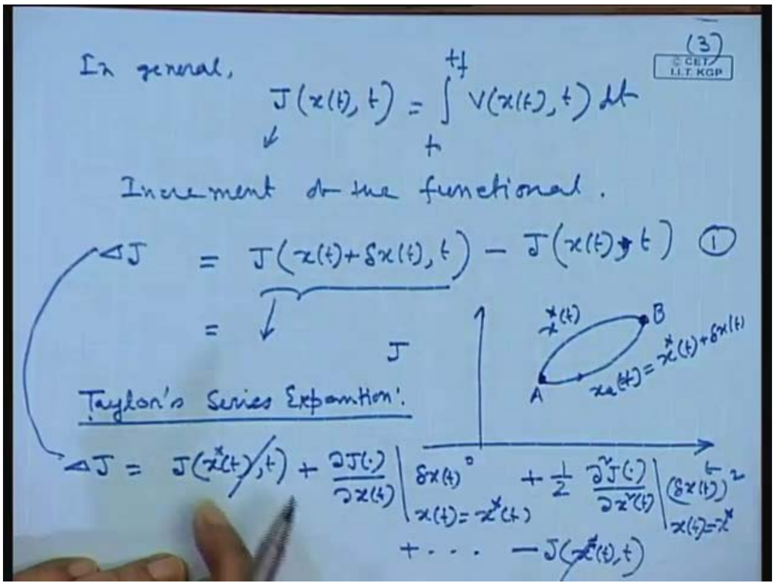

Let us call, we have a function J is a function of time and t which is nothing but a just how it is t f of t, then it is a v x of t of t d t. Again this is the functional J is a functional is a functional of a function. Then this is the our functional again. So, what do mean by the next question how to find out the increment of the functional, increment of the functional. By definition increment in one exam, let us call dell j is equal to del x plus delta x t of t minus the functional value when x is part by delta x plus delta x and difference of these two functional value is called the increment of the functional. Incremental function, the difference of the two function value and what is called x is equal to when x is part of an x plus delta x t.

hen original is x of t of t this functional incremental functional. Let us see if we just do that Taylor series expansion of this one because delta x is very close to the x of t. Again you can think of it this equation one, the equation one you can think of it like this way.

Suppose this is t this is our j of, so this is our A and this is B and this is the x star of t I will explain the what is x star of t and this x A which is equal to x of t is equal to x star of t plus delta x of t. This point is let us call B and this point is A. So, this is t and we have to this point A is moved from A to B along trajectory of x star and x star is what is called optimal trajectory.

That means to reach from the point A to B what is the optimum path of this one transverse that is if you move the along the x star. I have consider which is not a optimal path, but sub optimal path around the path around a or a never root of x have consider an another path, that is x star of t plus delta x, but this is not the optimal path, it is a sub optimal path. So, our problem is let us call J is this one. So, if you see this one, then I can write it that that one what is called J. If you do that Taylor series expansion of that one is a function of x and t, Taylor series expansion.

So, delta J I can write it is del J x of t of t plus and again from this equation plus del J and differentiation of x of t del x t and this x t is equal to x star of t into delta x t. I am considering this is a single variable case this again and plus then half del square J del x square of t x star of t and delta x of t whole square plus add terms minus of that term minus J x star of t of t. Now, this and this cancelled. So, we have considered a object what is this one function is the function of x t. Let us call function.

So, incremental functional value J value is nothing but a that along x star near about the x star there is a another trajectory is x plus delta x t. So, what is the incremental function value of this one. In other words you can say, if the function value of the functional the functional value is x which a function of x t of t. If you put up x t by delta x then function value is this one, what is the difference of between two functional that is called incremental terms conventional, where the partivation around x t is very small. Then Taylor series expansion, if you do it is like this way. We assume that after the what is called third ruler terms, after second rulers terms we are neglecting that contribution in the expression because the delta x is very small.

