Optimal Control
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Lecture - 29
Multi Objective Optimization Problem

So, we will start now multi objective optimization problems.

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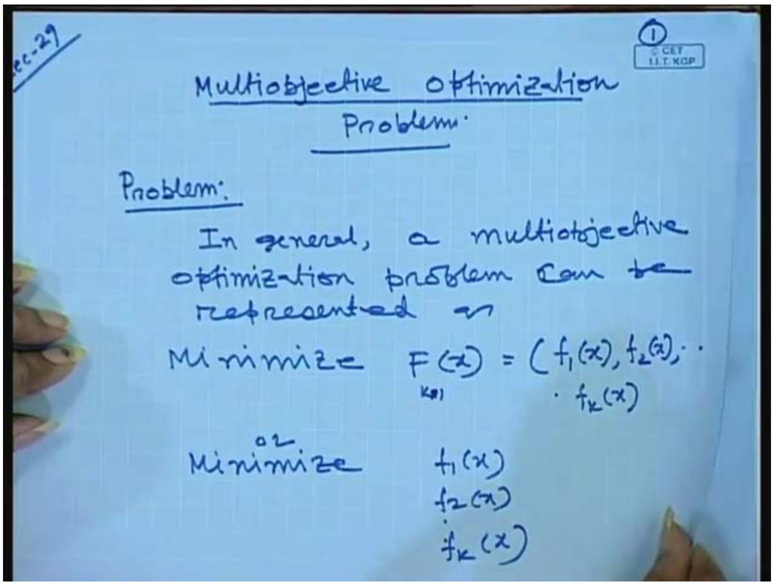

Multi objective optimization problem, so if recollect that, so far we have discussed the solution of a single objective function by using what is different techniques would equality constraint and inequality constraint. However, in real practices many designers are willing to optimize two or more than 2 who are more than to optimization problems. So, that problem is called multi objective optimization problems or multi criteria optimization problems or sometimes it is called betrail optimization problems. So, if want to minimize objective function more than 2 who are more than 2 with equality constants, inequality constants then we will call it is a mutinously objective optimization problems.

So, basic definition of this problem like this in general the objective function a multi objective function optimization problem can be represented. As minimize f of x is a vector of dimension k into k into 1 which we have a n k objective function f 1 of x, f 2 of x and dot f k of x and each is a function of x design variable and x is a. Then and then initial design variables or one can write it in other words minimize f 1 of x, f 2 of x dot f k of x we have k objective function, we have to minimize subject to the constant our constants are equality constants and inequity constants are there.

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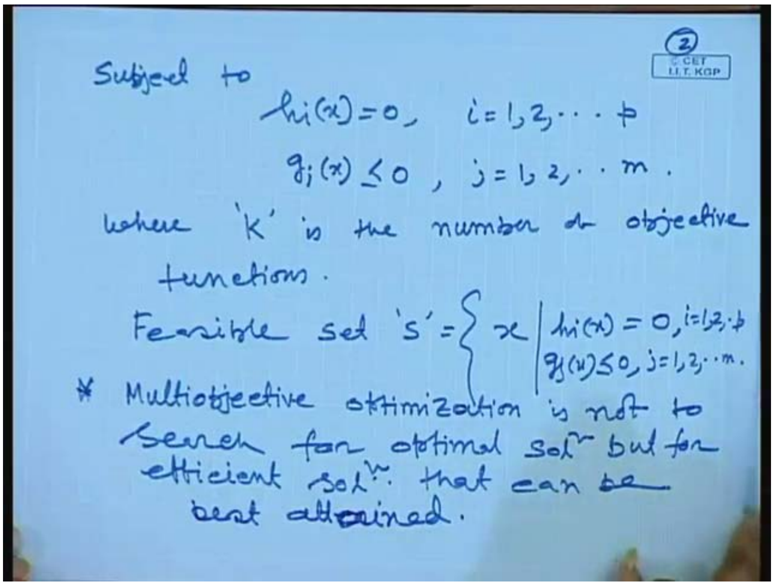

Subject to each i of x is equal to 0, i is equal to 1, 2 dot, p equal equality constants that inequality constants may be linear non linear and our objective function also can be linear or non linear. So, our inequality constant x g j of x is less then equal to 0 for j is equal to 1, 2 dot m, where our k is the number of objective function involved in the optimization problem, k is the number of objective function to be optimized. So, from this two inequality in that p equality constants and m equality constants will get the feasible reason or feasible space.

So, feasible set or space is defined set or space is defined as x is the feasible reason or set where we will get it from x i is equal to 0, i is equal to 1, 2 dot b and g j of x less then equal to 0 j is equal to 1, 2 dot m. So, this are s is the feasible set, so in general in multi objective optimization problem would cannot get a single solution which will simultaneously minimize both the functions. So, in multi objective optimization problem, which cannot get a single optimal solution again a single optimal solution that could optimize the both the objective function simultaneously.

