Optimal Control
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Lecture - 27
Solution of Non-linear Programming Problem
Using Interior Penalty Function Method

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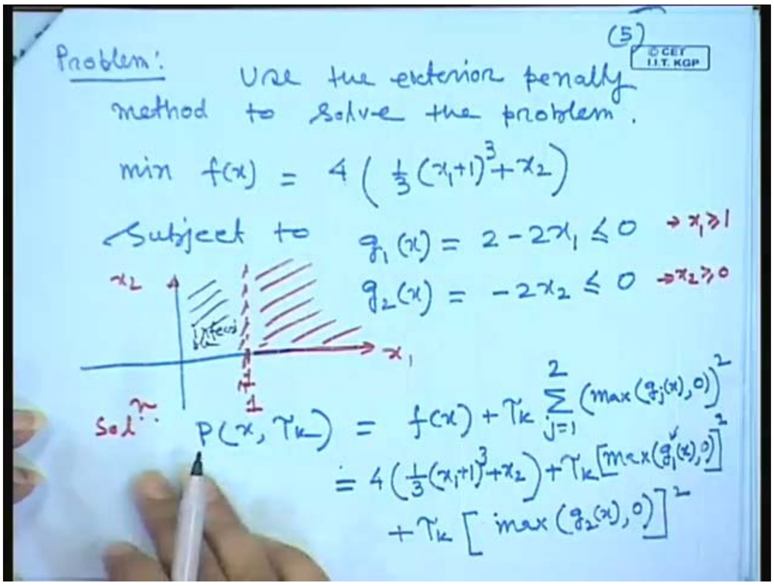

So, last class we have taken one example for a non-linear programming problem that f x is equal to 4 one-third x 1 plus 1 whole cube plus x 2 subject to this constraints and we have to solve these problem by using exterior penalty function method. Then we have seen this one that x 1 is greater than 1 and x 2 is greater than equal to 0 these are the feasible region. So, initially we have to form a penalty function method with the objective function and with the constraint like this way. Once you form this one then your job is to find out the necessary condition for the function to be optimised that we have seen it that first.

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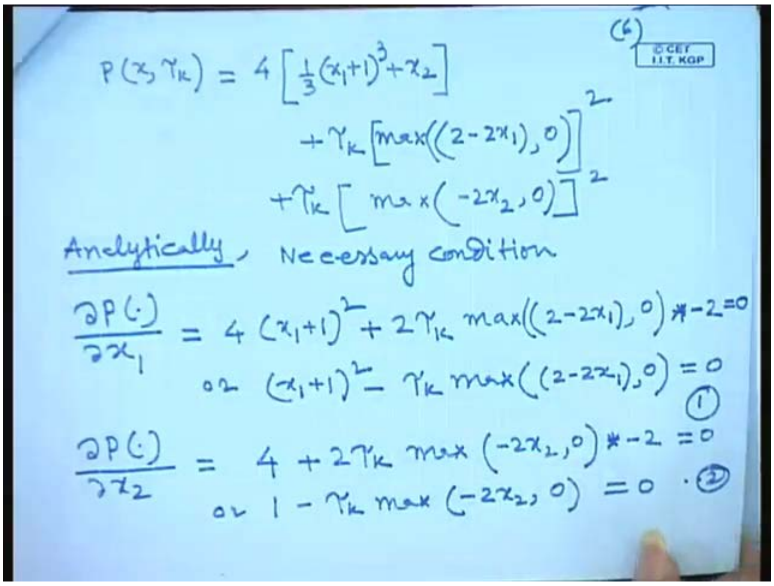

This is the analytical solution by finding the necessary condition del p del x 1 del p del x 2 is 0. So, after solving this after details expression for 1 and 2 will get a set of 2 simultaneous equation.

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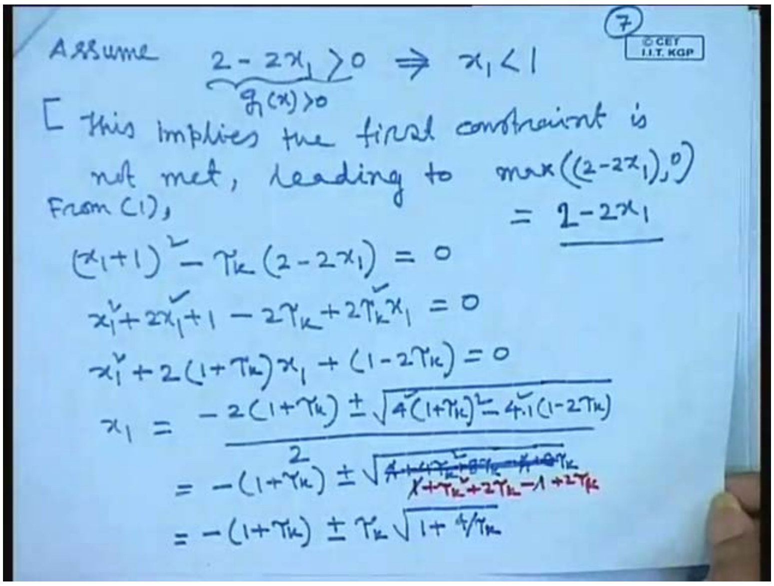

And our job is to solve this one and ultimately after solving this one we got x 1 value with this expression.