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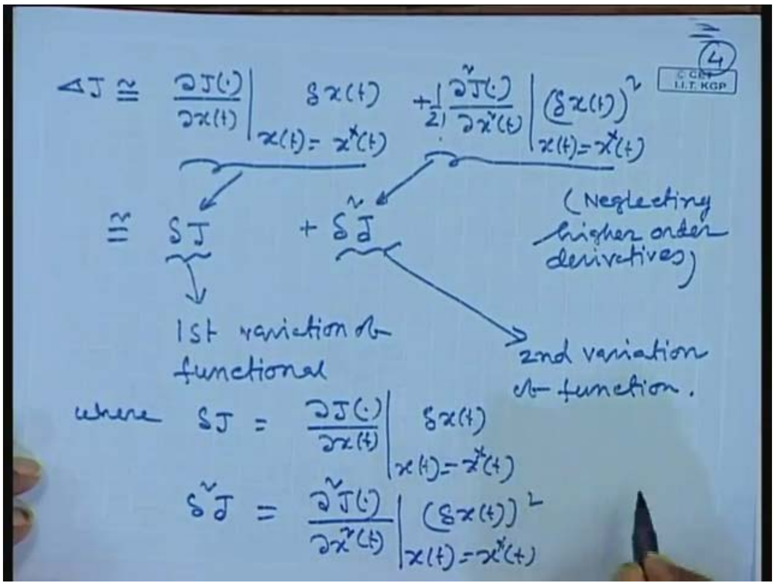

So, we do this one I can write delta x delta g nearly equal to this one is delta J of this delta x t is equal to x star of t into delta x t plus delta square of dot of this delta x square of t. Half is there then x of t x star of t delta x of t whole square. We have neglected neglecting higher terms higher order derivatives. If you see the example though I have shown you it is an example it is an example where A point is there. We have to move the B point along the trajectory x star is the optimal trajectory for which the functional value of this one is minimum.

Then I am telling the around this path and when very never route of this x star is that up to the path there is another path is there. Then what is the functional value of this one this one then difference of this functional optimal path and sub optimal path difference of this one optimal trajectory and sub optimal trajectory, what is the difference of the functional value, that is coming like this way, while we have neglected the higher order derivatives terms.

So, our condition is if you recollect the our basic necessary condition for the single variable case is nothing but a first derivative of this one must be 0. So, this I mean just for simplicity I am just using delta J is the this part and delta J square for this part. So, this is called the first variation of functional and this is called second variation functional. These two variation is important to take a conclusion, whether the functional is a minimum or maximum. So, necessary condition a functional will be extreme that delta J must be 0.

So, our necessary condition are a first we write it to as a delta J, delta J is nothing but a delta J little j delta x of t x of t is equal to x star of t delta x of t and delta square of J this is the second variation of the functional is a delta square J delta x square of t is x of t is equal to x star of t delta x of t whole square again. So, keeping in this mind we know the necessary condition for this one is you delta J must be equal to 0.

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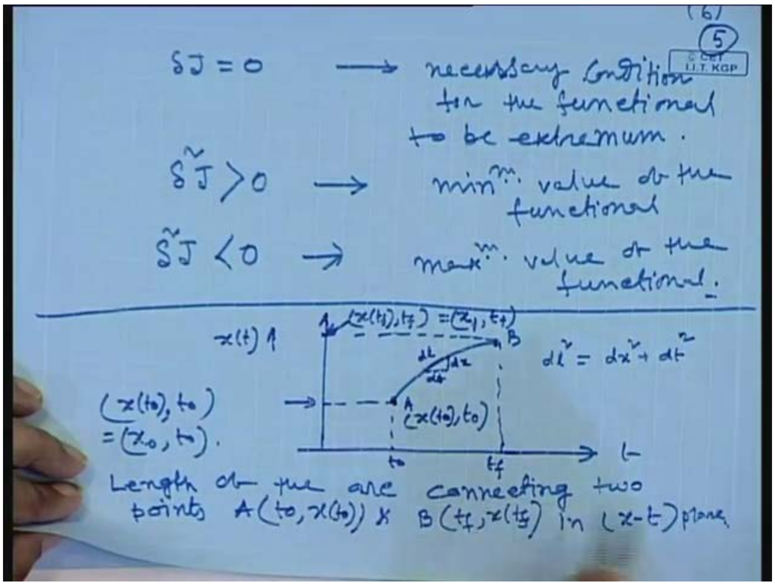

So, our necessary condition is delta J is the, this is the necessary conditions for the functional to be extremum for the functional to be extremum. In other minimum or maximum then delta square J should be. If it is greater than 0 this implies the functional is the least obtain is minimum value function and del square J is less than 0. 0 means negative, then this is the condition for minimum, minimum value of the functional. This is greater than implies the maximum value of the functional A.

