Optimal Control
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Lecture - 21
Relationship between Primal and Dual Variables

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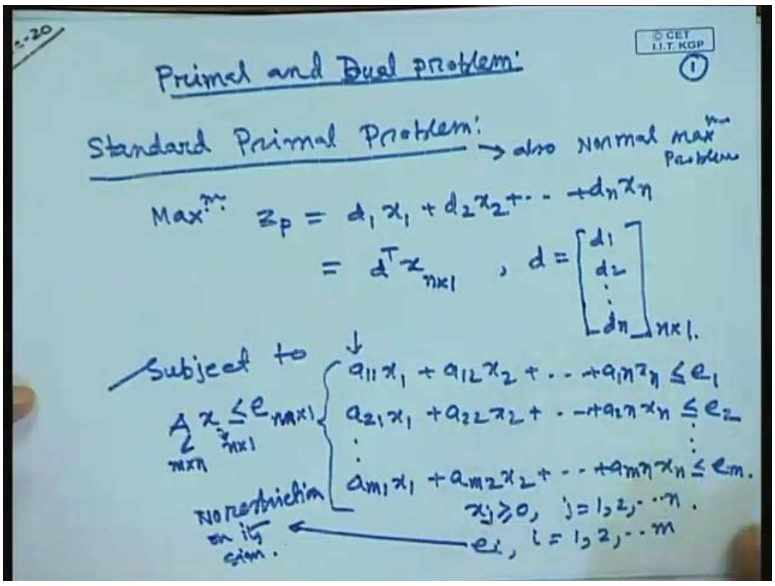

So, last class we have discussed the primal and dual problem. The standard primal problem is like this, way make maximum a linear function, which is where x 1 x 2 dot dot x n are the design variables which you can write it vector form like this way. Subject to a set of linear inequality constraints, which you can write it a x less than or equal to e. So, this type of form of linear equations either in objective function, or in and in what is called in inequality constraint, if it is like this way then it is called standard primal problem.

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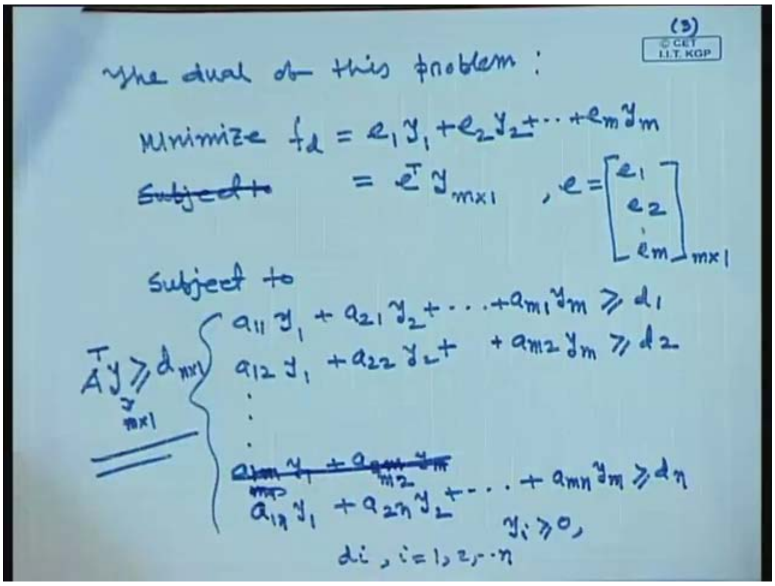

And dual of this problem is the structure is like this way minimize the function f of d, whatever the right hand side of this equation was there inequality equation right hand side of the this will come, as a coefficient objective function linear objective function of the dual problem. The e 1 y 1 e 2 y 2, so how many variables will be there equal to the number of inequality constraints in final problems. And subject to that this that this type of inequality conditions, inequality condition in on form of linear equations that is we when you are converting into dual problems and we know to how to solve this one.

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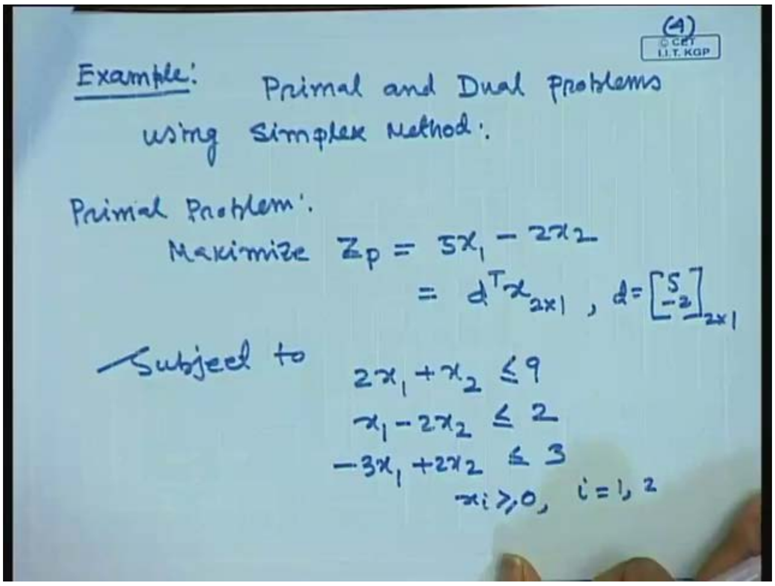

If the final problem is given then convert into a standard first convert into dual problem then dual problem you convert into a standard L P problem, then you solve it by either phase 1 method or phase 1 and 2 both methods are depending upon the type of inequality involved in the set of linear equations. And we have worked out the simple example and we have seen that the values…