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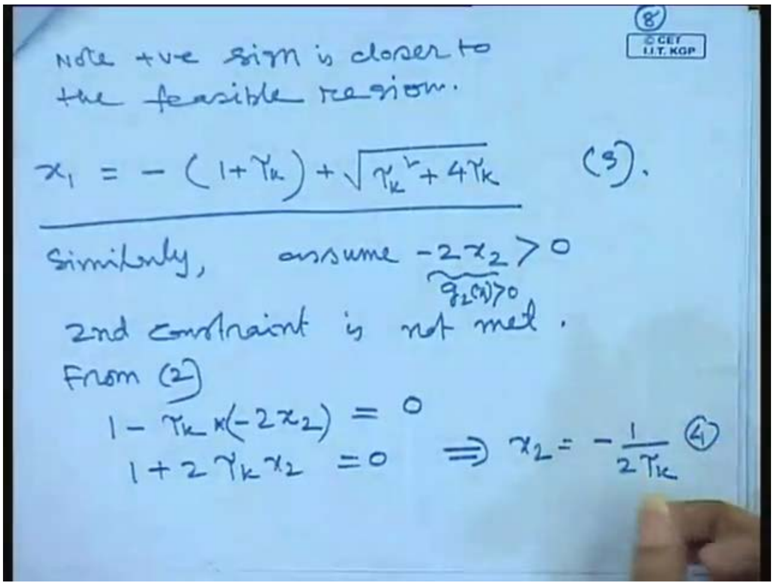

But, you have to take the proper sign of when we are finding out the roots of these quadratic equation of that one you have to take the proper sign of this one, because x cannot be negative because of feasible regions solution. So, that plus you have to consider this on were this plus this term is greater than this one that we have explained last class. Similarly, x 2 when you solve this x 2 we have to take proper sign chain.

But, our x 1 x 2 if you see this one x 1 x 2 our things should be a what is called that x what is called infeasible solutions is it not because infeasible solution exterior point meaning infeasible solution we are taking and then we are finding out the function value. Which is function value is decreasing with each iteration and ultimately it will converge to a some optimum value of the function which is a feasible. So, that we have seen and we have got the results this one...

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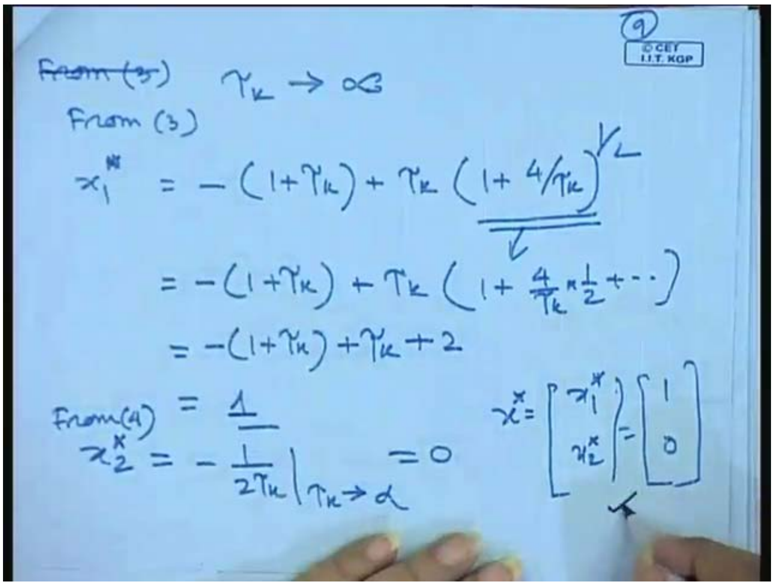

x 1 is equal to 1 x 2 optimum value is 0 when tau k that penalty coefficient tends to infinity.

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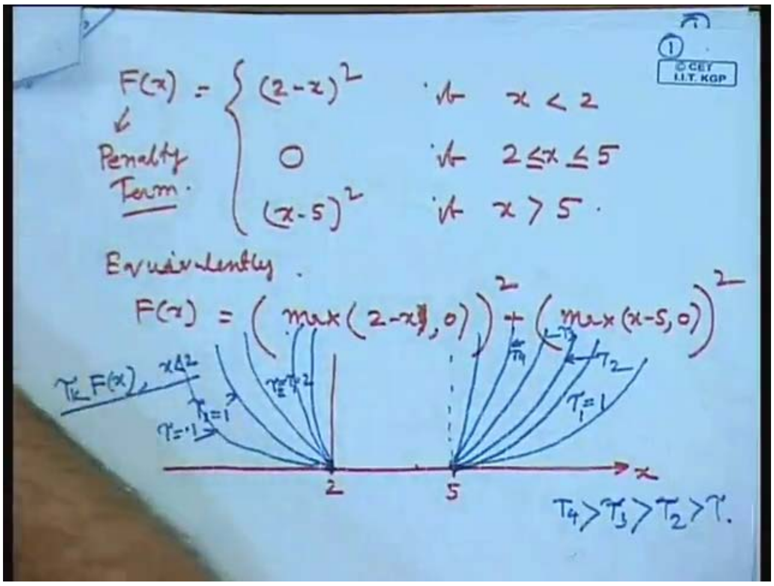

Let us solve this problem by using what is called iterative method because we are using the necessary condition expression necessary conditions what we have got it ((Refer Time: 03:05)) equation number 1 on 2 we express x 1 and x 2 in terms of tau k agree. And now tau k I will change from small value to very large value the infinity and if you use the iterative method how you will solve this problem that we will see.