So, let us take an example and see that how. First will see this is the just established the necessary condition for the single variable case. More details will go if it is a multivariable case, x is a variable of n variables of the x1 x 2 x 3 dot dot x n. The functional is function of x t, we have shown it may be in J is a function of what we called x t and x dot t comma t or secondary derivatives also there is the of functional, may be function of x and x dot x double dot t and so on.

So, let us see a simple example that how to obtain the functional, that a just mention it just. Now, suppose we have a point A and point B and this point is time t is this one, that here is you x of t in this direction. Then initially the point a has coordinates x of t 0 of t 0. This is t 0 and here is you t f and this co-ordinates is equal to this coordinates is x t t f, t f is the co-ordinate of this point and this coordinate is you x t 0 t 0. In short this is equal to x of 0 and t 0 x of 0 and t 0. That is in short you can write x of x t f x of you can write it if you write it x f and t f.

So, our problem is that, what is the length of this arc optimum length of this arc A to B. I have to mean what is the optimum length of this path while we move from A to B that is that means. So, in order to that let us call this is the optimal path of this one have to find out let us take one point is here this is delta x d x. You can see it is A d x and in time d t it is in time d t. It is that have position of the x is changed d x and this length is d l this arc length is d l.

So, we can write it this, our problem is length of the arc connecting two points A and B which coordinates are x of t 0 and B t f x of t f in the x t plane. Find the length of the arc connecting the point this and this obtained. So, let us form in the mathematical form let us call this you see we can write it this one. This small element in d t seconds, this distance covered is d x again. So, I can write it this is almost it 90 degree this is for. So, far we consider the small incremental time what is the change in x. So, I can write it d l square is equal to d x square plus d t square this one.

So, from this equation one can write it that d l square is equal to I am taking the common of d t square, d t is the small incremental time that squared. So, if it take it to a this 1 plus d x and d t whole square again. It is nothing but a one plus d x d t it is nothing but a velocity or you can write it as x dot of t this square then this d t square. So, what is d l, d l is that small incremental the arc along the arc d l this is I am writing d l is nothing but a square root of 1 plus x dot square of t and it is d t.

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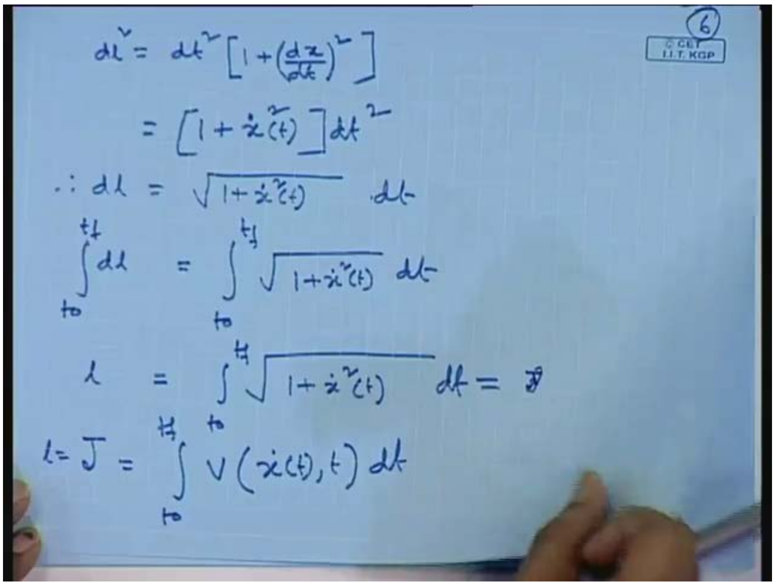

So, what is the length when you when you moved from time t 0 to t f, what is the arc length of this one. We have integrate both sides that dl we have to indicate from t 0 to t f again is equal to integration t 0 to t f a root to bar 1 plus x dot square t d t. That length this in length let us call is d f and this is t 0 to t f root to bar 1 plus x dot square is this again this is.

So, I can write it that this is nothing but a function of you can write it function of if you write it this one is a function of we can write t 0 to t f, v function is a function of x dot t of t d t this d t is that. Now, we are if you see a time t is equal to 0 what is the distance covered from A point to B, that means what is the arc length of this one. This is nothing but a express arc length of the l which is defined by one is equal to this nothing but A is a function of a functional again. So, this is the basic any problem is giving to this one. We have to formulate into this form again. So, that once you get it this one this optimal length of this one, you say from A to B.

