Optimal Control
Prof. Dr. Goshaidas Ray
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 20
Standard Primal and Dual Problem

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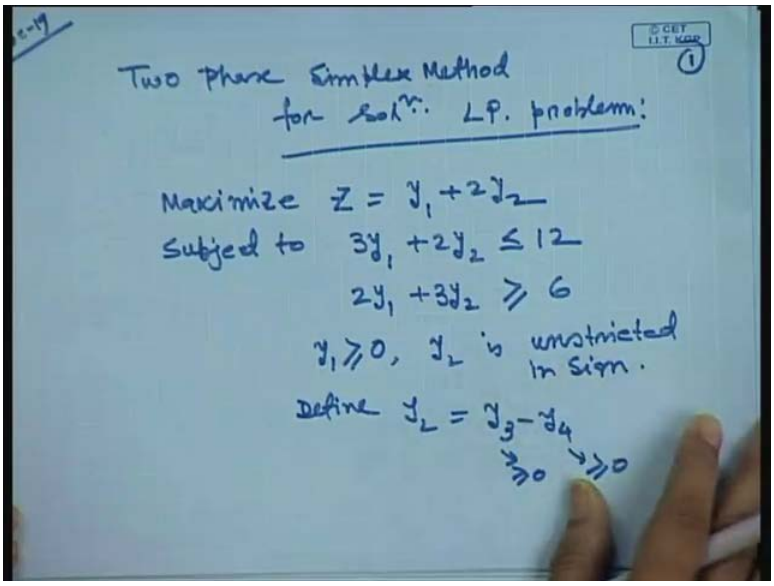

So, last class we have discussed how to solve the LP problems using two phase simplex method, we have taken this example maximize this function subject to this constrains. Here y 1 is greater than equal to 0 min non negative number and y 2 is unrestricted sign, so this y 2 we have to convert into a by interested sign. Two new variables we have defined y 2 is y 3 and y 4 which is individually that y 3 y 4 is greater than equal to 0, but this problem we have to convert into a standard LP problems.

We have to see how to convert the standard LP problems after converting this one, then you see if this type of equation when you will get this type of inequalities greater than equal to equalities. Then we have to introduce a variable which is call artificial variable, this artificial variable is that x 6 in our case and that artificial variable we consider we have to minimize.

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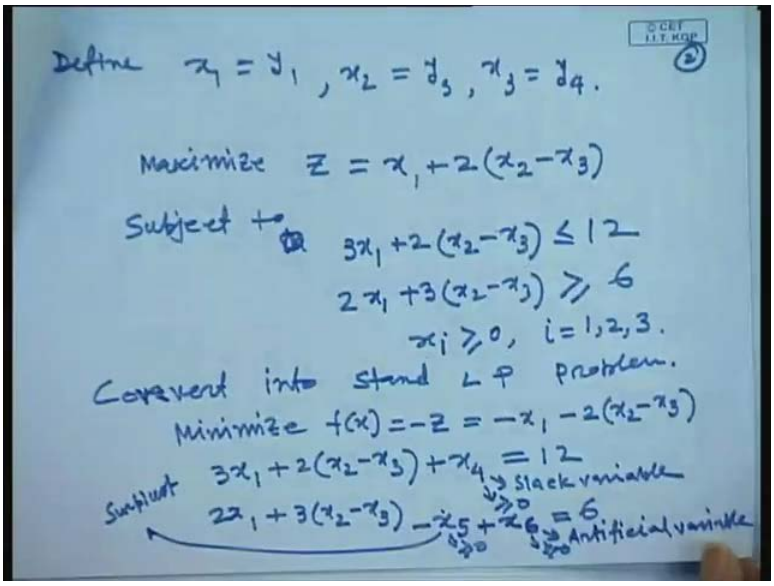

That means that we have consider the artificial variable x 6 as a w 6, a w and this simultaneously you have to optimise and that minimize two function objective function. One is x 6 which we denoted by w 6 and other is y, our original objective function and this first phase what we optimize that x 6 is a w, this we have optimum and minimize this one.