So, because of this the multi objective optimization is not a, is not to search the optimal solution. But, to find out an efficient solution which will make the both the objective function is minimum as minimum as possible, so multi objective, we can say note multi objective optimization is not to search for optimal solution. But, for efficient solution that can be best at any or that can provides the best solution or that can provide best solution. Since, we know in multi objective optimization problem we cannot get a single solution that will simultaneously optimize the both the objective function.

If it is 2 objective function it cannot simultaneously optimize that 2 objective functions, if it is more than 2 let us call n objective function k objective function is there a single point solution it cannot optimize, it cannot optimize all the objective function simultaneously. That is why we are looking for a multi objective optimization problem is not a search for optimal solution, but a efficient solution can be obtained can be attend efficient plus solution can be attend, so let us see with an example what we made this statement.

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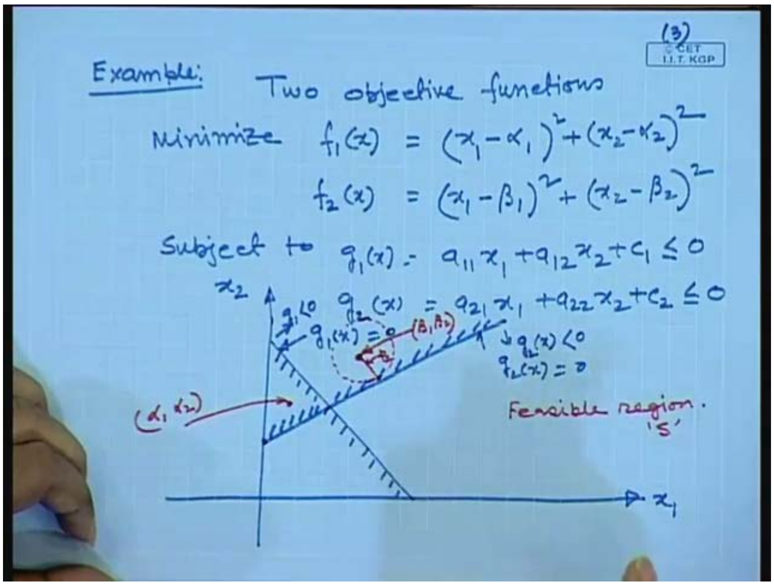

So, let us call will take an one example, start with an simple example 2 objective functions, minimize f 1 of x is equal to x 1 minus alpha 1 whole square plus x 2 minus alpha 2 whole square this is one objective function. Another objective function f 2 of x I am writing x 1 minus beta 1 square plus x 2 minus beta 2 whole square, if you look both the objective function what did it form and our subject to the constant. Let us call just we are consider only for inequality constants that g 1 of x is equal to a 1 1 x 1 plus a 1 2 x 2 plus c 1 is less than equal to 0 and g 2 of x is equal to a 2 1 x 1 plus a 2 2 x 2 plus c 2 is less than 0.

Look these two objective functions, this objective function are non linear in are in more specifically, it is in quadratic form where as our constants are in linear equation form. Now, if you plot this things in a graphically look what we will see this one suppose this is the x axis x 1 axis, this is the y 1 axis, sorry x 2 axis this are the two equation linear equation. Let us call g 1 of x i denoted by that 1, this was our g 1 of x and its inequality condition satisfy in this region that above this line or on the line let us say.

So, this cont this one is you can say g 1 of x is equal to 0 or it is g 1 is less than 0 in this side, on the line it is g 1 is equal to 0 and we have another line g 2 is let us call it is something like this say. This feasible region is either on the line, either any point on the line when it is equal to sign is where and if it is a less than 0, it is let us call in this sign g 2 x is less than 0 or on the line this is g 2 of x is equal to 0. So, if you see carefully the feasible region of this optimization problem for multi objective optimization problem is our region, feasible region or s feasible set is as this one.

Now, let us see what is this our 2 objective function then 1 objective function the center is, here let us call this center is here whose alpha 1 and alpha 2, this one another is let us call it. Here, this is call is this 1 center is beta 1 and beta 2, now you see this objective function f 2, this is called full into f 2 objective function.

This objective function will be minimum when u draw a circle which touches this line g 2 of x which touches this circle g 2 of x that means the perpendicular distance from here to here it is should be perpendicular distances, let us call this is p 2, this is p 2. So, p 2 one can easily find out from the equation of this g 2 expression, so our if you see this one I will read out the circuit, read out this figure because I have to make more conclusion from this corp, so I will just read out this one once again.