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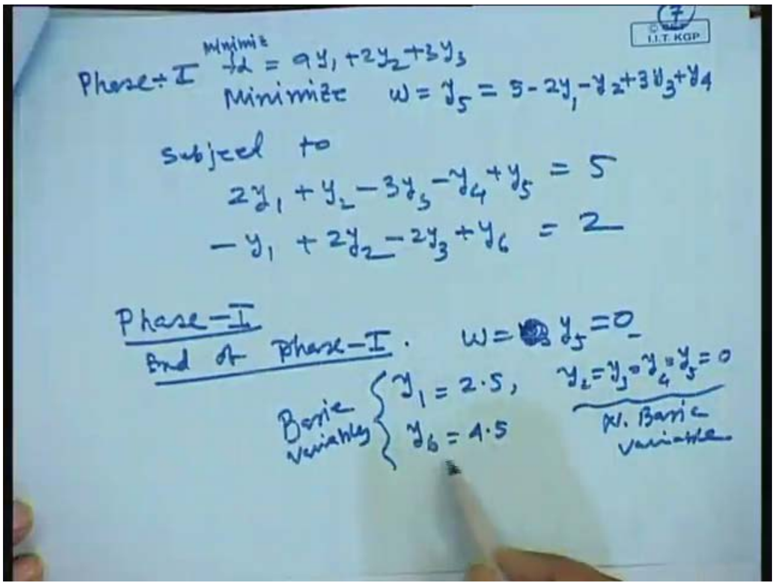

That is y 1 y 2 dot dot y 6 variables which is dual variables values are there then question is how to go back to our final variables, which is the problem is original problem is given that way now. Today's lecture is we have to find out the relationship between the primal and dual variables or what is the relationship between primal and dual problems? So, first relationship you can say if one problem has a feasible solution, if one problem is feasible solution and unbounded, what is if one problem has a feasible solution and has a bounded objective unction value then other problem should have a feasible solution, this is one thing. Next is if one problem is next, next point, if one problem has a feasible solution.

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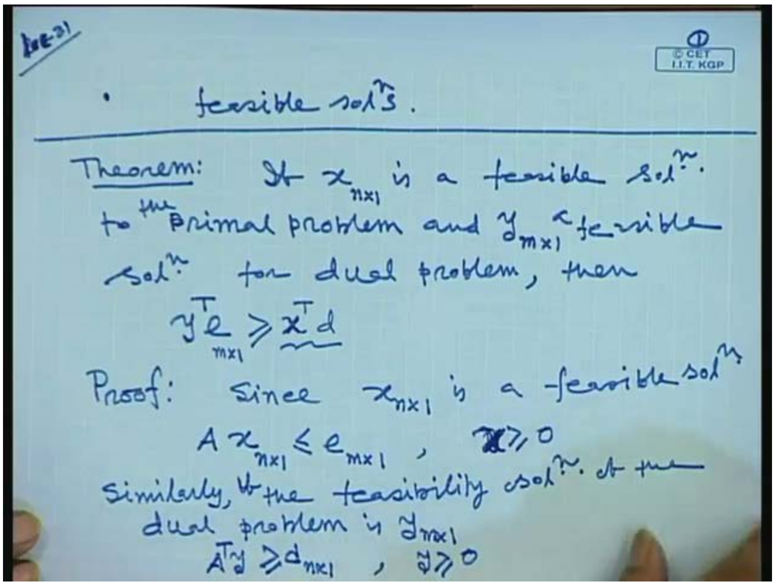

If is this is feasible solution and unbounded objective function value and then other function is also do not have any feasible solution, feasible solution. That means, if one problem has a feasible solution, but unbounded objective function that other problem having a infeasible solution, this is a second observation you can see and the third one. If one problem has no feasible solution and other problem other do not take feasible solution or they have what is called unbounded objective function value. So, this...

Now, next we will see it how can convert the feasible variables, how one can convert that our feasible variables let us call what is called dual variables, how one can convert dual variables in the final variables and vice vice versa, before that we will study the first see the few theorems, that theorem if x having a dimension n cross 1 is a feasible solution to the to the primal problem, and y is our m cross n feasible a feasible solution for the dual problem. Then one can write y transpose e which is objective function of the dual problem y transpose z or e transpose y is value objective value function is greater than equal to x transpose d.

The x you write it d and you know the dimension the dimension e you know what is dimension of e, e is dimension is m cross 1 that primal problem how many inequality constraints are there, there are n constraint are there. And same number of variables presence present in the dual what is dual problems, so this is a scalar quantity and this is also scalar quantity. So, our theorem tells if x is a feasible solution of a final problem and y is a feasible solution of a dual problem, then the objective function of the dual problem is always greater than or equal to objective function value of primal problem. That is one lets us see the proof of this one.

So, since x is a feasible solution is a feasible solution of the primal problem, feasible solution then what is the kind it must satisfy the our inequality constraints of primal problems. That means A into x to the dimension is n cross variables are there A is less than or equal to e, that is dimension e m cross 1 and what is the variables in dual problems? Y is greater than or equal to 0 and also that x or I write it x is greater than or equal to 0 for this one.

Similarly, in dual problems the feasible solution of the dual problem is y m cross 1. Similarly, if the feasible solution of dual problem is y then we can write it immediately we can write it A transpose y is greater than or equal to our d, and d dimension n cross 1 and y and dual variables values are all greater than 0. Now, see from this one this equation if I multiply by both side by x transpose then one can write it if I multiply the both sides x transpose for previous equation then we can write it.