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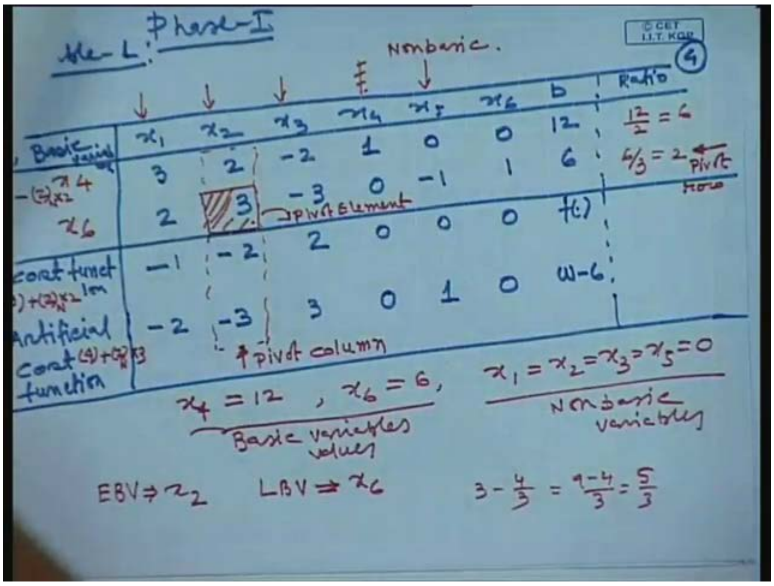

In first phase of the table form after the first phase we got the solution.

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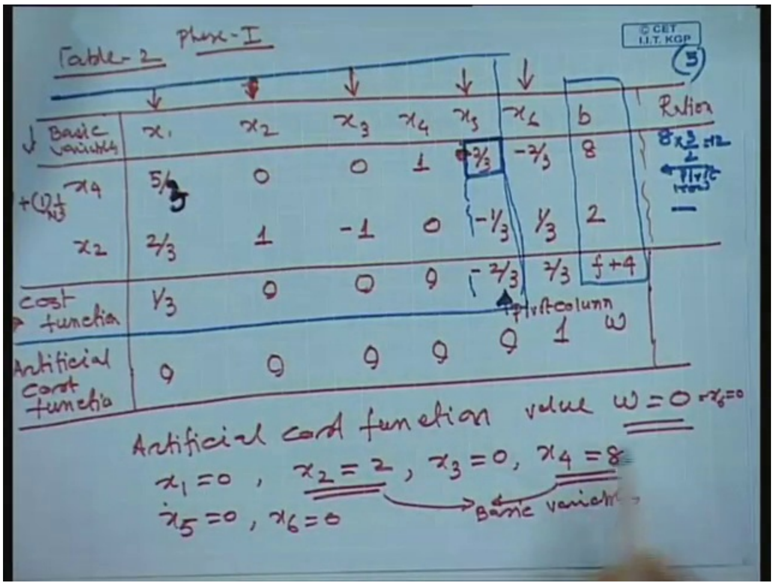

After completing the first phase we got the solution of this that is our artificial objective cost function value is omega is equal to 0. That should come omega is equal to 0 or x 6 is equal to 0 and corresponding the design variables what we consider x 1, x 2, x 3, x 4, x 5 and x 6 is this values, we got it. Then next set that phase to since w x of 6 is optimized and minimized, then we involved the control what is call calculation in the table x 6 variables. So, you just ignore x 6, so in second phase it will just find out the cost function of the original systems original objective function we could find out.

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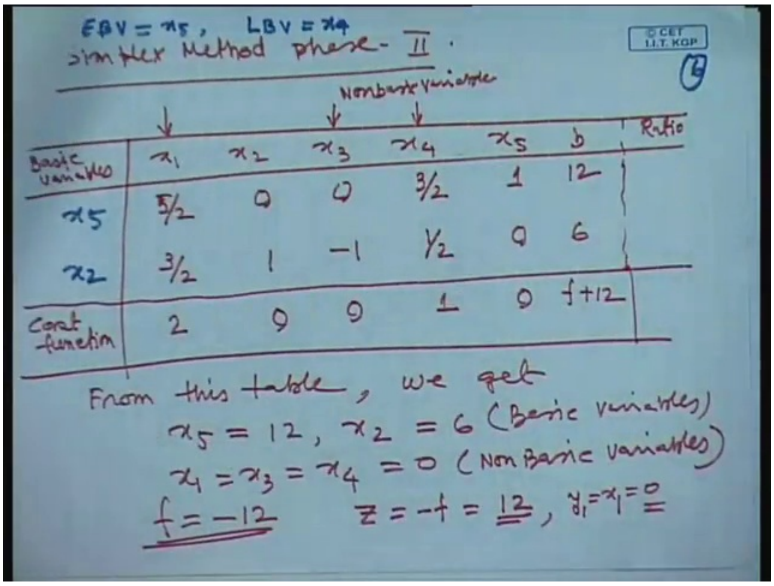

After completing the second phase, we got the function value is f is equal to 12 and corresponding the variables value x 1, x 2, x 3, x 4, x 5 we got it this values x this. So, if you look carefully this one, we have this is our basic functions of that one if you plot it that one, we have seen that this basic function if you plot it graphically.