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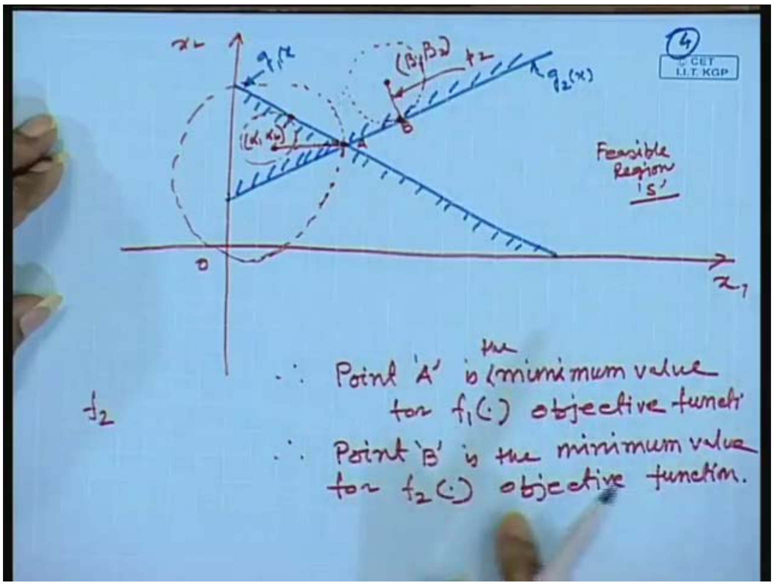

To make it more clear, I just read out this one once again, so let us call g 1 just straight and this is our g 2, g 2 of x, this is g 1 of x and our feasible region is that we have assumed this feasible region is that one for feasible region. For this one is above this line just we have a point, here the circle of this one is beta 1 and beta 2 center of this one and we have a another is, here we can say alpha 1 and alpha 2. Now, if you look at this one, this the our feasible region and feasible set if you draw a circle which touches this line g 2 x then it is that one and let us call this distance from here to here is p 2, p 2.

We can easily find out that is the if you see the p 2 is the minimum value of the function, f 2 is the minimum value of the function f 2 that one and it and this point is b. let us call and what is the minimum value of the function from the center of this objective function is that one, let us call this point is something else let us call it is a this. But, when this it is the minimum note out this function is minimum at that point that point which is lying on the g 1 x is it satisfies the first do a circle.

So, this if you draw a circle at this one then this point a is the minimum value of the function f 1 and which is a lies in the feasible region because it satisfy both equality condition of g 1 and g 2. Similarly, that this one I told is satisfy the point b, satisfy the equality condition of g 2 the inequality condition of g 1, now the objective function of this f 2 is minimum. But, objective function of that one is we cannot say the minimize both are that at this point is simultaneously it minimizes that one. So, you can point out, now the point a is the minimum value for f 1 function, f 1 objective function if you just draw another circle which radius is greater than this one, then this is a feasible in this region in this region that in this region.

But, that is not the minimum value of the functions whereas this below this below is the minimum point, but they are not conceding with the same point. Therefore, next is I can tell you this one the point b individually if you think point b is the, is the minimum point value for f 2 objective function. Now, because this at this point this is the minimum value of the function where as the function f 2 has a minimum value of the function is b 2, but at a single point both the function are not simultaneously minimized. So, we cannot expect that there is a single point for which both the function will be both objective function will be simultaneously optimized, so we are looking for not for a.

We are not searching for an optimal solution for a single point optimal solution, but we are looking for best efficient solution that will satisfy that will give you the best optimal solution. So, let us see, so we can make an conclusion, now but which look at this point at as I mentioned, here this point is minimum for the point of this one, now which point will give you the simultaneously both the function. If you just add it will give you the minimum value of the function, so we have a so many points are there, from there we have picked up which one is the minimum of objective function value minimum is the some of the two objective functions.

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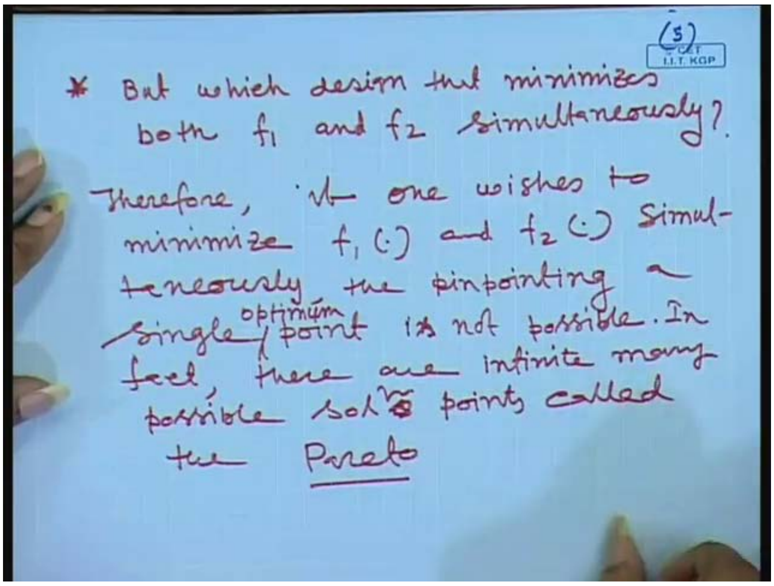