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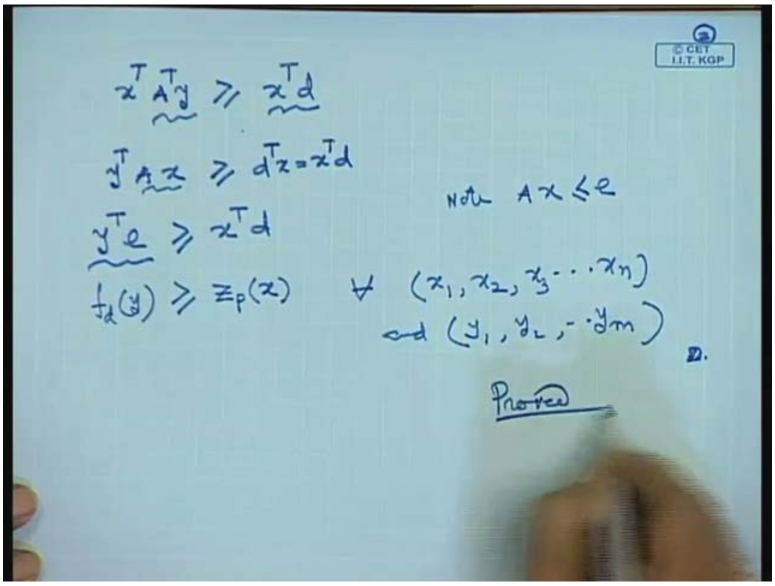

x transpose a transpose y is greater than equal to x transpose d this is a scalar quantity this is a vector column vector we have multiplying this which is nothing but a case is d, if you see it and this is a vector of dimension if what is dimension n cross 1 and this is the dimension of 1 cross n x transpose. So, this is a scalar quantity if you take a transpose both sides because scalar quantity turns for both sides remains same. So, I can write it y transpose A x is greater than or equal to d transpose x, which is nothing but x transpose d same because scalar transform scalar value same values is there.

Note you can note that that A x in the final problem this A x is based on is less than or equal to e. So, if I replace this 1 by greater number this inequality sign still valid. So, I can write y transpose e is greater than or equal to x transpose d and what is this one, this is nothing but a objective function value of the dual problems. So, this I can write it f dual problem objective function which is a function of y is greater than or equal to the objective function value of the primal problem. That is function of x, this always proves for all x, x components is x 1 x 2 x 3 dot dot x n, n for all y 1 y 2 dot dot y n. So, this proves the theorem of this one proved.

So, if you tell one thing if x is the feasible solution of the primal problem and y is the feasible solution for dual problem then we can write it the objective function of the dual problem is always greater than or equal to the objective function value of the primal value. That is the our conclusion which we have proved here. Next we will see that another theorem that is theorem.

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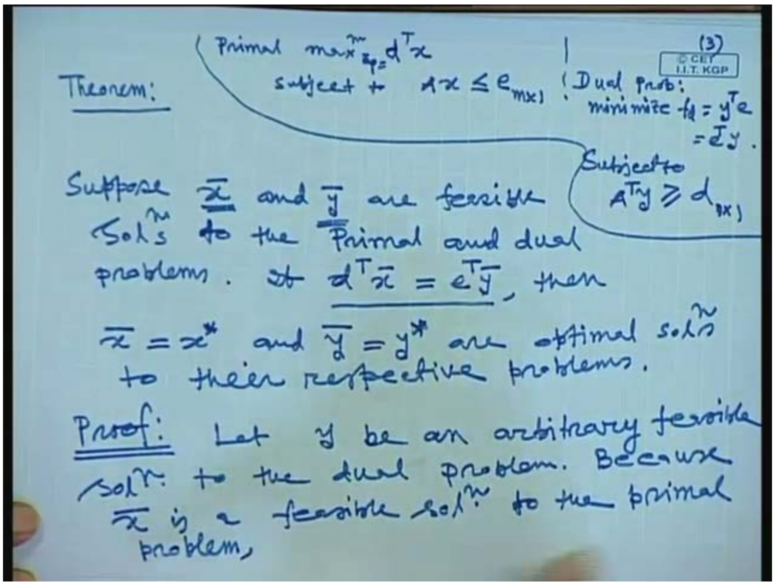

And if you remind the just writing it here primal problem is what I am writing maximize z transpose of x which is nothing but a we have written it z of p is equal to this one, subject to A of x is less than or equal to e. That e is number inequality constraints in the dual primal problem is correspondence in dual problem is what, minimize f of d and what is the right hand side of the primal variables that will come as the coefficient of the objective function in the dual problem.

So, it will be a if you see this one that is nothing but y transpose, y transpose e or you can write it e transpose this. Subject to A transpose y is greater than or equal to d, what is coefficient in the primal problem are coming in the objective function that will come that right hand side of the inequality constraint of dual problem. That is dimension is that is our basic statement of the primal dual problems.

The next theorem is this suppose x bar and y bar are feasible solution solutions to the primal and dual problems to the primal and dual problems. That means x bar y bar are the solution of the primal or dual problems, if the feasible solution of the dual problems, then if d transpose x bar equal to if it is equal to that y transpose or e transpose y, if it is this y bar that are feasible solution of the primal improvement. If this is equal to this then x bar which we have considered feasible solution that will be x star, means optimal solution of the primal problem, x bar and y bar will be equal to y star.

That means, optimal solution of the dual problem means maximization of the function variables, that optimal points will get it here if this equal to this as this, this indicates this is the optimal point of the dual and this is optimal point of the primal problems. These are optimal solution to their respective problems, so please try to understand what is this if x bar and y bar are the two feasible solution, if x bar are the two feasible solution. If you get the objective function of d objective function value of the dual problem value and the primal problem objective function value are same at that two feasible points, then we call that two feasible points are the optimal solutions of their respective problems.