By using what is called that our necessary condition, necessary condition of the in quaint first, first you find the inquaintal of the functional. From there you find out the first variation of the functional assigned to 0. Second variation of the functional will give you the, what is the whether the length of this is maximum or minimum.

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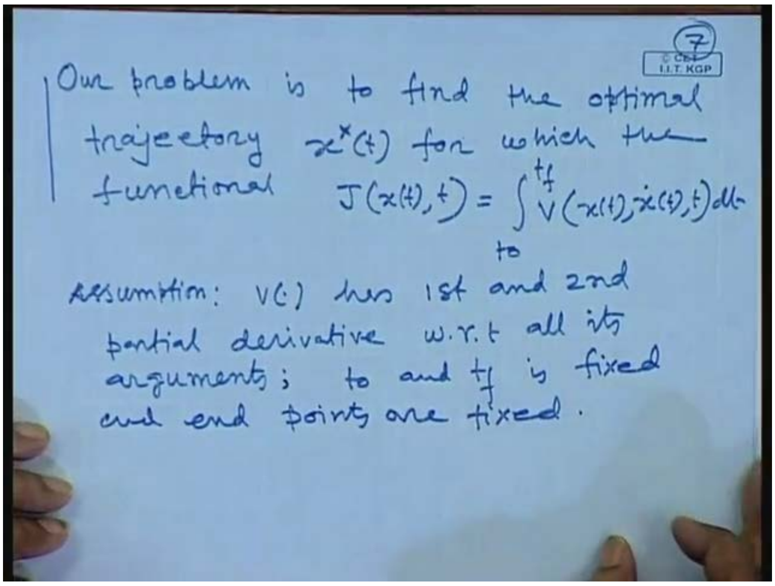

So, let us take this specific problem to establish that in general that what is called the necessary and sufficient condition for these problems. So, our problem is now, our problem is to find the optimal trajectory x star of t for which the functional J is a function of t, t this will be t 0 to t f, v is a function of x, x dot of t d t. Let us follow I have given u a functional which is function of the integral is function of x t x dot t and t. So, you have to find out the optimal trajectory x star which in turn will get x dot star. You have to find out the optimal trajectory x star, such that this functional is maximized or minimized or extimum.

So, this is our problem statement solution of this problem. We made an assumption that V of this has first and second derivative second partial derivatives with respect to all it is arguments. Again and t 0 and t f is fixed, fixed and they are the end points are fixed. The more specific this we are considering this means that we having the functional. The functional value we have to integrate from t 0 to t f and it is nothing but I told you I mentioned I told you have to look for a optimal trajectory x t, such that this functional value is optimized.

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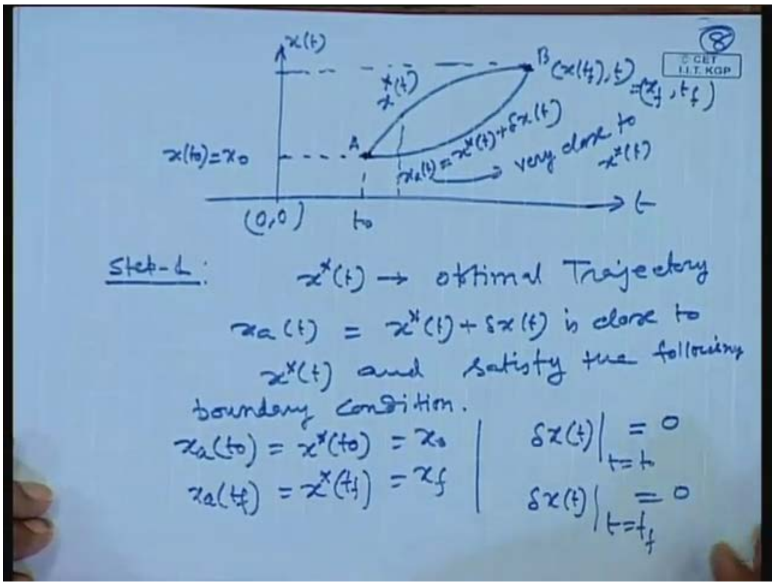

So, let us call that optimal value of the function optimal trajectory of the function x t is like this way when you move from A to B this two end points A to B and x star is the optimal trajectory for the function for which the function value is extremum. Near about neibourhood of this one there is another sub-optimal trajectory x t plus delta x t. So, x t and delta x t this one and this is I denote it by this x a of t and x of t very close to x star of t.