But, which design that minimizes both f 1 and f 2 simultaneously, but this is not clear even for a simple two objective functions you see two objective functions we have consider. But, it is not clear which at which point will get the f 1 and f 2 simultaneously minimum simultaneously minimizes this one, therefore what we are looking for. Therefore, if one wishes, if one wishes to minimize f 1 and f 2 simultaneously, then pin pointing a point is not possible then particular point or pin pointing pin pointing a single point, single point, a single optimal point you can write single optimum point is not possible.

So, you have to do some compromise, so in fact there are infinite many possible solutions, possible solution points called the perato, the perato look at this point suppose I am telling you at this point a point this function is f 1 function is near. But, it satisfy the all constants at this point this function is minimum, so if you increase this one this function is increasing function below is increasing.

But, this if you make it same, but this function is below, so function below is, now increasing, so you have to find out some point. Some point which directly not minimizing the both the function simultaneously, but some of the function some of the two function below will be minimum.

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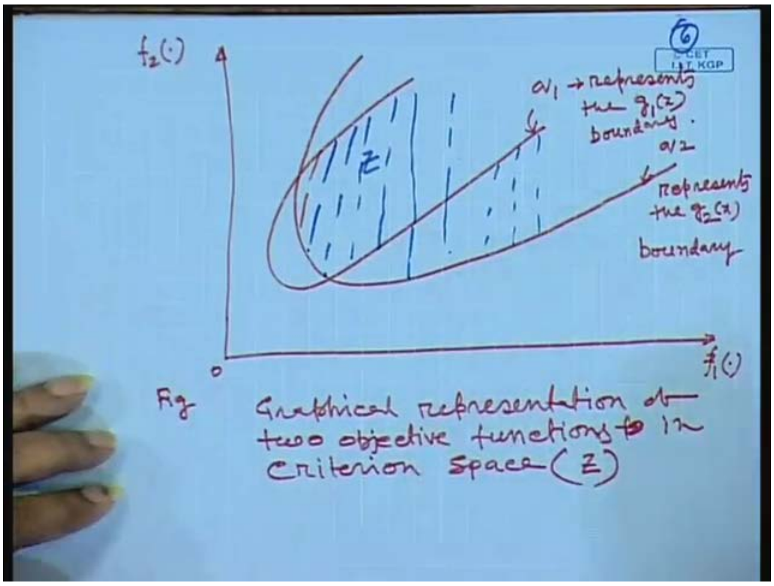

Now, let us see this one what is called we plot it, here criteria space that is f 1 of x in this direction then f 2 of x in this direction 0 the criteria. This figure represents the graphic, the graphical representation of two objective functions in criteria space the graphical representation of two objective functions. We have taken the example of two objective function two objective function functions in criteria criterion space and criterion space is divided by Z. What is mean by criterion space that we have a two inequality constant will move along this line g 1 and also g 2, hence will plot the value of function corresponding to g 1. If you move by along the g 1 along g 1 and we can easily find out what is the function value of f 1 and f 2 let us add this point you x 1 is what.

So, you find out the value of f 1 and f 2, plot it f 1 versus f 2 then next another point you take it you know x 1, x 2 find out the value of f 1 of x f 2 of x plot it in that space. So, that space what will get it f 1 versus f 2 that is called criteria space, so if you plot it this one let us call we are getting this curve agree another curve this is for, let us call it is a q 2. So, q 2 represents the g 2 boundary, g 2 of x boundary what is the mean this is the g 2 of x, so any point on this one let us call if you take this point I know x 1, x 2. Then correspondingly if f 1 f 2 you plot it and along this line when g 1 is 0 along this line, you plot it, if plot it that is q 2. Similarly, if you move the along the g 1 x is g will straight line on the line, you will get q 1, q 1 represents the boundary of g 1.