Then let us see the proof of this theorem, so let y be an arbitrary feasible solution, y is an arbitrary feasible solution of dual problem this arbitrary feasible solution not optimal it is at this moment to the to the dual problem, because x bar because x bar you see is a feasible solution to the primal problem, then one can write immediately we can write it.

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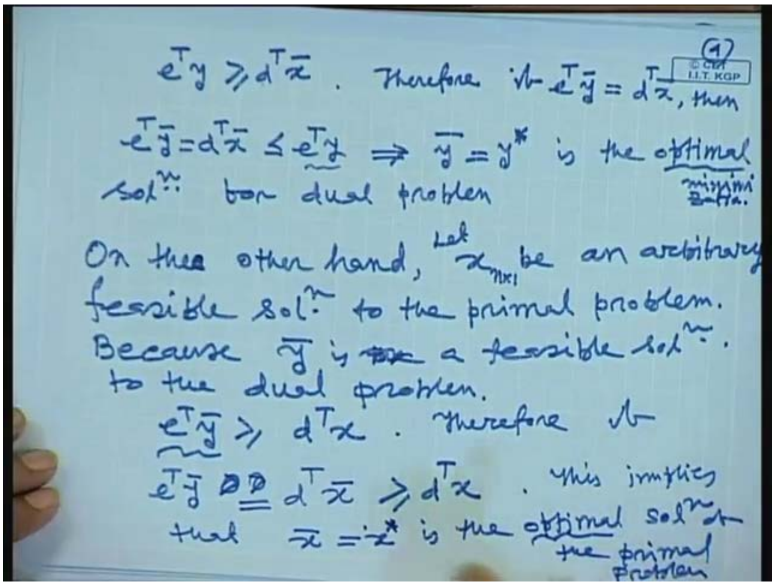

e transpose y objective function of this one is greater than or equal to that d transpose x bar, this we can write it in a. Therefore, if e transpose y bar there are values of y bar some feasible solution if this is equal to that d transpose x bar, this indicates then this employs e transpose y bar is equal to d transpose x bar, x bar is less than equal to less than equal to e transpose y, you see d transpose x bar is less than d transpose is there. So, some other than y is a there is a feasible point, which equal to this, this indicates this quantity of objective function value of primal and dual is less than e transpose of that one. So, what does that mean?

This employees that our y star y bar this employees y bar is nothing but a our optimal solution of the dual problem is the optimal solution for the dual problem, optimal means optimal solution dual problem, dual problem is minimization of the problem minimization. Similarly, on the other hand on the other end with the same reason on the other hand, if let x on the other hand, let x enclose 1 be an arbitrary feasible solution, arbitrary feasible solution x greater to the primal problem because according to the statement of the problem because y bar is the feasible solution, is the is a feasible solution of the dual problem to the dual problem.

Immediately you can write it that y transpose e transpose y bar is greater than or equal to d transpose x this is a objective function of the dual problem of the dual problem, where y bar is the solution in the feasible solution in the dual problem is better than this one. So, this is greater than this one.

Therefore, if e transpose there are some choice of x is some of x some value let us call it is a d transpose x bar some choice of x is equal to x bar, this objective function is same. And this is this quantity will be greater than or less than or this quantity will be that this is a quantity d bar of this will be greater than or equal to d transpose x, e of transpose this and now telling their exist let us call some feasible solution x bar for which this value is increased, objective function is it and it becomes equal to the e transpose y bar.

And since e transpose y bar value is greater than d bar of x this indicates that x bar this employs, this employees that x bar is equal to x star is the optimal solution of the primal problem. Optimal means, there are maximum that optimal if x is the optimal solution of the corresponding to the maximum value of the function x star will give you, and that proves the theorem. So, there are 2 theorems just told you If x is if x bar is a feasible solution or primal problem, and y bar is a feasible solution of a dual problem.

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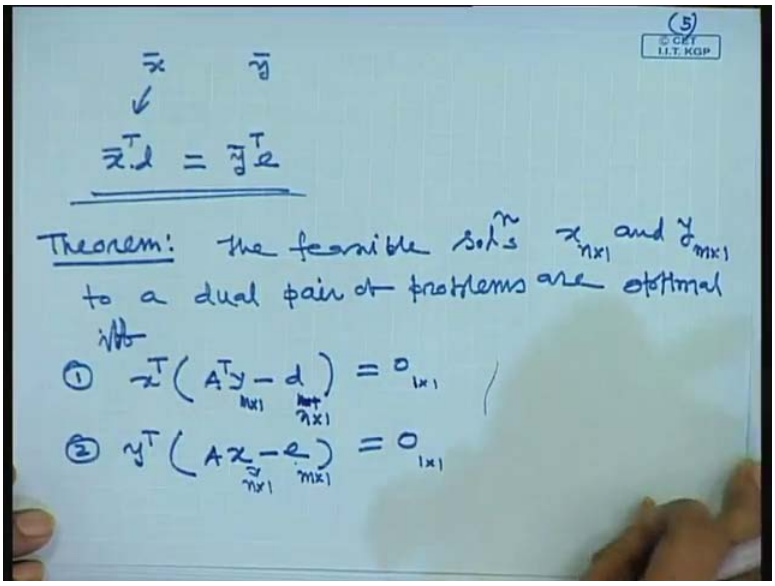

If this that means, what is this objective function of x of this one if x, x bar transpose into d is equal to that y bar transpose y bar transpose into e if it is same or vice verse d transpose x is equal to e transpose y because it is a scalar quantity, I can take transpose same then this indicates x bar at y bar at the optimal solution of the dual, what is called primal problem. Optimal corresponding to the optimal solution of the dual problem and x bar which is equal to x star the optimal solution of the primary problem, if their objective functions are value is same.