So, in this direction t and in this direction x of t, so what is this coordinates at time t is equal to 0 that co-ordinates is x t 0 is written as x of 0 and this coordinate is x t f of t which is denoted as x f, t f this is the co-ordinate of that one. So, what we can write it first step to solve this problem, but still I am. Now, considering x is a single variable, but our functional is a function of x and x dot t and t.

In general it can be a function of x x dot x double dot and so on. Not only this that x may be a multi variable function that means x may be a vector x 1 x 2 dot dot x n. Here is x dot is x 1 dot x 2 dot dot x n dot. So, in such situations how to handle this one for the time being I am considering of x is a quantity. So, step one x star of t optimal trajectory x a of t is the trajectory which is very close to the optimal trajectory. It is the sub-optimal trajectory for which the functional value is not optimal is close to the x star of t and satisfies the following boundary conditions.

What is the following boundary conditions, that means this condition x I can write it x star of t 0 is equal to x 0. So, this boundary condition I can write it x a t 0 is nothing but x star of t 0 is equal to x sub 0. You see this value along this value for this curve and this curve is same. So, x of t 0 is same as x star of t 0 this. Another is x a of t f is equal to x star of t f is equal to x f. These are the boundary conditions, not only this you can see x delta x of t at and t is equal to t 0 that means any time t equal to 0 this is the delta x of this.

Let us call t is equal to t 1, what is the change in optimal trajectory and sub optimal trajectory this is the change in value. So, at this point both the function value and the optimal functional value are same. So, their change in the function value will be 0. Same as it is delta of t, t is equal to t f is equal to 0. This boundary conditions are satisfied that, we have to satisfy that this function that means we are considering A and B, both are fixed point. So, before we go to the second step, we have a one lemma we have to use it for solution of this problem.

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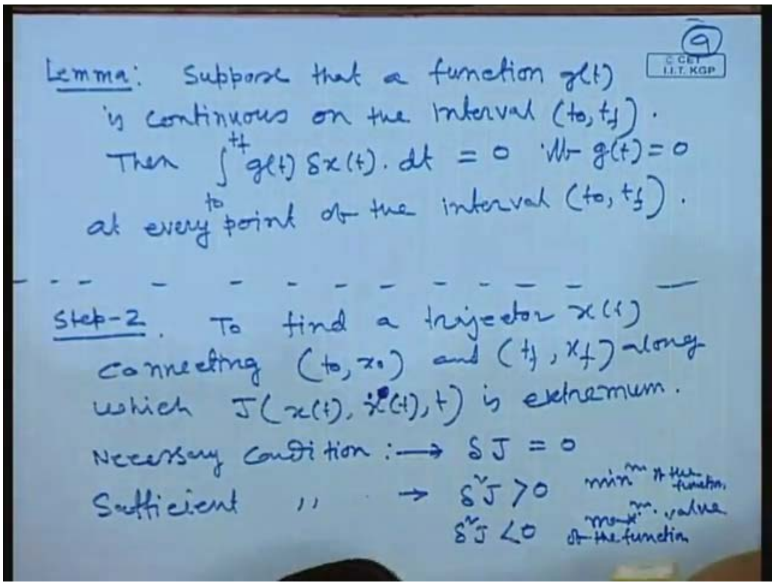

So, this just this lemma is like this way and before the lemma. I will skip it and time permits I will just wait at the end of this lecture, suppose this is the important lemma when we establish the necessary and sufficient condition for a functional. Suppose, that a function g t is continuous on the interval t 0 to t f, then one can write it then t 0 to t f g t delta x t d t is equal to 0. Then integrate the g t multiplied by delta x t d t will change in x about the interval 0 to t is 0 if and only if necessary and sufficient condition g of t must be 0 over the interval, anywhere above the interval is 0 at every point of the interval t 0 to t f.

So, this the important lemma of this one. Now, we will use this lemma when we establish the necessary and sufficient conditions and for the functional values. So, the proof if time permits at the end of this class I will do it otherwise next class. Step two, so what is the necessary and sufficient condition that we have to write it to find a trajectory x of t connecting t 0 x 0 and t f x f between t 0 x.

Zero this point connection trajectory to this point along the along which the functional value will be extremum x dot of t is extremum that is our problem. Find the trajectory x star of t, find the trajectory x star which is optimal. So, our necessary condition is to the functional to the optimal is dell J is equal to 0 is the necessary condition, but sufficient condition dell square J is greater than 0 is minimum and dell square J if less than 0 is maximum value of the functional.