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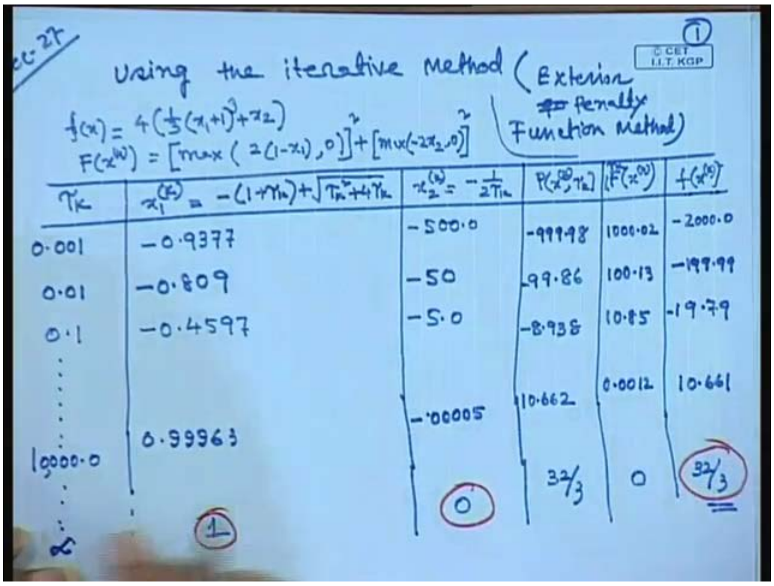

Using the iterative method that is we are considering the exterior point, exterior penalty function method. The same problem what are the we got the necessary condition for this problem and x plus x 1 in terms of tau k, x 2 in terms of tau k that we are writing now. So, our iterative method make the tabular form and this is the tau k penalty coefficient then will expression for analytical function expression for x 1 of k, k superscript k indicates iteration.

Which expression we got it 1 bracket minus 1 plus tau k see the expression plus root of tau k square plus 4 tau k under square root then x 2 k is equal to minus 1 by 2 tau k then penalty function value you write it which is a function of x superscript k into comma tau k. And then you find out the f if you can like you can find to x of k capital x of k then you find out function value f of x superscript of k this. Note that our f of x small f of x is equal to 4 one-third x 1 plus 1 whole cube plus x 2 this is our f x and capital f x, x of superscript k is equal to max 2 1 minus x 1 0 this then whole square for g 1 constraints. And for another constraint max minus twice x 2 comma 0 max of this and this square so this is our x x.

Let us calculate this one first we will start from small value of tau k 0.001 then this value will put the value of tau k her in this expression you will get this value is 0.9377. Then x 2 1 by this if you put this value you will get minus 500 and once you know x 1 and x 2 that function value you can compute that function value is minus 2000. Then once I know x 1 x 2 f f of x I can compete that value will be 100 thousand 0.02. So, what is p this multiplied, this is multiplied by tau k so I included tau k mind it this is also I have included tau k and got it you can find out only f k, but I multiplied this tau k into f of k, tau k into f of k is that one.

So, this is equal to what this minus tau k of this one that p then you will get this value is this then tau k see this expression that what we can get that p expression. P expression f of k tau k into this whole thing is capital f of k so this and this you add it this and this if you add it you will get it minus 999.98. Then next choice of tau k is 10 times of this one that is 0.1 if you take because lambda k for exterior point method if you recollect lambda k plus 1 divided by lambda k is equal to c, c is greater than 1 and we say that C value is 10.

So, next is this value if you put this value say this you will get 0.809 minus 50 then you will get f is you see you will get minus 199.99. Then lambda k into f of x is here f of x I know multiplied by lambda k you will get it this value is here that is you are getting 100.13. Then this plus this will be your 99.86, but minus sign again 10 times of this one 0.1 you see the function value is decreasing 2000 to now it is coming near about 200 minus 200. So, next value is this is 0.04597 minus 5.0 and this value will be minus 19.79 and this value will come 10.85 because when you put this value lambda x 1 x 2 value here and multiplied by lambda k this value is representing here. So, this plus this will be minus 9 minus 8.938.

Now, if you proceed like this way and increase in the value of lambda k you will see at 1, 2, 3, 4 up 10000 lambda k value is 10000 the value what we will get it the x 1 value by applying lambda k. Here, x 1 value will come 0.99963 then x 2 value will come minus 0.00005 and f x value will be coming is that is 10.661 and lambda k into f of k is coming 0.00012. So, that value will coming this plus this is equal to 10.662 now this see this point our feasible region is x greater than equal to 1 and x 2 is greater than 0, but both are not satisfied this one still this is a in feasible region. So, we need further more iteration if you go on iterate like this a very large value of lambda let us call say infinity.

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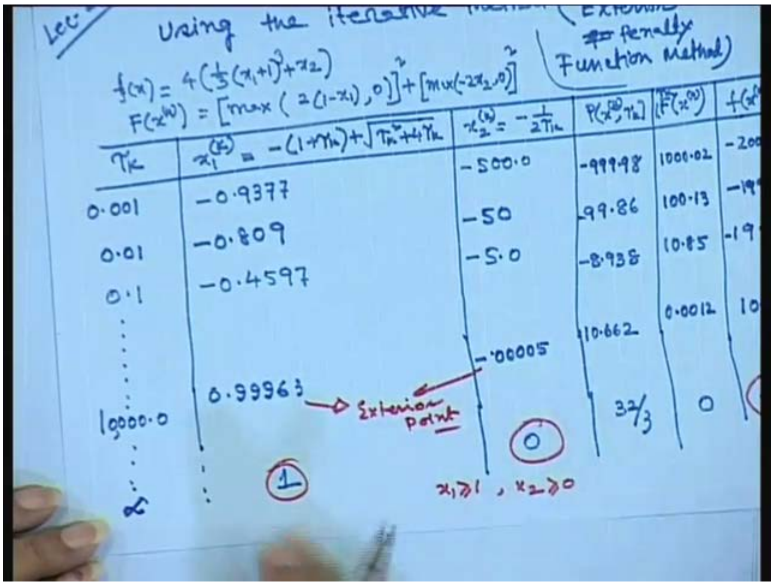

Then this will get 1, this will get 0 and this will get these value we will get it here 32 by 3 this is 0 then this is the 32 by 3. So, this contribution of the penalty function is become 0 when it reaches to the optimum value of the function so our optimum value of the function is 32, value 32 by 3 and corresponding point is x 1 is 1 and x 2 is 0. These are the optimum point for which the function value is this. So, this we have and you see even though these two points still is not interior point it is exterior point still here point. And since we know the our feasible region x 1 if you see the feasible region x 1 is greater than equal to 1 and x 2 is greater than equal to 0. So, both condition is not satisfying so it is not interior point of that one.