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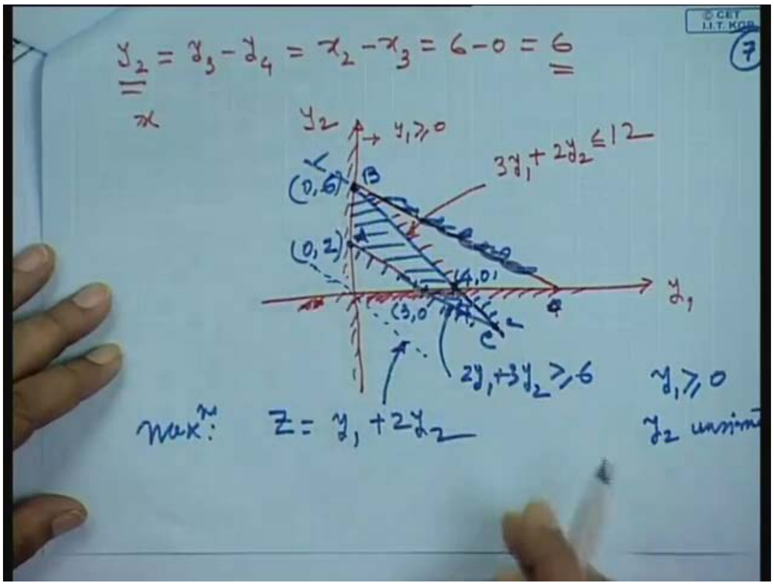

If you see this is our equation one of the constraints is this this one when you equal to y 1 is equal to 0, x 2, y 2 is equal to 6. So, this is our six when equal 122 is 0 y 1 is 4 that is our 4, 4, 0 coordinate this is 0, 6 coordinate.Then, another constrain is 2 y 1 plus 3 y 2 is greater than equal to 6, our original problem if you see putting it just to brought this equation straight line y 1 is 0 y 2, we will get it 2. So, this value is 2, then y 2 value is 0, then where it cuts, the x 6 is this one that is equal to our 3. So, this quadrant is 0, this quadrant is 0, 2, so it cuts the x 6 if you see this cuts the x 6 is y x 6.

Let us call point B, this is point A and this is cross x this intersection of two straight line is C. Now, according to our problem if you see y 1 is greater than equal to 0 that means y 1 indicates the right half of the this vertical line y 1 the whole vertical right half of the vertical line and y 2 is unsigned. That means y 2 value can be positive and negative that means this may be above this line above this horizontal line or below this horizontal line.

So, if you see carefully this one our physical region is that portion only of our corresponding problems. In this case our objective function if you see our objective function is nothing but a z maximize z is equal to y 1 plus twice y 2 and that is we have to maximize. Graphically, you see when this function value will be 0 this is equation of just and this function value is 0 when this passing to this origin. So, this the equation means we are finding out the word value of y 1 y 2 in the physical region the function value will be maximum this is the maximum, we want maximum value of this function.

So, if you put, if you just move this line parallel to this that line which is passing through the origin you will see maximum value of this function value will get at this point b parallel to this one parallel to this objective which is passing through this one. So, that one will give you this one, let us see in first phase of our problem what we got is if you see and recollect that first phase. In first equation, we got the value of if you see x 1 0, x 2 0, x 3 0, x 5 0 and from this one what we can say y 1 y 1 is nothing but a x 1 that means x y 1 is 0 y 2 y 2 is what we are divide y 2 is nothing but a x 2 minus x 3.

So, x 2 minus x 3 both are 0, so this point origin or origin point is the in phase one first iteration. We got it this point y, but this point is not in the physical and next iteration what we get it in the first phase next iteration. If you see we got x 1 is 0 x 2 is 2 and x 3 is 0, in other terms because y 1 is equal to our x 1 that means our y 1 value is 0. If you see y 1 value is 0 and then y 2 value is x 2 minus x 3 means 2 that mean we got this point.