So, next is very which stated theorem, which will give you the relationship between the primal variables and the dual variables the theorem will tell you, how they are related to primal variables x and dual variables y, how they are related the theorem. And keeping in mind that primal standard primal problems statement and the dual corresponding dual problem statement keeping in the mind, we can write it now the feasible solution. The feasible solution x and y to a dual pair of the problems are optimal if and only if, if and only if necessary and sufficient conditions.

The feasible solution x, x corresponding to the primal problem and y corresponding to the dual problems there is a dual pair are optimal if and only if, 1 that x transpose then you write the inequality constraints associate to the problems. That means a transpose y minus d and that dimension n plus 1 similarly, this dimension is what should be the dimension of this one m cross 1, this is m cross 1 this dimension is n cross 1 this equal to 0. Another condition is that we write y transpose then the dual problems we can write the inequality constraints, that you got it what is called in the primal problem that A into x minus e is equal to 0, this is scalar quantity this is scalar quantity. And this dimension is m cross 1, this dimension is n cross 1.

Similarly, you can find out dimension as you earlier so our theorem tells if x and y are the feasible solution of the dual problems, this are optimal this dual problems are dual problems x 1 x 2 are the dual problems are optimal, if and only if this condition is satisfy. That this indicates that if you know the solution of the dual problems one can find out what is the solution of in primal problems, when a original given into a if it is primal problems, what is the solution that we can find out using these two relationship. So, let us see the proof of this 1 proof is very easy to see to how to get it this proof, prove it this one.

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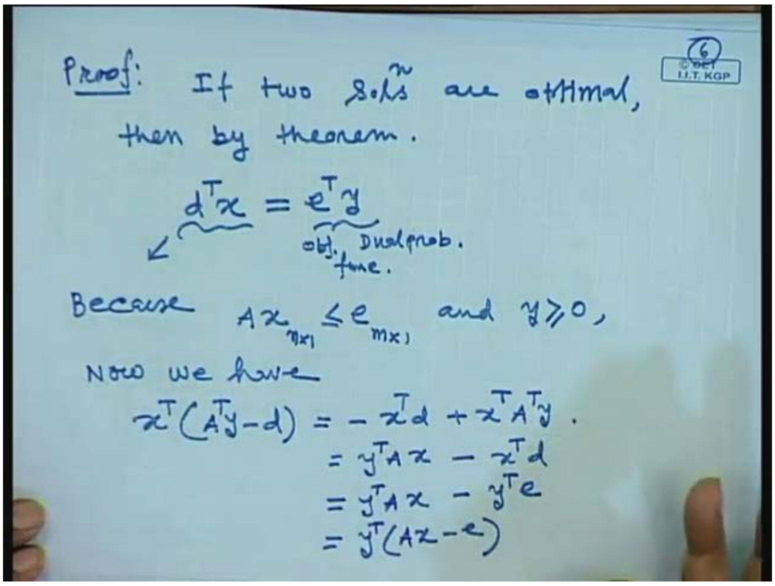

If the if two solution are optimal then by theorem, the optimality theorem what we gain d transpose x must be equal to y transpose or e transpose y, agree d transpose x is equal to the, this is the function of the dual problems, dual problem objective function objective function of dual problem. This is the objective function of primal problem, so if the solutions are optimal then only this equal to this optimal, this equal to this we can write it. We know that if x is the optimal solution of this one that means x is a feasible as a feasible solution that because we can write it because x of k A into x with the is less than or equal to e. Since, x is a feasible solution x is also optimal and feasible solution and then is to not only this in the dual problem this y is greater than or equal to this.

Now we have now we have let us see x transpose A transpose y minus d, what is what we get it this equation? This equal to you just expand this bracket you open it, then x transpose d plus x transpose A transpose y, since it is a scalar quantity I can write it this inverse y transpose A and x minus x transpose d, if you take x transpose d or d transpose x is nothing but y transpose d. So, you can write it first term as it is second term I can write it easily y transpose e, if you take y transpose common then this is x transpose of e. So, x transpose a transpose is nothing but y transpose x is y, now look at this expression individually the left hand side and right hand side then what we will get it see this one left hand side.

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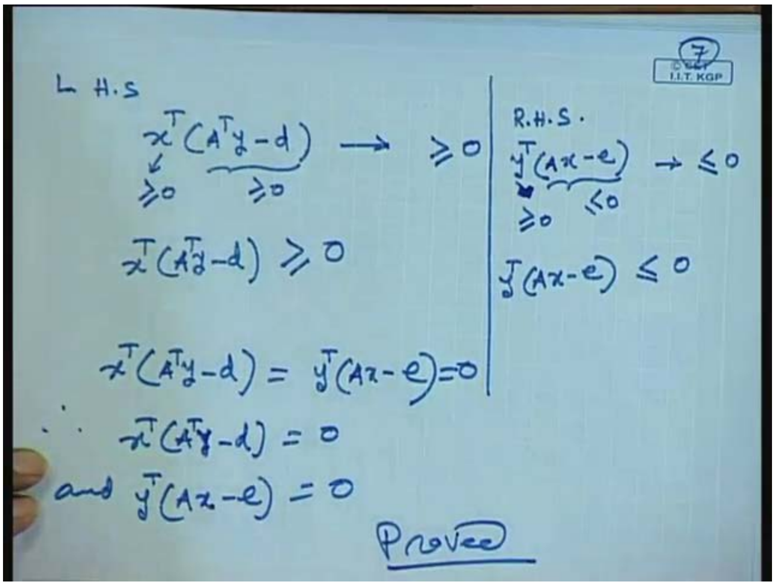

x transpose A transpose y minus d, what we can write it x is the all the decision in the primal problem and then in that values each element of x there are n variables are all are greater than or equal to 0, whereas the equality constraint in dual problems, this quantity if you see the inequality constraints this quantity is greater than or equal to 0. So, this employees the results of this one is result of this one is becoming less than or greater than or equal to 0.