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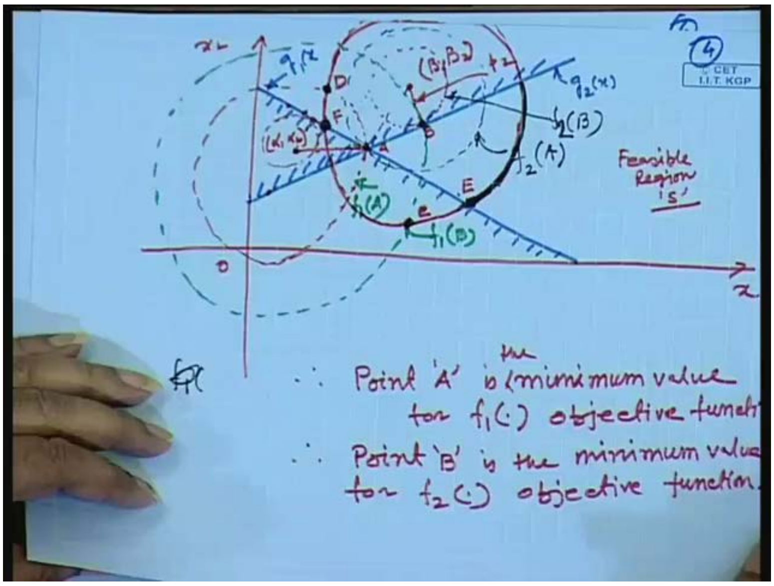

So, let us call q 1 is like this way, so this is q 1 that is represents the boundary represents the boundary of represents the g 1 x boundary, so this is the boundary of this one. So, inside this g 1 is this region and inside the g 2 is the, is this region, so what is the common space of that one if you see, so our criteria space is that one only.

So, that is the criteria space, now look at this point that is what I am telling it, here if I make a circle, here if I make a circle, here with a radius b. This function will, f 1 function will look I can I will call this is the function value of f 1 at b and this function is f 1 at a, so this function will be what this is the common point for both this are common point for both the objective function of this. You can say this is our optimize, this our one of the solution, but whether solution is the correct optimal or not we do not know and another thing we can draw it here.

But, centering this one which this point of this one, so this is the f 2 of a and this point is f 1 of b point, now you may say that this point whether is it is a, some of the two objective function value is less than the sum of the objective function of. So, what we can say the f 1 a, sorry we can write it, now just on this one that f 1 a and f 2 a, this point some of this function objective function whether it is less than the f 1 b plus f 2 b. This is f 1 is, this is f 2 b, sorry this is f 2 b and this is the f 2 b this and this is the f 2 b this is f 1 b which one is less we do not know, but we have to search for this one for which will get that.

There are infinite numbers of points that for which the function value you say the function value is increased, but this function value is decreased, whereas if you see in this case. At this point, the function value of what is called f 1 is minimum whereas the f 2, f 2 a value f 2 a value is f 2 value a is increased, so we do not have a clear cut answer at what point weather a point b point or other point will get some of the two objective function is minimum. So, if we take another point let us call this point is the point is if you can take it you draw a circle of this one this point which crosses this one and that that is if you see this circle.

If you draw a circle our feasible region is from here to here and this portion is in feasible region, this region in feasible region so let us call this circle I am considering this one. Now, so we have a, now let us see this one if you want to represent this thing in here, so let us call our d point is here or let us call our b point is here, a point is here. We draw a circle which is passing through this one another point is that one it cuts here lets us call this is the cuts is here is d point d point is here which cuts that circle which passes through the a point of f 1 of a d point. Then we have a, it cuts that point here that is we can say this point is our c point agree is the c point and this is another point which cuts is here let us call this point is this cuts here.

Here, that is the e, let us call this is the f point, now see this one and I told you, as I mentioned here that this region this region is the our feasible region this region as our feasible region and this is this point e point and f point is satisfy is satisfy the g 1. But, it does not satisfy the f 1 f does not satisfy the g 2 condition where is e satisfy the g 2 conditions and if you see this d point and as well as the c point d point satisfy the what is call g 1 condition but, c point does not satisfy the g 2 conditions. So, in this way we will see that there are some points are there which will give you sum of the two objective function value is minimum.

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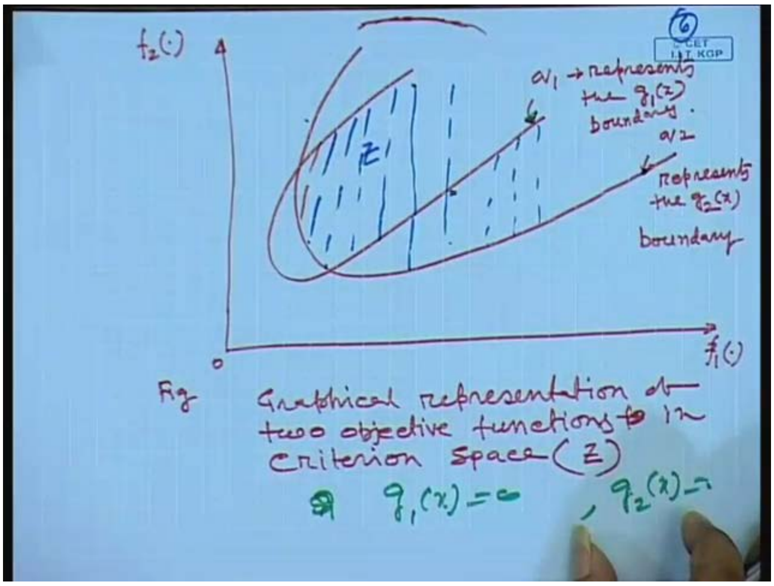