So, immediately we can say the function value is increased from previous value previous say what at this point function value of 0 at this point one can find out what is the value of this. It will be increased physically also you can say this straight line is moving parallel to this, then cuts y x is some value of this instead of 0 proceed it is cutting this one, so thisandthis is the first phase after completion of the first phase.

Now, see after completion of second phasewhich will minimize the function of that one in other words that our original function will be maximized because we have converted into standard LP problem that is why minimization term we are using. So, in the after the completion of second phase of this one our x 1 value is you see x 1 value is 0 z 2 value is our six so this indicates x 1 0 means x 1 in y 1 is equal to x 1 that y 1 is 0 y 2 is what x 2 minus x 3 the way we have define, so 6 minus 0, 6.

So, this is a, this point on this point is a, I will give you all the maximum value of the function and this one agree. So, graphically also one can represent this one, so today we will discuss what it is. Then you can solve the LP problem by using primal and what is call dual problems considering the standard primal and dual problems, so primal and dual problem.

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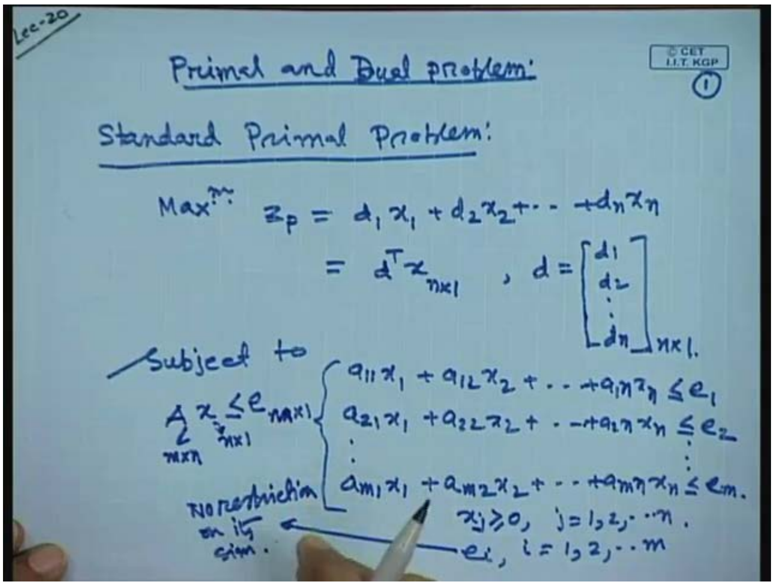

So, what is primal problem first standard primal problem is what we will define standard primal problem. So, first you convert this problem into standard prime primal problems, so what is this is the standard primal problem maximize z p is an objective function is z way. We have defined x 1 d 1 x 1 d 2 x 2 plus dot d n x n, we have a n variables at there you can write into vector form this d transpose x, x is a dimension n cross 1 and d is a vector column vector is your d 1 d 2 dot d n and this dimension is n cross 1.

So, this is subject to a 11 x 1 a 12 x 2 plus dot a 1 n x n is less than equal to e 1.Similarly, second equation constant equation and constrain isof this type less than equal to x 2 1 x 1 a 22 x 2 plus dot a 2 n x n less than equal to e 2 and so on. In this way, we have a n inequality constrains less than equal to type, so it is a m 1 x 1 plus m 2 x 2 plus dot a m n x n dot is it less than equal to e n. So, x j is greater than equal to 0 for j is equal to 12 dot n and e j e is equal to 12 dot, we have a m constrain of their dot m.

This e i is unrestricted in sign it can be positive negative 0 whatever may be it is unrestricted in sign. So, no restriction you can just write it no restriction on its sign agrees, so this is call standard primal problems. So, this you have to if you give the LP problem to convert into a standard primal problem. You have convert into this structure where mind it from the standard LP problem difference is there, the right hand side of this one is not is not restricted, it can be unrestricted.

It can be any sign positive negative all this things, so a set of equation I can always write in terms of matrix x a into x less than equal to e that e dimension n cross m. So, it is m cross one this dimension is n cross 1, immediately I can write it this dimension is m cross n and e is a vector components of e or e 1, e 2, e 3 dot e m. Suppose, in this equation if you get it you are given a LP problem which having an equal design and we have discuss earlier if you get that equality sign say like this.