That means I can write it that x transpose A transpose y, y minus d is equal to greater than or equal to 0 the left hand side. Now, see the right hand side y transpose A x minus e, what is this one? This is dual problem variables this values each elements of this values vector y is greater than or equal to 0, and whatever the constraint in the primal problems a x minus a is always less than or equal to 0. So, this employees that this quantity is greater than or equal to 0 or less than equal to 0, this indicate that results is less than or equal to 0. Means, y transpose A x minus e is less than or equal to 0, just now we have already seen 2 equivalent equal and one is it is telling that is equal to greater than 0.

Another case it is telling this equal to less than 0, but both are equal value so this cannot be true this is only if is 2 only if x transpose A transpose y minus d equal to y transpose A x minus e equal to 0. In other words we can write x transpose individually this two terms y of d is equal to 0, and y transpose A x minus e is equal to 0. So, this is proved to the theorem.

So, using this theorem one can get the values of dual values dual variables values from the primal variable values and vice versa. I can get the primal variables values after solving the dual problem by using this theorem and vice versa. So, this is the theorem let us workout one simple example and see how 1 can solve such type of problems let us see that example.

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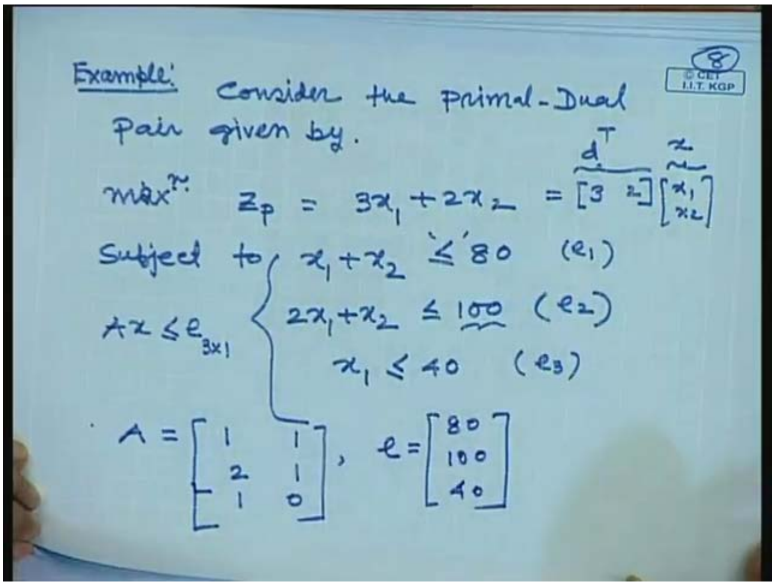

Consider the primal dual problem pair given by just here we will solve here, we know the solution of dual problem how to get the solution of primal to solving the what is the primal problem in solving using L P problems simplex method. So, let us call our primal problem is maximize z of p is given 3 x 1 plus 2 x 2 subject to x 1 plus x 2 is less than 80. So, this is the inequality constraints of this type if you do not have at this type, you have to convert into this type this is equal to this type of form by some manipulations, we have to do even if you have a equality sign, equality sign then we have to convert into this form that you know we have already discussed in the details earlier.

So, that equals this part we have written is e 1 if you recollect then we have a another equations inequality conditions to x 1 plus plus x 2 is less than or equal to hundred is e 2 this is the our e 2 right hand side of the inequalities. And we have one equation are there x 1 is less than 40. That means it e 3 40 is equal to right hand side of equality is 40, so if just write it in terms of our notation that is d, d transpose of this is 3 2 I can write x 1 x 2. So, this is nothing but a our g transpose this is nothing but our x so this set of equation I can write it into matrix form, that is A into x is less than equal to vector e whose dimension is 3 cross 1. And now identify what is our A, A is in our case if from this 3 equation one can write 1 1 2 1 1 0 this is our A, and this is what is e, e is our if you see that 80, then 100, then our 40 is our e. So, it is a standard L P problem, what is dual what is primal problems?

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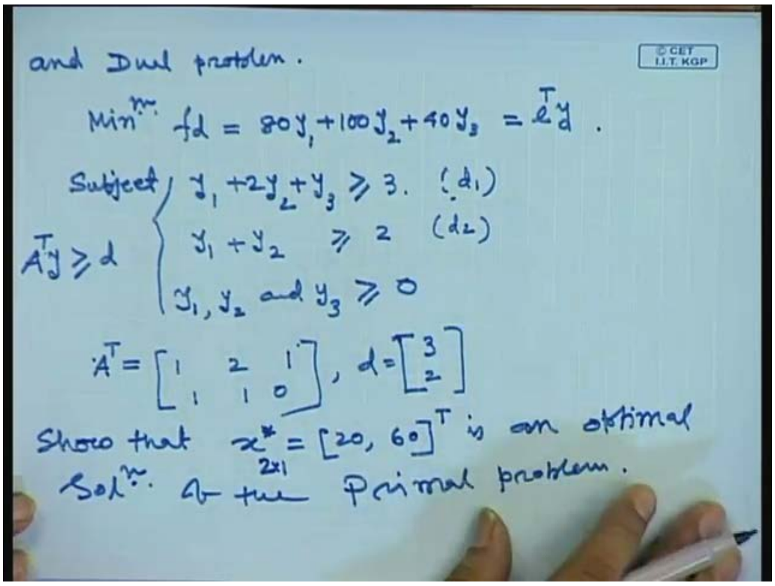

Now the corresponding dual problems, dual problem minimize f of d now you see what is this? This will come as a coefficient in the dual problems of the objective functions. So, that is 80 y 1 plus 100 y 2 how many variables will be there in the dual problems saying as the number of inequalities present in the primal problems, plus 40 into y 3 and subject to so this coefficient first column of this coefficient multiplied by y 1 y 2 y 3. So, it will be a y 1 plus twice y into plus y 3 is greater than or equal to first coefficient of the primal objective function 3. That is then next is y 1.