So, next we will see some of the problem associate with this one or you can say more general expression we will discuss the problem later for general form of that one. That how to solve the problem using that what is called exterior penalty method, penalty function method. We have seen that minimize the function and the subject to inequality constant suppose if you have a equality constant than how to tackle that problem.

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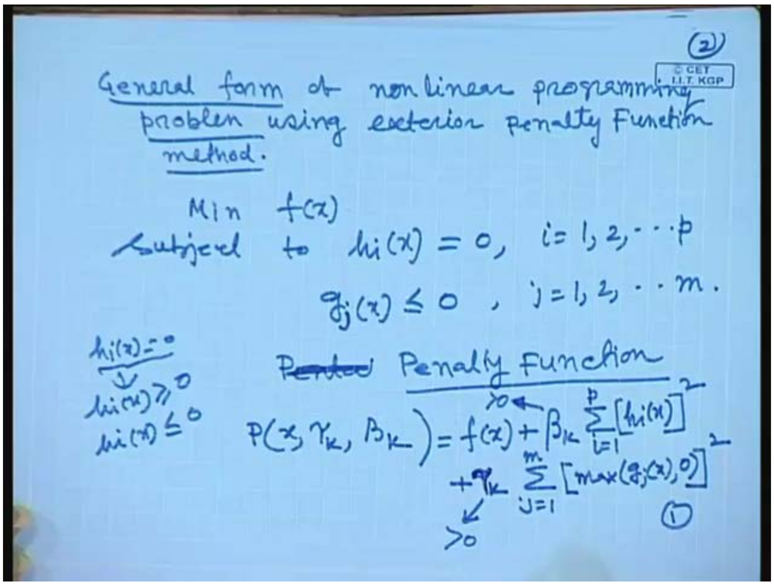

So, it is a general form, general form of non-linear programming problem using exterior penalty function method. What since I am telling a we have not consider the equality constant so our problem is like this way this is the minimize f of x subject to h i of x is equal to 0 and we have a such equality constant, p equality constant is there an equity constant g j of x is less than equal to 0 we have m equality constant. Now, how to introduce this one, so this one can write it there are two ways of doing one can write it h i of x 0 this one can write it into a inequality constant h of x greater than equal to 0 h i of x is less then equal to. So, this equality constant I can write it like this and we can solve similarly, as we have discussed earlier agree.

So, in each equality constant we get a 2 inequality constant agree so in turn we will get 2 p inequality constant and there is already m inequality constants are there. So, there will be a m plus 2 p inequality constants we will get it, but in generally we can write penalty function, penalty general penalty function like write it. So, this p which is a function of x tau k and beta k, tau k is penalty coefficient associated with the inequality constant and beta k is the penalty coefficient associate with the equality constants. So, this equal to now we are writing objective function as it is plus beta k summation of all 1 to p equality constant h i of x this whole square.

That is just like a inequality constant we use to do gamma k summation of j is equal to 1 to p. Then we used to write it max of this max of g j comma 0 whole square in place of this one j of j x square plus gamma k summation of j is equal to 1 to m max g j of x 0 whole square and this is let us call equation number 1. And these are this and this are penalty coefficient gamma k tau k let us call we have written tau k throughout this one, tau k is a penalty coefficient associate with the inequality constant and beta k is the penalty coefficient associate with the equality constant. Both quantity is greater than 0 agree so we make some comments of this one that.

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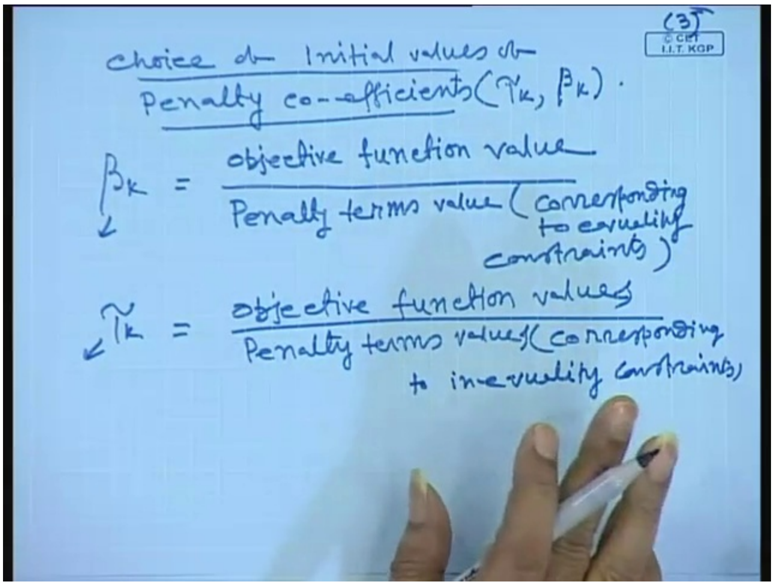

Now, how to choice, choice of initial values of penalty coefficient that means tau k and beta k 1, one of the simplest choice is there you consider a point which is outside the feasible region that is called the exterior point. Then beta k selection of beta k initial choice of beta k after that what we are doing we are incrementing we are increase the value of beta k or gamma k by 10 times. So, initial value of beta k penalty coefficient associate with the equality constant and tau k associate with the inequality constant how to select it.