So, if you plot it this one from there we can find out the region where is our, what is called criteria space of that one. This is we got from the design space by finding out the objective function value along the constraint that g 1, along the constraint g 1 x 0 and g 2 of x 0 this is corresponding to g 2 x and this is corresponding to g 1 of x 0. So, we will just give some definition of that one now some of the important definition of that in context to what is called multi objective optimization problems.

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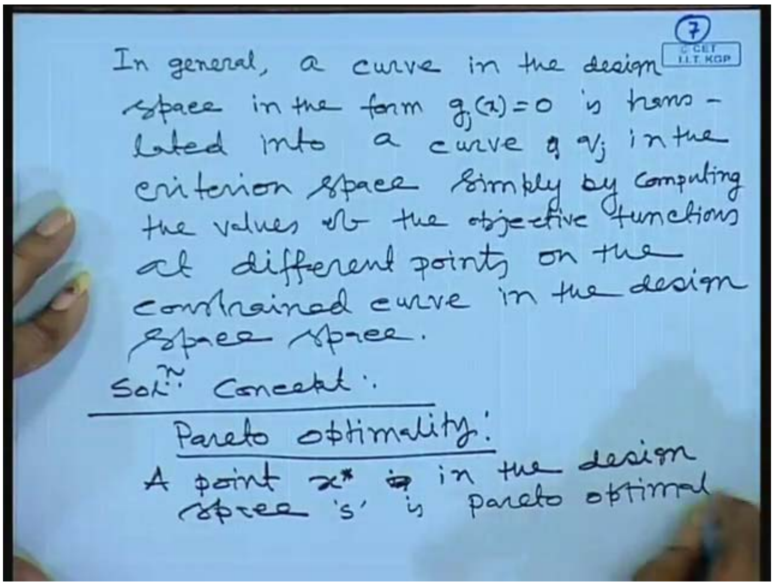

In general a curve in the design space in the form of g j x is equal to 0 is translated into a curve q j in the criteria space, simply by evaluating the value. Simply by computing the values of the objective function at different points along the g 1 on the surface of g 1 or g 2 functions at different points on the constraint on the constrained curve in the design space.

That is what we have drawn it this q 1, q 2, so there are some of the terminology generally used in objective function what is called multi objective optimization problems that is called solution concept. The first is pareto optimality what is this, suppose a point x star is a point x star in the feasible or design space in the feasible, in the feasible region or in the design space, s is the feasible or design space s is the pareto optimal pareto optimal.

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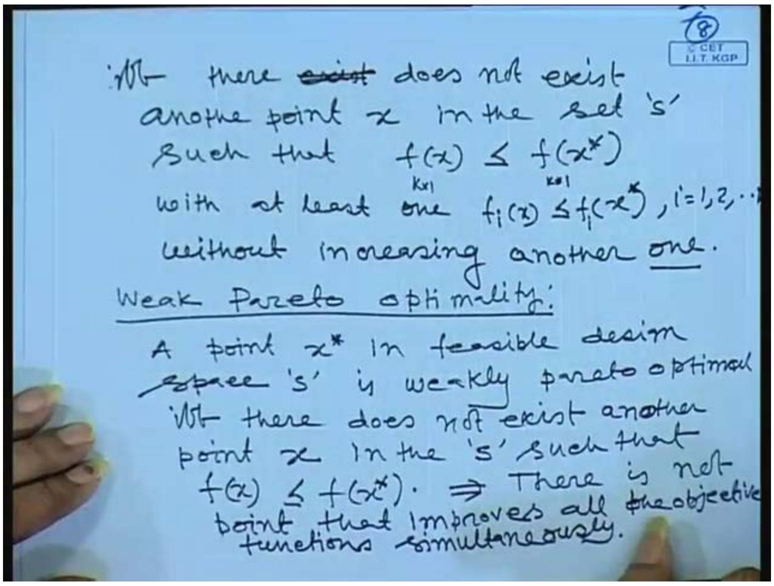

If and only necessary sufficient condition, if and only if there exists there if and if only if there does not exist does not exist another point on that in that feasible region. Another point x in the set s, s is the feasible design space such that f of x is less than equal to f star of x this where as no such x is there other than x star that whose function below will be less than this. We have a ,how many functions are there k functions are there any function of that f 1, f 2 should not be less than this if it is less, then this should not be less than this one with one f 1 of x is less than equal to f 1.