Similarly, I can write it y 1 plus y 2 greater than or equal to 2 that we denote it by d 1, d 1 coefficient is this is d 2 and then we have a because we have a two coefficients see here so two inequality constraints will be there in dual problems, and y 1 y 2 and y 3 will greater than or equal to 0. And what is our A transpose if you write in this set of dependent is nothing but A transpose y is greater than equals to our d, and then you see our a transpose is nothing but A what is A got it that transpose 1 to this matrices, if you write this matrices y into form is 2 1 1 1 0 and it is nothing but a what are the A is got it that transpose, agree? Than our y we know y 1 y 2 y 3 and d our case is these again case is 3 2 and it is nothing but a we can write it this one, if is see this one we can write it e transpose y, agree? So if you see the what e is this 1.

So you can solve this formula and by suppose this is here given primal problems and if you are asked to solved, show that whether I can say show that, that x transpose x which dimension is 2 cross 1 is equal to 20, 60 transpose is an optimal solution of the primal problem, that is you show it. Once in a optimal solution of the primal problem that x 1 x 2 x 1 is 20, x 2 is 60 must be feasible solution of the primal problems.

So, let us see how one can solve this problem by that is first you solve be to this one. Suppose, you want go back to from one solution to the what is call the I want to go this is the final solution you have solve it that is I have to show it, so end no the solution. Suppose, I want to know the what is dual problem solution that means y 1 y 2 y 3 start, what is it solution of this one that I can once I know the solution of final problem, I can get the solution of dual problem by using the our that theorem plus theorem, we have discuss or if you have know the solution of the dual problem. We can go back with the solution of our final problem by using that theorem.

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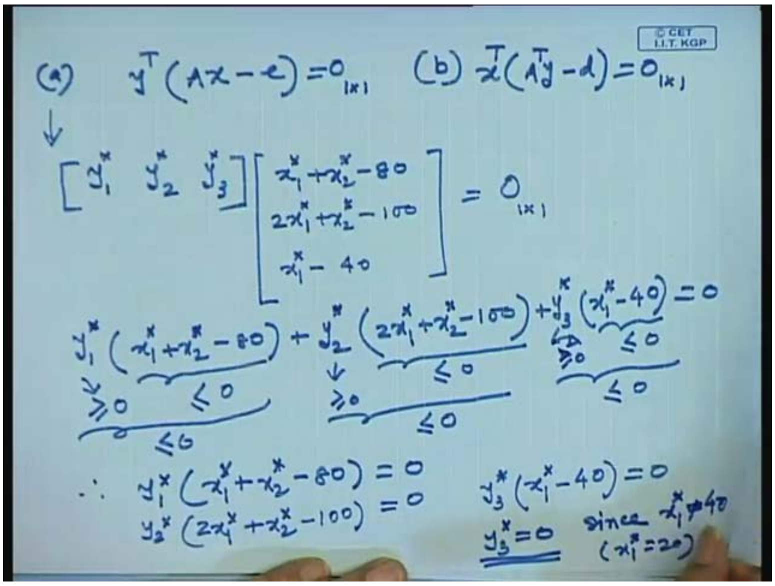

Now, see our the complementary condition that implies conditions is relationship between the dual variables and the primal variables are this two equation. If you see a y transpose ((Refer Time: 43:22)) A into x minus e is equal to 0. And this is 1 cross 1 and another condition is x transpose A y of this minus d is equal to 0, this is 1 cross 1. So, this now let us see that once you know this one, how can you find out y start that means, if you know the final solution how you get the dual problem solution y 1 y 2 y 3, let us see. So, you can write it that y has a 3 components y 1 y 2 y 3 is a called as transpose is a becoming a row vector.

And the theorem tells if you see this theorem it tells that if you have a what is called here if you x and the solution of the dual problem, or primal or optimal then this equal to this optimal solution. And our problem is given show that the optimal solution is that the find out dual problems y 1 y 2. So, I can write it this or I minute like this is in star of y star, star indicates the optimal solution of this one, multiplied by A x minus e and that already we have seen our problem this A x minus e problem if you see this is A x minus if take this is that side is that that one only.

So, x 1 plus x 2 minus 80 so x 1 star plus x 2 star minus 80 is 1, first equation then second equation 2 x 1 you see 2 x 1 plus x 2 star 2 x 1 star plus x 2 star minus 100 then the third 1 x 1 star minus 40, this equal to 0, 1 cross 1. Now, look this one if you expand that that 1 y 1 star multiplied by this into this x 1 star plus x 2 star minus 80 plus y 2 star 2 x 1 star plus x 2 star minus 100 plus y 3 star plus into x 1 star minus 40 is equals to 0 look at this expression y star value is what? We expect that value must be equal to get to an equal to 0.

And what is this values this values, if you see from the what is called in equality constant of primal problem x 1 plus x 2 minus 80 is less than equal to 0. So, this quantity is less that equal to 0, so whatever resultant of this one we will get tell me, greater than equal to 0 and this quantity is great than is less than equal to 0. So, the results will be less than this products of this one is less than is equal to 0, this one. Now, what about this one similarly, this is greater that equal to 0 and this is 2 x 1 plus x 1 minus 100 is less than equal to 0 less than equal to 0.