We have to derive it you see this one earlier in our optimization problem that what are the necessary conditions and new functional case of the single case, we have also derived dell square J is greater that 0 is minimum value of the functional. So, let us see what we can get it from equation number one. Now, our function is functional is function of x and x dot. So, from equation one that is our starting with this one you see this equation is called equation number one and referring this equation and our problem states in this equation.

From equation one incremental functional value J is equal to J x star of t plus delta x t x dot star of t plus delta x dot of t this minus j x star of t x dot star of t, t. If you do the Taylor series expansion of x function of this one, what is the expansion x star of delta t just now you have seen, if it is optimal trajectory of this one never of this one or very close to this optimal trajectory exit an another path which is a sub optimal that is it.

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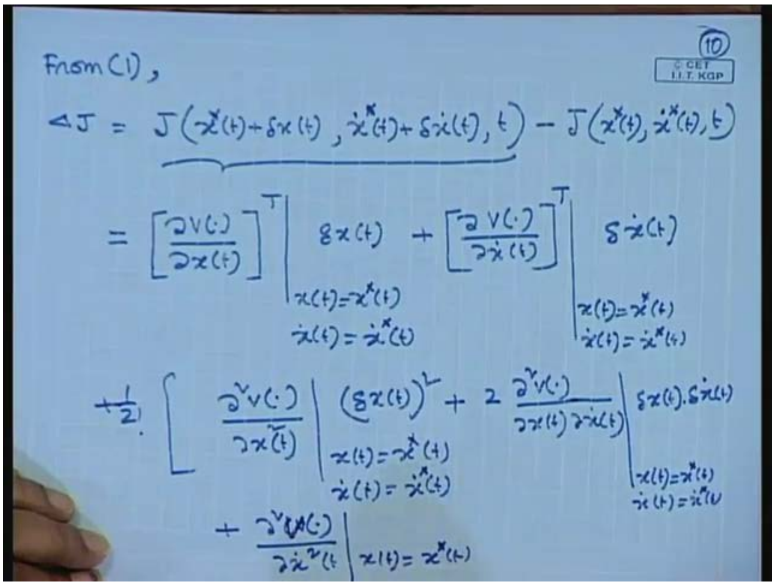

So, if we do that Taylor series expansion this one I will skip the detail steps of this one then you will get it dell v dot dell x of t, this transpose took the below x star of t around x star I am doing this Taylor series expansion. Like x t is equal to x dot t x dot t is equal to x dot star of t into delta x of t plus dell v dot dell x dot of t whole transpose. This is a row vector transpose the row vector and multiply by this is a column vector is a scalar quantity.

So, that again you find out x star of t x t x dot of t is equal to x dot star of t into delta x dot of t. This one then second order will be half then find out dell square second derivative of this one with respect to x, x star of t x dot of t x dot star of t into delta x of t whole square plus twice dell square v dot del x of t dell x dot of t. We have done this in several times in our static optimization problems. So, we find these values x of t is equal to x star of t and x dot t is equal to x dot star of t and it is dell x of t dell x dot star of t this plus another term dell square v dot of t dell dot square of t to evaluate the value second derivative of this. This is what would be it is B is a scalar quantity differentiating with respect to x will get a vector again differentiating with respect to x will get a matrix.

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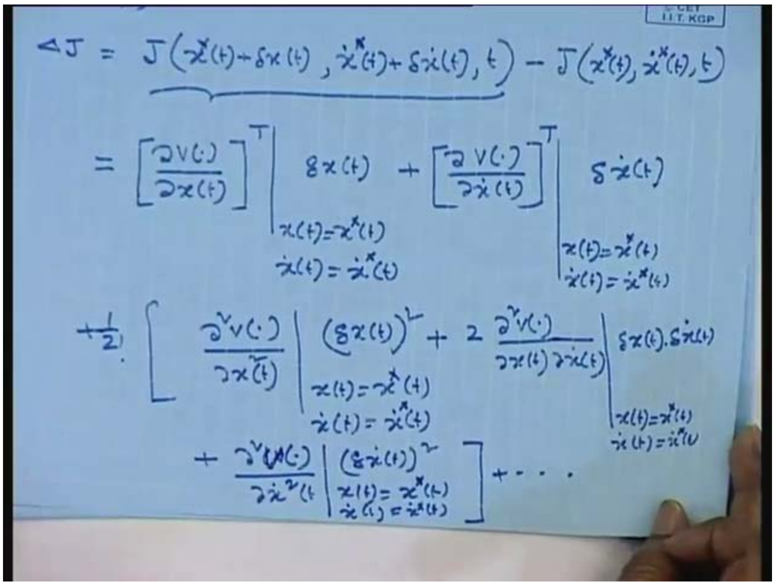