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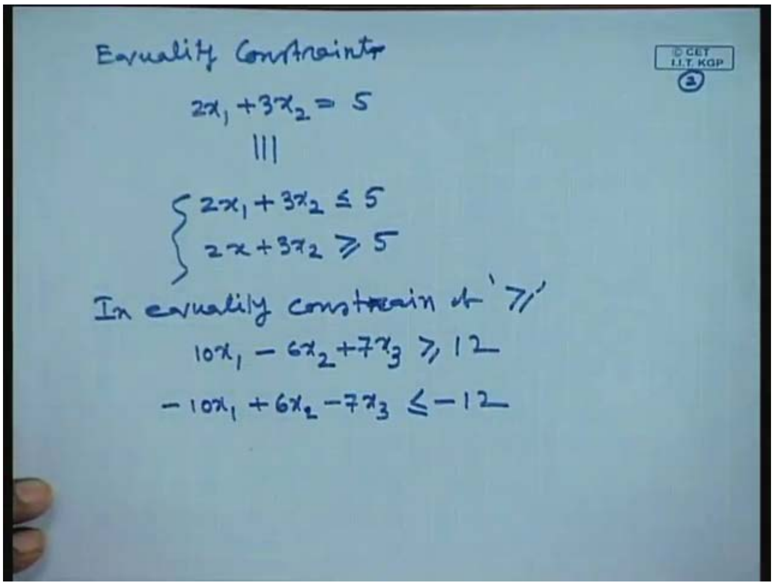

If you get it, let us call something like two x 1 plus 3 x 2 is equal to 5 that equivalently because I have to convert into if you want to convert this standard primal problem.You have toconvert into less than equal to agree, so this can, I can do it immediately,I can write by set of two equations 3, 2 x plus x 2 is less than equal to 5. Another, I can write 2 x plus 3 x 2 greater than equal to 5, equivalently I can write it this 2, equality is same as that one.

If you have an equality of this type, let us call inequality if you have an inequality constrain of this type the how to convert into a standard primal problems this one. So, suppose you have a equality is like this way 10 x 1 minus 6 x 2 plus 7 x 3 is greater than equal to 12. So, whatever you do it both side you multiplied by minus, so if you multiplied by minus both side minus that it will be 10 x 1 plus 6 x 2 minus 7 x 3 is less than equal to minus 12. So, this has converted into a standard primal inequality form, so keeping this thing in mind any LP problem, I can convert into a standard what is called primal problems.

Now, what is a dual part of this one that means standard dual problems, this problem is also referred as normal maximization problem this standard LP problem also referred as normal maximization problem. The next is what is the dual problem of this one, next is dual of this problem, what is the dual of primal problem, this is you can maximization.

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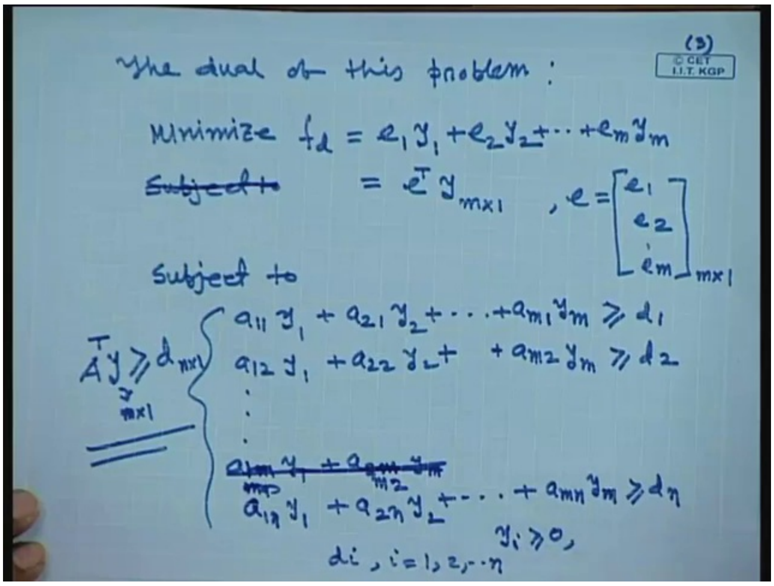

I will write minimizations, minimize that let us call a, I have denoted by f d that I have denoted by z p, p stands for primal problems means original problem and dual stands f of d is this is minimization. Then how you write it see dual problems whatever the inequality constrain you have you have m equality constrain and right hand side of the inequality constrain e 1 e 2 e dot e m. That quantity will come to the objective functions of the dual problems, so you will write it this e 1 multiplied by some dual, some new variable y 1 variable. You can say e 2 multiplied by y 2 that new variables in this way, so if you have a m equations are there m inequality equation.