Whether it is more than f i of x star i is equal to 1, 2, dot k in any one of this is by let us this should not be less than this one without increasing another one. So, a point x star is said to be optimal what is paired optimal if there does not exists any feasible point inside the feasible region such that this condition does not satisfy, this condition does not satisfy. Again, without increasing any other function values then we will call it is a pareto optimal solution, pareto optimal solution another this one is weak pareto optimality.

This definition is exactly same as this one only inequality sign only it is a strictly inequality sign will be replaced, so a point x star in the feasible space in the feasible design space. The feasible design space s is weakly pareto optimal if only if and only if necessary sufficient that does not exist there does not exist any x another point that does not exist another point x in the design space s such that f of x is less than equal to f star of x that is that is no point. In other words what is it mean there is no point exist in the design space for which the objective function value improves means less than that one this implies that there is no point exist.

No point that improves means the value of the function value is reduced for that that improves all the objective function, all the objective function simultaneously the difference between this one is here the if you further, if you, if you further decrease this one. But, the other case is without increasing, the another one it may be same but not increasing this one, but here it does not improve all the function values simultaneously is the weekly pay to optimal.

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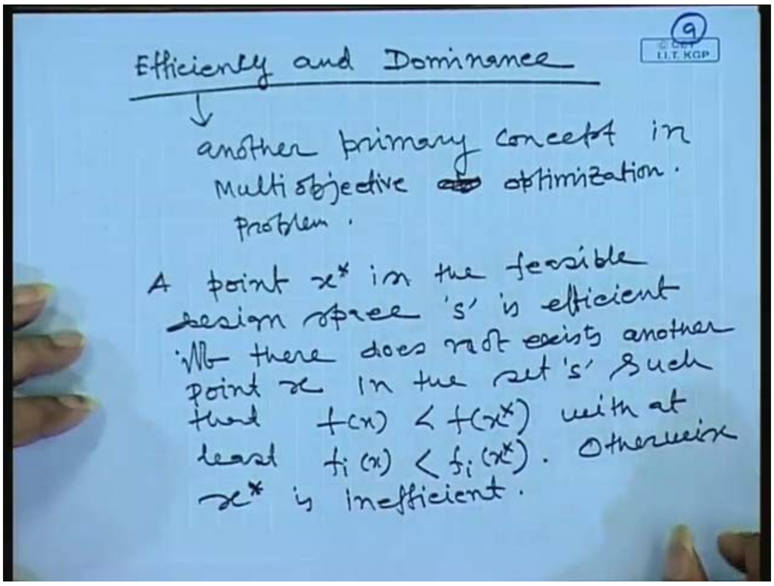

Basically, there are two more definitions are there which are called efficiency and dominance, efficiency is another primary another primary concept in multi objective optimization problem. This is another you can say another primary concept in multi objective optimization problems, so let us call what are this efficiency indicates this one definition. The point this basically efficiency and pareto optimal definitions are exactly same a point x star is the in the feasible design space s is efficient if and only if that does not exists.

There does not exists another point x in the set s such that f of x is less than f star of x with at least f i of x f i star of x that does not exists which we have at least, otherwise x star is inefficient. So, this next is dominance, so I just mention is that there are two concepts are generally used in efficient and dominance. So, another common concept is the non dominance and dominance points in a multi objective optimization problems.

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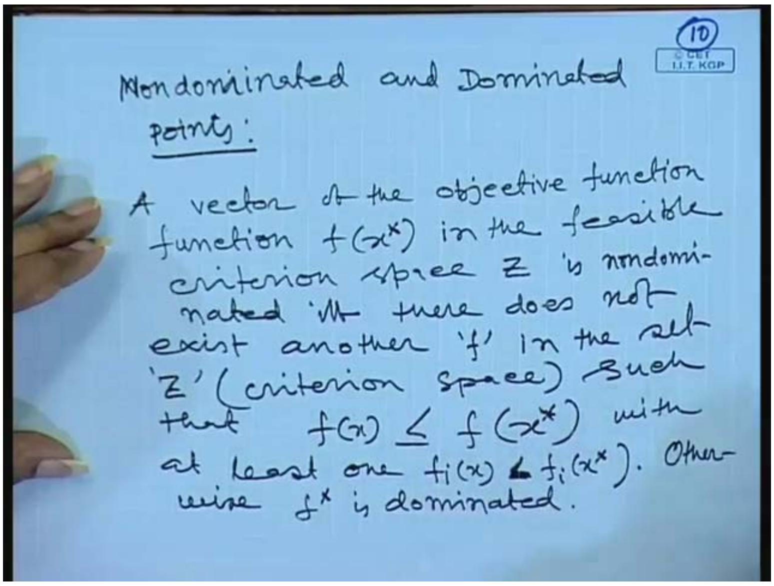

So, non dominant, non dominated and dominated points, so this are defined in based on what is called criterion space a vector because our objective function is a vector f 1, f 2 dot f n. A vector of the objective function f star of x in the feasible criterion space z star z a vector objective function f star in the feasible criteria z is non dominated when you are talking about the feasible criteria space. That means we are going along the that area if you see this one this is the criterion space this boundary is indicates the along the g 2 constant and this boundary constant the g 1 constants and along this side is the criteria this.