So, this matrix and that matrix is a symmetric matrix, so that x of t is equal to x star of t and x dot of t is equal to x dot star of t into delta x dot of t whole square whole bracket of this. Then higher terms this is the terms are left over in given problem function. So, and because what we will do it with this one you write this expression for J and then do it what is the J expression.

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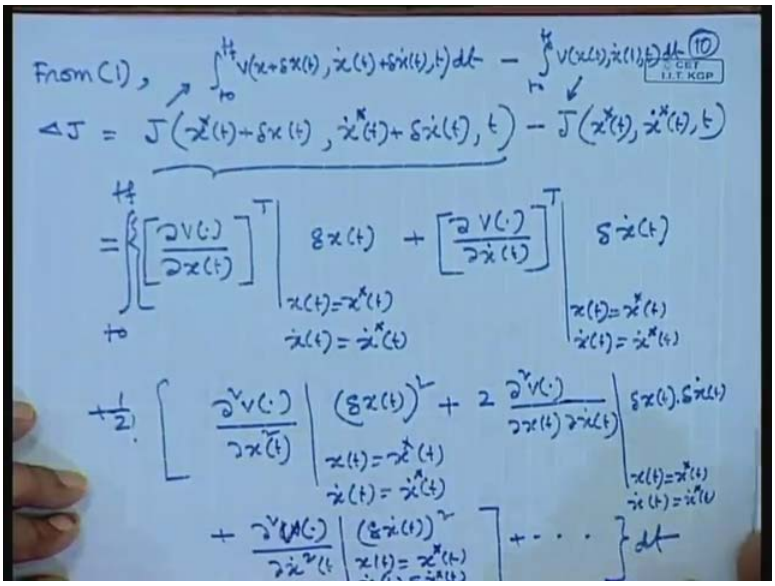

If you see this expression is what integration to t 0 to t f, v x of t x dot of t of t d t. Similarly, this one so if you do this on. Now, doing the what is Taylor series expansion of that integral part on this one. So, the whole thing if you look at this expression the whole thing, you have at 0 to t f and the curly bracket of this one and after this the curly bracket ends d t, so if you see that J expression t 0 to t f v x dot d t. Similarly, this expression is what t 0 tot fv x plus delta x of t plus x dot t plus delta x dot of t t this one into d t that one.

So, this after writing this one that this Taylor series expansion we are doing t 0 to t f in both cases it is common. So, I will get it this expression. So, our basic here are you see what is the necessary condition, we will write it step. Let us call this equation what we are writing this equation number two, then step three.

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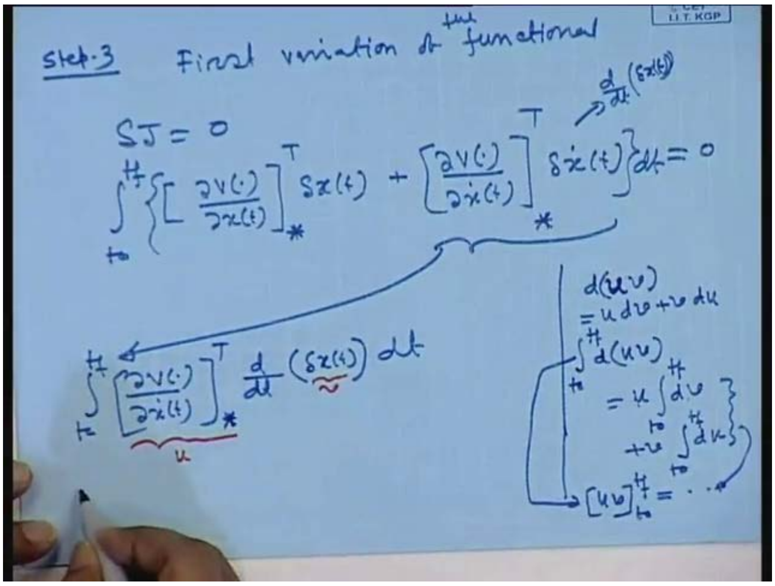