If there in standard primal problems that dual problems, you we will have a new term variables instead of x variables inspired that x n variable in the primal problems in dual problems, you will get m variables new variables you have to introduce. So, I am writing this one e 1 into y 1 e 2 into y 2 again e 2 into y 2 and this e 1 e 2 is known to us. Then in this way e m y m, so in the standard primal problem inequality constrain we have a m inequality constrain in dual problem we have to make new m variables y 1, y 2, y 3 and objective function which is a minimization of F d is form with the coefficient.

You have got in the right hand side of the primal problem agree that coefficient that values we have to introduce in the objective functions. So, this is minimized subject, so this I can write if you write the in a vector forme transpose y and y is what m cross 1 and where is e is that e is a vector of a dimension m cross 1 agree our problem is now minimized. This one subject to see this one this that a 1, a 1 column wise, I am reading a 1 multiplied by y 1 a 2 a 21 multiplied by y 2 a m n.

Similarly, m n multiplied by y m, so Iam writing first constrain equation a 1 1 multiplied by y 1 a 2 1 multiplied by y 2 and dot a m one multiplied by y m and whatever the coefficient was there in the no standard primal problem. In the objective function coefficient that coefficient will come in the right hand side of the inequality constrain in the dual problem. So, our inequality of this type greater than equal to and this is the first constrain correspondence to first one it will come d 1.

Similarly, take the second column second column multiplied by second column is my 12 multiplied by y 1 a 22 multiplied by y 2. In this way, we have a m to multiplied by y m is will be greater than equal to second coefficient associate in the standard primal problem agree in the objective functions. So, it is equal to I can write it next a 1 2, y1 a 2 2 y 2 plus dot a m 2 y m is greater the equal to d 2. If you continuous like this way last one will be a 1 m y 1 plus a 2 m 2 m agree a 2 m here is 2m a 1, sorry the last will be a 2 m element, see a 12, 21 m 21, this is m 2 y m.

So, last one will be m this will be m row m 1 plus four last row you see what will be the others. This I have written this a 1 n sorry, a 1 n, I am writing this on last one that how many equations will be there will be m equation. So, last column is a 1 n into y 1 plus a 2 n, this 12 n y 2 plus dot a m n m n into y m that equal be equal to d n.

So, you see that how many coefficient instead thatprimal problems are there in objective functions.There are n coefficients are there and that n coefficient will come in the dual problem inequality constrain. That means n inequality constrains will be coming here, so we have written this things if we say a 1 a 11 y 1, then a 1 1 a 1 2. Then this a 1 m y m and similarly these are second equation we will get, we say d 2 and all a n th equation it will be d n. So, this is we will get it set of inequality n inequality constrain with the help of standard primal problems of objective function coefficient will come to the right hand side of this inequality constrains and you see this inequality constrains.

This if you reexpress matrix from it is a transpose of y is greater than equal to d is your dimension n cross 1 agree and this m is m cross 1, so this we can write and here y i is greater than equal to 0. Here, d 1 d d i varies from 1, 2 dot n, this has no restriction on its sign now in short what is standard primal problems structure is like this way maximize z p d 1, d 2, d 3 are the coefficient associate in the objective function and x 1 x 2 dot x n is the dual variables. That you convert into a this type of inequality form agree and this e 1 e 2 e m are there is no restriction in sign, but this design variables are all greater than inequality of 0.

In turn, I can write matrix vector form in to this form this how will you convert into dual problem of this are dual problem of that primal problems that whatever the coefficients are associate in the objective function of standard primal problem. That will come to right hand side of the inequality constrain of inequality constrains of dual problems. So, it is coming right hand side d 1 d 2 d 3, there are and what are the expression we will write it in equality constrain this column once consider this column a 11 into y 1 a 21 into y 2 plus a dot plus a m 1 into y m will be greater than equal to d 1.