So, resultant of this one will be product of this will be less than equal to 0 similarly, this one in the same logic you can write y 1 this y 3 is greater than equal to 0, and this one less than equal to 0 and the product is less than equal to 0. So, whole thing is 0, so there is no possibility of cancelling one and other either to be negative if was what situation, if all be negative which does not satisfy this right in this side. So, this only satisfy in each component of this one is 0.

So, I can write it therefore y 1 star into x 1 star plus x 2 star minus 80 is equal to 0, and then y 2 star is 2 x 1 star plus x 2 star that component minus 100 is equal to 0. And the last 1 is your y 3 star x 1 star minus 40 is equals to 0, and this set of equation I got it from this equations, this condition and this condition is getting from what is called the theorem that if x and y are the 2 optimal solutions of the dual pair, then you can write this and this equalities here only then if it is rare.

Now, look at this expression we have what solution is given and final solution is given x star is 20 and 60. That means x 1 is 20, so in this equation if you put x 1 is 20. That means this quantity is not equal to 0 then it denote what to make this is 0, y 3 must be 0 so y 3 star is equal to 0. Since, x 1 star is not equal to 40 again our case in your case x 1 star is what? X 1 star is 20, that why we got it x 3 is the dual problem solution x 3 star is 30 agree? But now look at this expression that x 1, x 2 now from the set of equation this set of equation what you can write is from b conditions.

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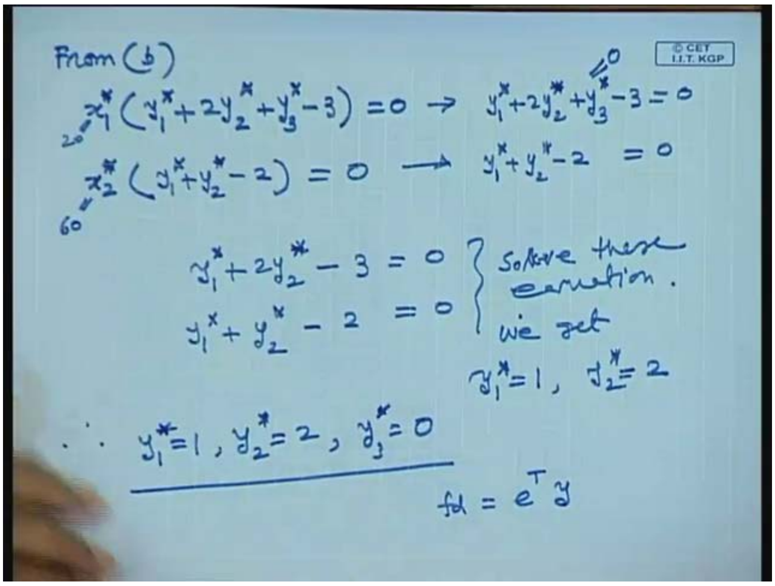

From b conditions similarly with the same logic If you expand that one you will get it from b condition you will get it x 1 star, y 1 star plus twice y 2 star plus y 3 star, minus 3 is equal to 0. You see I am just applying that x 1 x 1 star is x 1 stands and x 1 transpose then x 2 x 3 and then multiplied by A transpose y minus d you will no A y transpose minus d that is our A y transfer minus d is this equation. So, I am writing this equation now, so x 1 star into twice y 2 star plus minus plus y 3 minus 3 is equals to 0, because if you expand in the similar manner with same logic, I can say that in the individual component must be 0. So, this is 0 agree? And another condition I am getting it here x 2 star into that equation A transpose minus y minus d that one that y 1 star plus y 2 star minus 2 is equals to 0, agree?

So you the value of x 1 is 20, this value is 20 primal solution and this value is our 60 or you see this 1 is given that problem is given 20 and 60, so this is 60 so this implied that y 1 star plus twice y 2 star plus y 3 star minus 3 is equals to 0. And this implies this since it is 60 cannot be only possible that y 1 star plus y 2 star minus 2 is equal to 0. And now this value just now you have found out from condition A is 0, so we have to solve the two equations now y 1 star plus y 2 star minus 3 is equal to 0, another is y 1 star plus twice y 2 star minus 2 is equal to 0. So, solve this equations we get we get y 1 star is equal 1, y 2 star is equal to 2. So, our solution in the solution on the dual variables the optimal solution of the dual variables, we got it y 1 star is equal 1, y 2 is equal to 2, y 3 is equal to 0.

And clearly if you see the graphical representation of this problem it is a feasible corresponding function values is what? You can get it value in this equation, in which equation we have to put in this value that f of d our nothing but e transpose y our e is what 30 into 18 into 1, 100 into if you see y 1 value this is the our dual problem 18 into 1, 100 into 2. So, 200, 280 and y 3 value is 0 similarly, if you in the primal problems here, primal problems our solution is just now we have assumed the solution is 20 n, 20 n 60 what is this problems, just a minute.

So, this our this is our primal problem is that one, if you just, if you just apply to use the 20 and 60 is an optimal solution of the our primal problems this in the objective function, if you put the objective function x is equal to 20 and 60, you will the same objective value function as you got of the dual problems.

So, I will just show you this results now here, here you see here you see optimal solution of this one what do you got 20, 20 into 3 20 into 60 and this is 60, 60 into 2, 120, 120 and 60 180 and dual problems. What the results we got it, x 1 y 1 is our 1 y 2 is 2 and our that, that value just if you see this one that value will got also 180 just now we told it you got it 180 in the dual problems here 80 y 1 is 80 y 2 is 2, y 280 here you are getting 60 into 2, 120 into 60 180 you will get it the same as you did in the both cases. So, we will stop it here ok.