When this indicates the value of the function value of the objective function inside this region indicates the value of the objective function which is, which is inside the feasible region. So, it is dominated non dominated if and only if there does not exist another vector another f, f in the solution set in the set z means criterion space such that f of x is less then equal to f star of x. But, it is in the criterion space with at least one, f i of x is less than is less than f i of x star, otherwise it is dominated. Otherwise, f star is dominated and last point associate with the multi objective point is contest to plug the way this utopia point.

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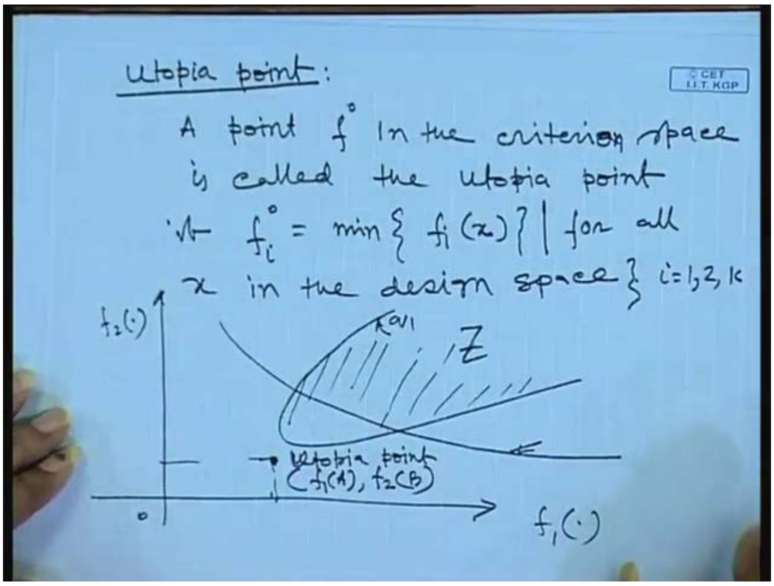

Now, you see this one utopia point is if you look at this expression now let us call this one if you see, if you see this function f 1 when it is passing through this one. The function value is minimum when it is function f 2 when it is passing through the point b it is an optimal. But, that two points are not the solution of this one is not the common point that solution of this one, and that point you can easily see it must be the outside the what is called criterion space how let us see this one.

This point, this point is on the surface of the on the boundary of q 2 and this is on the boundary of, if you see this one this is on the boundary of what is called q 1, one is on the boundary of q 1 and another is on the boundary of this is on the boundary of q 2 and this is on the boundary of q 1. So, where is the utopia point will be definition of this one a point f 0 in the criterion space the n space is called the utopia point f. We have a how many function objective on k functions are there f i is minimum of f i x i is for this indicates for all I am talking about f i for all x in the design space s for i is equal to 1, 2, k.

So, if you see this point of this one let us call this is our z 2 and this is our q 1, and this is our q 2 point this 1 and our utopia point is outside the what is called our feasible region. This is outside our feasible region, this in this direction f 1 of dot this is the f 2 of dot, see this one if you see this one that is at this point it is lies on the q 2 and this point lies on the what is called q 1 point. So, q 1 point and q 2 point, in other words what you say this one this is the minimum value of the function and this is the minimum value of the function.

If you plot it here you will be here minimum value of the function and this is the utopia point of this one by definition it is nothing but a point in the criterion space is called utopia point. When f i is minimum, f i is minimum for all x in the design space because this is the minimum for all feasible region this is the minimum point and this is the minimum point for all feasible region. But, that point is outside the what is called our z space which is called criterion space, so we will stop it here and this is correspondingly this point you can say this point f 1 of a and this is f 2 of b point this u 2 point.

So, from next class we will start with a what is called the dynamic optimization problems, so far we have discussed last few lectures is the study optimization problems starting from single variable case non linear. How to solve the optimization problem using the num, what is numerical techniques, then linear programming, non linear programming, all these things we have discussed.

Lastly we have given some basic idea of multi objective optimization problems and related some definitions we have seen that there is not possible at all to find a single optimum point. Single point for which both the objective function and all the objective function multiple objective function will be simultaneously minimized so there must be a, what is called part optimal point. So, here we will stop here.