First variation of the functional dell J is equal to 0 and invert of dell J is what, if you see this one t 0 to t f this is the dell J delta v dot x dell x of t whole transpose. So, from now onwards I write it x t is equal to x star dot t. Simply star means star indicates x t replaced by x star of t and x dot t is replaced by x dot star of t into delta x of t plus dell v dot dell x dot of t whole transpose star means x t is replaced b y x star t x dot t is replaced by x dot star of t this one and this must be equal to 0. That this d t is there this is the first variation of the function must be 0.

So, this one can simplify this all by integration by parts. Like this way if you see this one our basic integration, dell u dell v which we can write u dell v plus v dell u integrate both sides t 0 to t f dell u dell v is equal to u is constant now t 0 to t f dell v plus v constant integrate t 0 to tf dell u. So, what is this called this part is nothing but a dell u to d u v is then limit is equal to t f this part we can write it u v.

Limit is t 0 to t f this point and right hand side as it is this thing u dell t 0 to t f d v plus this one that means this terms are here right hand side. So, if we use this one here this expression here we can simplify further that one. Simplification you see I can write it to that one what is called from here to here that is, you can write it this is integration of this is you see this you are differentiating this with respect to x and put the value of x t into x star.

So, this is the constant term that is constant term you got it evaluated this one and dell x dot I can write it this as dell x dot I can write it d of d t into dell x of t into that our d t is there. So, differentiate of this with respect to dell x of t gives dell x dot of t in general. What we will write it differentiation of v dot is equal to d of d t x t. So, this is dell x dot is equal to d of d x t dell x this one. So, if you consider this part now.

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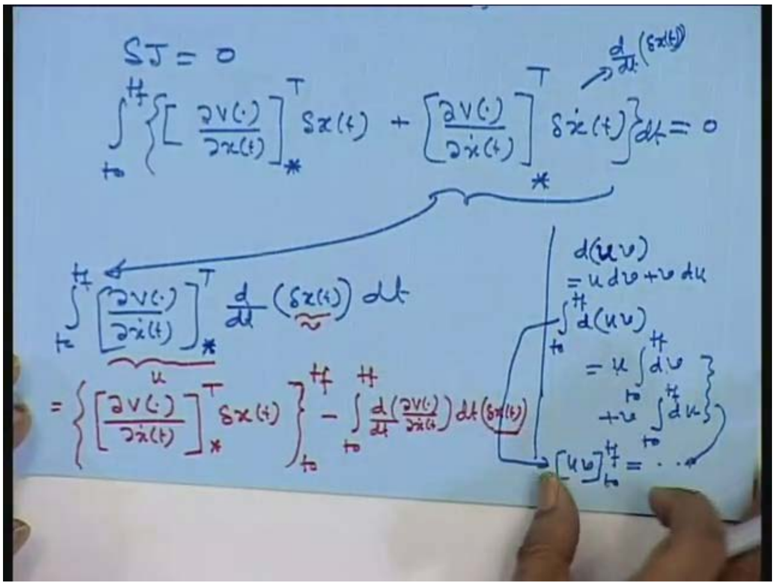

Of this expression then you can write it this is t 0 to t f dell v dell x dot of t whole transpose star. Then you can write it d of d t dell x of t this d t. Now, you consider this is as a v and this is u consider as a u. So, I is nothing but you consider this as d v. So, u d v you see u d vi can write it that u if you consider u d v this I can write it that dell v dot dell x dot of t whole transpose star into v is dell x t. This values is from t 0 to t f, so this is u v u v is equal to t f I have written and that is minus.

If you take it this is minus v means dell x dell x is the scalar quantity dell x an integration of dell u integration of d u d u integration is what if u judge this one integration t 0 to t f and you have to do that d u is dell v dot dell x dot v does not matter if u multiply like this. You have a d x that is your u v, so that will be your u dell x of t u is this one and your v is tour d v d v is nothing but a d v is dell v dell x.

This one you can write anyway we will discuss this for next class more clearly. My aim is this one this term I can using this expression I can make it similar form. Then I can establish the dell J 0 what is the condition for to be become 0. So, using that lemma what we have discussed, we can find out the necessary condition. So, here I will stop it. Now, and the lamda which we discussed also explain or prove in the next class.