So, that value is so objective function value then penalty terms value corresponding to equality constraint penalty terms because we have a penalty terms associate with the equality constraint we have the penalty term associate with the inequality constant. So, this penalty terms then tau k is objective function value similarly then penalty terms value corresponding inequality, corresponding to inequality constraints values corresponding this. So, this initial value we have to select this is the one choice of this one.

Next, is one remark important remark is there that analytical solution that what we did it that in order to find out the what is called first the initial that necessary condition we have found out. Then we have solved this problem for x 1 the solution we have solved this to search set of equation or in general n set of equation by say x 1 x 2 all this in variable we have calculated in terms of our tau k for our present problem for.

Now, you have a x 1 x 2 everything here to express in terms of tau k and beta k, but it is really very tedious job to get such an expression. So, we have to look for what is called the, what is called the numerical method solution of a set of equation which is obtained set of equation which are obtained from the necessary condition of the problems. So, let us what is this just mentioned remarks from the remarks from this one.

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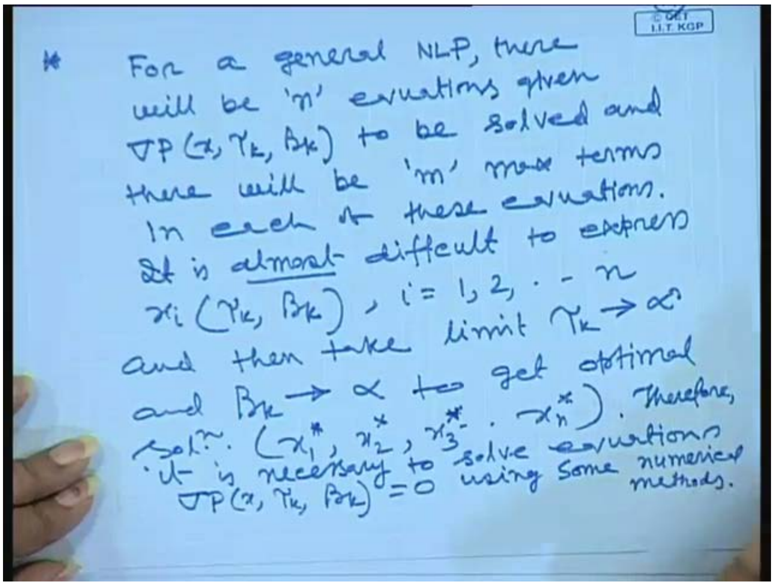

For a general non-linear programming problem there will be n equations because if the non-linear problem is a n decision variables are there. Then when we are taking the necessary condition that after forming the penalty function del p of del p which is equal to x 1 del p which is equal to x 2 we have a x 1 x n variables are there. So, we will get x n equations given by from where we will get it from gradient of penalty functions x prepared penalty function is a function of tau k and beta k to be solved and there will be m max term in each of this equation. In each of this equations because we have a m we have a m inequality constant in if when you have a inequality constant m inequality constant in penalty function you will get summation of max term of g 1 comma 0 whole square. Then g 2 max of g 2 comma 0 whole square in this way m terms will be there in each equation.

Again, it is almost that is what is the main thrust is there it is almost difficult to express x i which is a function of tau k beta k for i is equal to 1 to dot dot because we have a n decision variables. And then take limit tau k tends to infinity large value and beta k tends to infinity to get optimal solution of the decision variables x 1 x 2 x 3 dot dot x n. So, one should look for what is called the numerical method to solve a set of necessary condition if you have n decision variables are there you will get a set of n non-linear algebraic equation that you can solve by some suitable numerical methods that is our steps.

Therefore, it is necessary to solve equations del p of x tau k beta k is equal to 0 that is the necessary conditions using some numerical method technics, so this is the remarks for this one. So, next we will solve that what is called non-linear problem solution of non-linear programming problem using interior penalty function method or it is called barrier function method.

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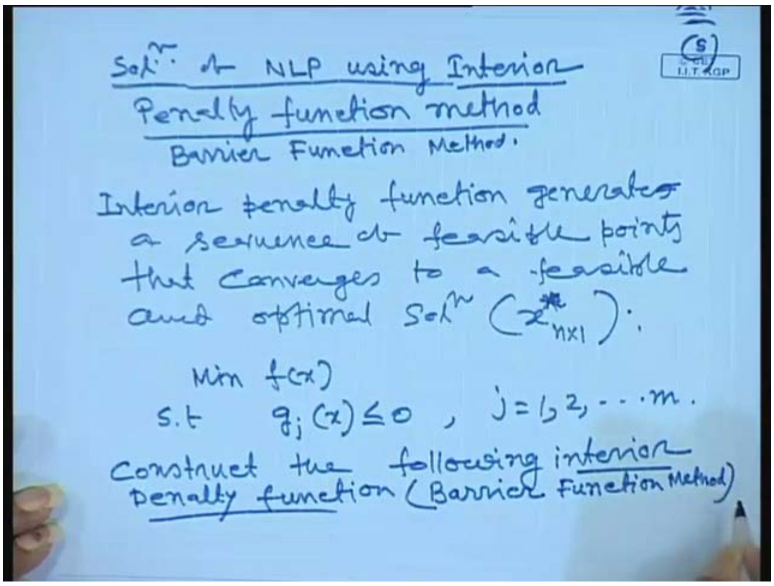

So, next is solution of next topics is solution of non-linear programming problem using interior penalty function method or this is called interior penalty function is method called is a barrier function method. So, what is this we will see first if you recollect this,

this one our earlier problems the exterior penalty function method generates sequence of infeasible solution that coverts to a feasible solution. In exterior penalty function method generate a sequence of infeasible solution which converts into a feasible solution and optimum solution.