Similarly, second columnthird column in this thing, so since we have a in primal problem n coefficients are there we will get n constrains are there and what are the right hand side in the standard problem. How many equation inequality equation, there are m equation are there, so that that variables will go in that dual problem objective function its going and corresponding m new variables are introduced. So, this is call the dual problem of the primal problems agree and observation if you see this observation of that that one it is clearly say that number of dual variables number of dual variables.

The dual variables is y 1, y 2, y 3 dot y m is same as the number of primal the primal constrains how many constrains are there m constrains and dual variables are let say 1 y 1, y 2, y 2. So, number of dual variables is same as number of constrains in the primal problems is the first observation second observation number of dual constrains is same as number of number of variables in the objective functions. Primal variables number of constrains in the dual problem is same as number of primal variables in the primal problems.

This is of just second observation, we got it and coefficient associated with the standard primal problem coefficient associated with x is a matrix what you coefficient matrix in the dual problem, primal problem. If you take that transpose that you will get you the coefficient matrix of dual problems this is the third observation we have seen thenwe can see this one that inequality constrains are reverse in direction the primal. Primal problem inequality is less than equal to type and dual problem is reverse in direction greater than equal to this is the third fourth one.

We will get it and minimization or maximization problem of primal problem becomes a minimization problem of dual problems. So, you see this is maximization problems, now become a minimization problem of dual problems this. So, these are the observation we got it this one and you have seen this coefficient associate with primal in the objection function coefficient associate with the variables that comes in what is called objective. That comes in the dual problem inequality constrain right hand side and vice versa which is coming in, thus standard problems inequality constrains in the right hand side, what is coming e 1, e 2 dot e m.

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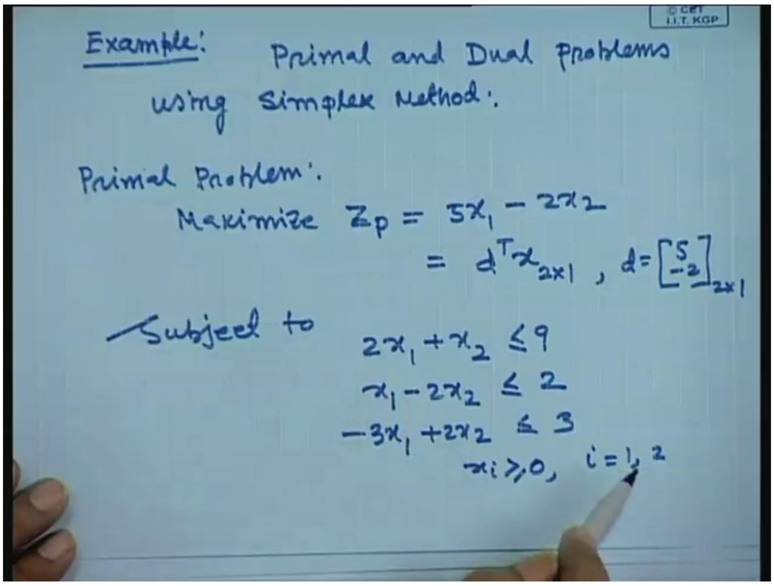

That will come as a coefficient in the dual problem objective functions and this primal and the dual variables are non negative that means x i is greater than equal to 0 and y i is greater than equal to 0. So, keeping this thing in mind we further see how to convert that a dual problem into primal problems and primal problem into dual problem, we can do it. So, let us take an one example and we will see how to solve it.

Here, let us call the examplethe primal and dual problems using simplex method, so let us just give you the primal problem convert into the dual problem. So, primal problem what is the primal problem maximize z p is given 5 x 1 minus 2 x 2, which I can write it d transpose structure into the x this is our 2 cross 1. If you see our d where d is equal to 5 minus 2, which dimension is 2 cross 1 subject to 2 x 1 plus x 2 is less than equal to 9 x 1 minus2 x 2 is less than equal to minus 3 x 1 plus 2 x 2 is less than equal to 3 and x i is greater than equal to 0 for i is equal to 12.

Now, see it is already in standard what is called primal problem, if it is not there in standard primal problem you convert into standard primal problems. So, this is the standard primal problems, first your job is let us call i, given the problem convert into a standard dual problems. Now, you see this if I ask you, let us call first I ask you this is the primal problem, solve this problem by using simple method. Then you can solve it because you have to convert this thing into a standard LP problems this is one way and if I tell you that this is a primal problem convert it dual problem.