Whereas, in case of what is called interior penalty function method it generates a sequence of feasible solution and ultimately it converts to a optimal solution feasible and optimal solution so this is the difference between the two methods. So, exterior you can write the interior point method interior penalty function generates a sequence of feasible points that converges to a feasible and optimal solution. That means x n x super star of this whose dimension is n cross 1 and if it is a n ((Refer Time: 29:27)).

So, our problem is we have a problem minimize the function subject to the constraint agree then interior point method generate is sequence of first initial guess is the interior point in the feasible region then it generates a another feasible point. Where the function value is decreasing and this way generate a sequence of feasible point and ultimately it converts to a feasible and optimum point, optimum solution. So, let us see this one the problem statement how we will do it.

So, minimize f of x with the objective function you have to minimize such that g j of x is less then equal to 0 j is equal to 1 2 dot dot m agree so this our problem. Now, we have to solve by using a interior point method agree, the interior point method means we have to take a initial guess which is inside the feasible region and satisfy this all this things feasibly region means it should satisfy the what is called all types of constraints.

So, corresponding to this your problem construct the following interior penalty function or it is called interior penalty barrier function. Barrier means obstacle function, barrier is ((Refer Time: 31:29)) understand the obstacle function the sense that it will not allow to move the decision variables boundary of the feasible region it will not allow to cross the boundary of the feasible region. Now, let us see this one what is the penalty function is considered for this one.

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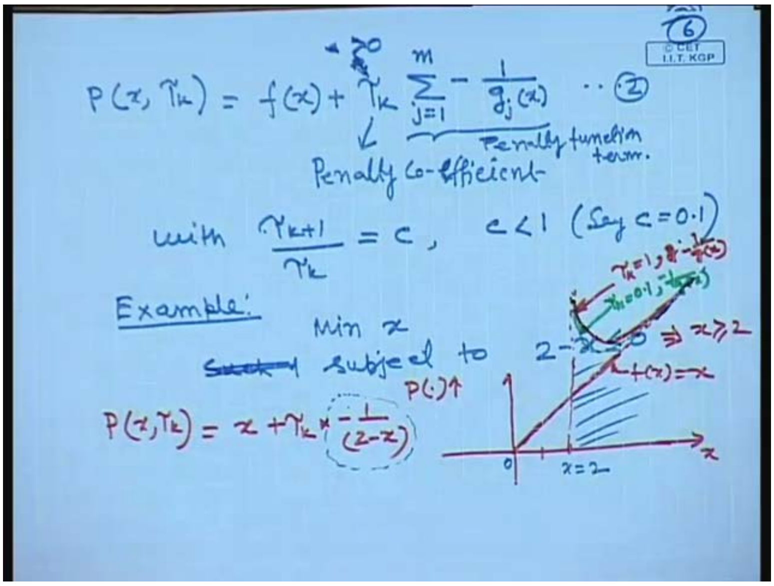

So, p x tau of k is similar as earlier also exterior point method f x plus tau k and tau k is our penalty coefficient. Since, we have a say since we have a m inequality constant we have to give the penalize this one j is equal to 1 to m minus 1 by g j x agree. Now, look this one our point is interior point means point is inside the feasible region when it is in feasible region g j of x value is negative. When it is negative, negative, negative is positive so this quantity is positive that means this quantity and also tau k value is greater than this tau k value is greater than 0.

So, this is our let us call equation number two and this set of equation what about the discussion of the problem it is a equation number one. So, with so this term is this whole term is penalty function term this and this is the penalty function term, penalty function terms. So, with tau k plus 1 next equation tau k value will be equal to c where c is c is less than equal to 1 say c is equal to 0.1. Now, we are decreasing the penalty function value penalty coefficients value one-tenth each iterations.

So, let us see the interpretation of this things why the tau k value will be decreasing and ultimately it is approaching to 0 get the optimal solution of the problem using interior penalty function method. Let us take this simple example, minimize x such that or subject to 2 minus x is less then equal to this so this implies, this implies x is this implies x is greater than 2. So, our feasible region is x greater than 2 if I plot it this x is in this direction and this direction is our prolonged function p penalty function in this direction if you plot it.

Now, let us see f of x is what first f of x is what is nothing but a x and that f of x you have to minimize so this is nothing but a is a straight line which is passing through a origin and our feasible region is if you see our feasible x greater than 2 so let us call this 1 this is 2 so greater than 2. So, our whole feasible region is that portion so this is x is equal to 2 this is 0 and this is you can write it this nothing but a f of x is equal to x and with this one there is a another function is added agree. So, our g of x for this one is 2 minus x so if you write it, if you write minus 1 by 2 minus x and our initial guess is we are considered in a feasible region.

So, the x value will be greater than 2 so let us see when x is equal to greater then equal to and x is equal to 2 the function value if you see this one the function value is x is equal to it is very large and then it will go on, go on decreasing and it will approach to the asymptotically it will approach to the objective function values. So, this so let us call this values is I am plotting is that this function value that means our inner case if you see the p in our case if you see p tau k is equal to f of x means x plus tau k only one inequality constant is there 2 minus x inequality constant is there.

So, I write it into minus 1 by 2 minus x so I am now plotting, I am now plotting this quantity again I am plotting this quantity so this value is infinity of that one and then if you go on increasing x then this value will be negative, negative plus. So, this will be a this way and it is approaching to a asymptotically it will approach along the objective function. Let us call you can think of it as if that our penalty coefficient value is 1 when you are plotting this one. Now, if the penalty function is made it half or one-tenth now what will be this the nature of the curve will be exactly same on only at each point of x the magnitude will be reduced by one-tenth if you are constraint that tau k value will be one-tenth.

So, this will be once again if you see this one once again this will be like this way. So, this is let us call tau k value is 1 again and I am plotting this is for thou k value I am plotting g of 1 by minus this I am plotting 1 by g of k g of x I am plotting, I am plotting tau is equal to 1 g of and this is corresponding to tau of k is equal to 0.1 and this is minus 1 minus g of x and plot in. So, in this way if you are reducing the value of tau k by one-tenth all these things ultimately you will see this value will be decreasing ultimately you will see this will be like this way and it will be approaching along this one agree.

And when x is equal to 0 you say when x is equal to sorry when x is equal to 2, x is equal to 2 or nearly equal to 2 this infinite value is this one. So, this will not allow to cross the value of feasible region, feasible region variables value cross the left hand side of this one that means it will not allow the this method will not allow to cross the boundary level of feasible region this method. So, that is why tau k value is slowly if increasing it will approach to the asymptotically to a positive function near to the boundary of the feasible region that is our main objective so I will just write it the remarks of that one.

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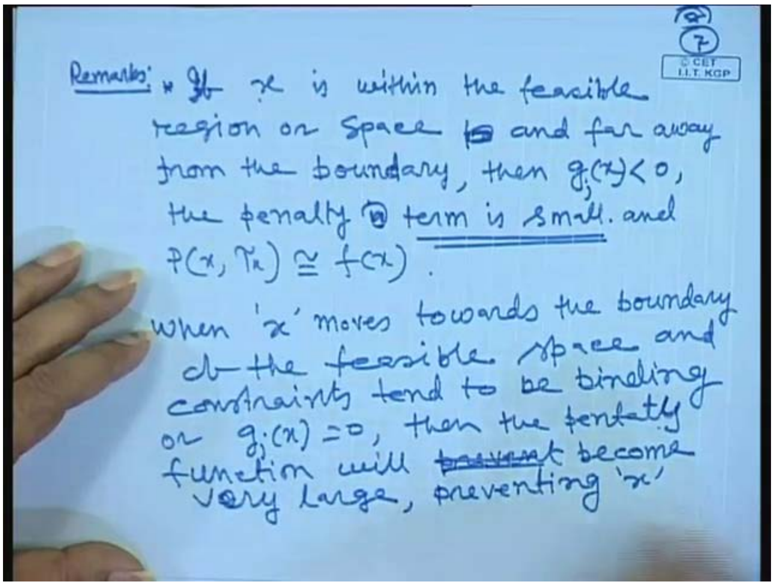

So, remarks so that is the basic idea of using the interior penalty function method. Now, remarks if x is within the feasible region, the feasible region or space for a feasible region space and far away from the boundary. Then g of x g j of x is less than 0 and penalty term, penalty term is small and agree. So, penalty term is small and p x of tau k will be nearly equal to f of x is first.

And second observation remarks is when x moves towards the boundary of the feasible space when x is moved towards the boundary of the feasible space and constraints tends to and constraints tend to be binding or g j x will be 0. Then the penalty function, then the penalty function will prevent or you write the penalty function will become very large and preventing a preventing x from moving out of the feasible point.

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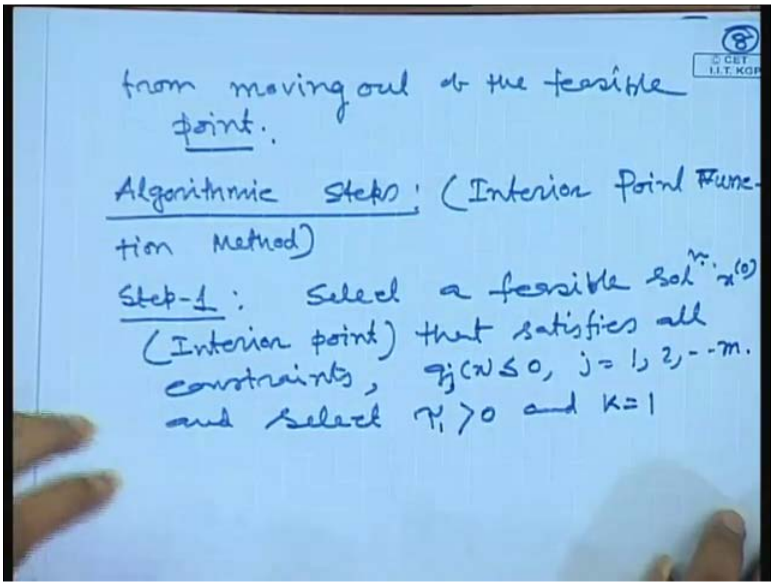

So, what we have discussed in this figure you see the first statement is made when x is far away from this one x is far away from this 1 you see this quantity is large that means 1 by this one is small it is small. So, in that situation penalty function value and objective function will be same that is that we told that one when the our variable x is closer to the boundary of that one when it is closer to this one then this quantity value is this quantity value that is 1 by x 2 value will be very large. That is when x is more towards the feasible region space tends to binding g x is equal to 0 then only this 1 by g x will be very large.

Then penalty function will be very large and will prevent the what is called the variable to go to the infeasible region. So, moving out from the feasible point so this is the basic idea behind using the interior point method so our algorithm steps, algorithm steps for interior point method, interior point function method same as earlier exterior point. First step is what we will select a initial point which must be inside the feasible region.

So, select a feasible solution or feasible point x superscript 0 and that must be a interior point that satisfy all constraints. That means g j of x is less then equal to 0 j is equal to 1 2 dot dot m and choose and select tau 1 greater than 0 and k is equal to 1 this is the first step. Once you select this one using the what is called necessary conditions of all these things you find the new improved value of the feasible point which reduce the function value from the previous value.

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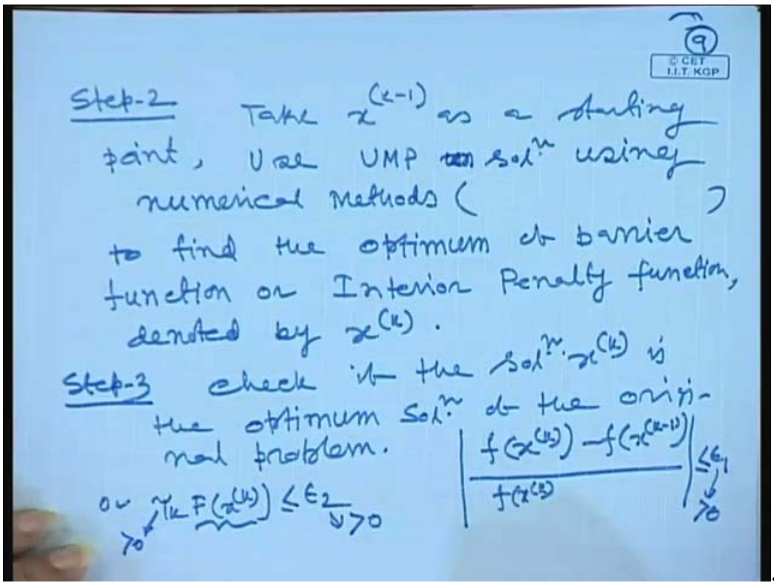

So, second step take x superscript of minus 1 as a starting point use unconstraint optimization, unconstraint optimization problem, unconstraint minimization problem if it is minimization problem m or unconstraint optimation problem and use unconstraint minimization problem solution using numerical methods. So, our numerical methods we have discussed what is called Newton Raphson method, ((Refer Time: 49:44)) method, conjugate gradient method modified Newton Raphson method we have discussed it earlier.

So, taking the initial starting point you get the first you obtain the necessary condition what the function to be optimised and you will get set of non-linear equation that you can solve by numerical method or analytically use optimum solution using unconstraint minimization problem solution using numerical methods. To obtain to find the optimum a barrier optimum of the barrier function or interior penalty function denoted by x superscript k from x k minus 1 you get it that one.

Next, step is step three is check if the solution is list to the optimum check if the solution x superscript k is the optimum solution of the original problem if the optimization is original problem. How to check it I told you one way of checking it you find out the value of the function at kth iteration then subtract wit k minus 1 th iteration value divided by f of x k value. So, mode of this one if you see if it is less then equal to epsilon, epsilon is a positive quantity, positive small very small quantity is this one or you can check it this or you can check it that tau k or f of x superscript k. That value is less then equal to epsilon 2, epsilon 2 is less then greater than 0 and very small quantity tau k is the penalty co efficient. Whose value is greater than 0 and these value you will see when it is in the feasible region that value is also greater than 0.

And these f of x is what that each term summation of f that is what is called summation of minus 1 by g j of x. That is f of x repeat summation of j is equal 1 to m minus 1 by g j of x is the f x check this one if it is satisfied stop the iteration otherwise you go to the next step.

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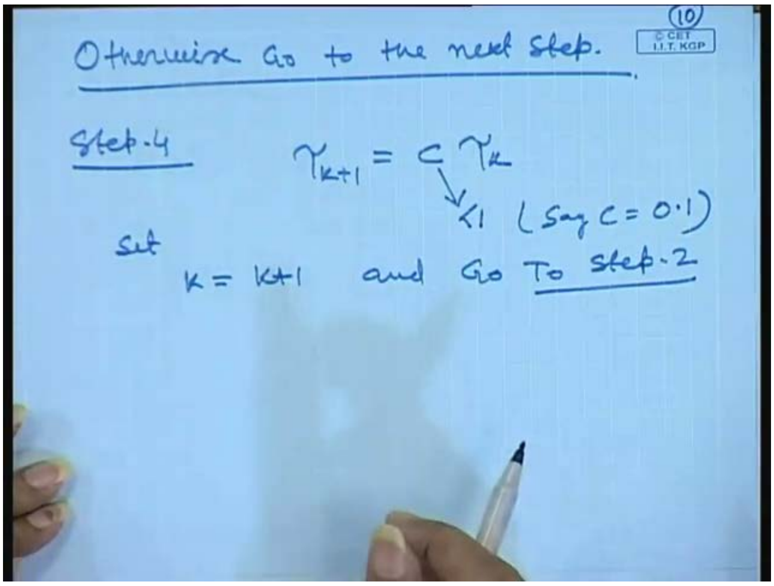

Otherwise, go to the next step so step four when you go to the step update your penalty function coefficient c by next year function method I told the value of c is greater than 0. That value of C is greater than 0 and mostly it is greater than 0 and less than 1, say C is equal to we consider 0.1 and we are decreasing the value with by one-tenth. That we are saying why we are decreasing this one it is the feasible region which is far away from the boundary of this feasible region, it will approach to the near the boundary when you do like this way so that things.

And set k is equal to k plus 1 and go to step two and step two is this one we will start the iterative process. Once again this one and in this way you will do until and unless this condition is stopping criteria condition is not satisfied either one of this is not satisfied. So, this way the interior penalty function method or barrier function method is solved and this two methods are efficient in solving the non-linear what is called optimization problems. Then we will see next that how to solve a problem by taking a numerical example embroiling that the technique what we have discussed the interior penalty function method or barrier function method. So, I will stop the lecture here only.