Then, solve it by using simplex method that is way why we have to convert into dual problem sometimes when you will convert from what is call dual here up standard problem to dual problem the computational advantage. You can gather you can get it while you will solving the multi objective optimization problems linear optimization problems that is may be the one of the advantage.

You can get it from this one, so let us see first our question is solve convert this things into standard LP problem and then solve it standard LP problem. Then solve it, but I will not solve this by standard LP problem because here I have already discussed how to solve a standard LP problem using simplex method means in a tabular form or it is an algebraic approach or matrix form.

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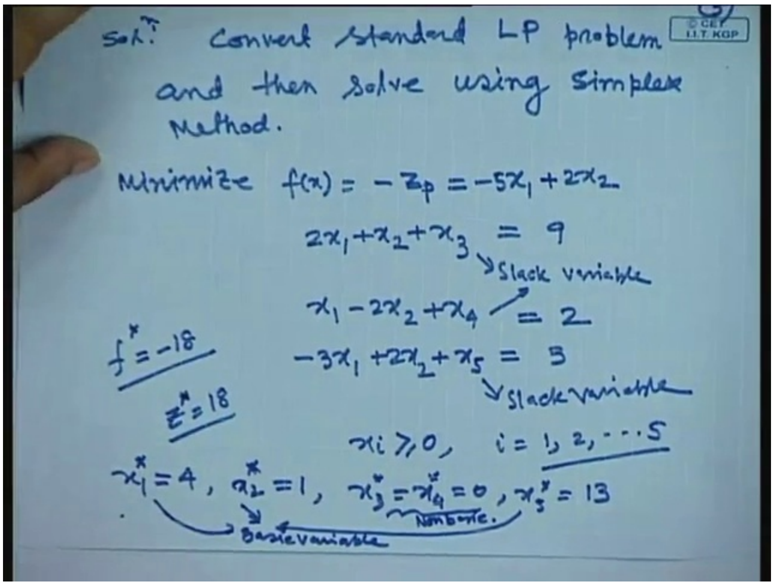

You can solve it, so solution suppose you are ask to solve i z problem is primal and dual problems using simplex method. So, this solution convert standard LP problem and then solve and then solve using simplex method. So, I will just tell you how to convert this how to convert the standard LP, what is this standard LP problem this we have to convert equal to sign. Not only that, right hand side of this must be positive right hand side of this equation must be positive. This is already in positive term, so we have to introduce in this equation one slack variable here one slack variable here one slack variable suppose. You have this is the primal problems your way suppose in the, so then after that you can solve it.

So, let us call how to convert this thing into a standard what is call convert I just convert into standard LP problem. So, our standard LP problem minimize that is z p f of x is equal to minus z p which is equal to set this 1 minus 5 x minus it is minus there. So, it will be plus two x and this you can convert by adding 1 by slack variables because this is less than this. We have to add some slack variable and slack variable value greater than equal to zero, so that slack variable i introduce as a x 3 so 2 x 1 plus x 2 plus x 3 is equal to 9.

This is the slack variable, and then we have taken and find out that next equation. Similarly, second and third I can write, so second equation is x 1 minus 2 x 2 plus x 4, so this is also slack variables is equal to your 2, then third equation you see minus three x 1 plus 2 x 2. I have add another slack variables with this one, so that it will be equal to three so this is also slack variablesand right hand side of this equation all are positive. So, our x i is greater than equal to 0 i is ready to 1 2 dot 5, so this is converting into standard LP problem.

Then, you will know in tabular form how to solve this problem agree by simplex method that you solve it if you solve this problem. Then you will get the solution of this one that our x 1 optimal value of this one you will get 4 x 2 star, you will get 1 x 3 is equal to x 4 this star it will get it 0, then x 5 star you will get it 13. So, this is the solution and in if you complete this solution in tabular form that our basic variables are x 1 x 2 and our x 5. These are 3, this and this are the basic variables basic variables and elements non basic variables this is non basic variables.

So, immediately what is our minimum value of the function if you just put it here that x 1 is 4 means minus 20 minus 20 plus 2 f d are f is you will get it f star is minus 18. Our problem is if you see our original problem is maximization problems, our maximization of problem of that one.

So, that will be a our maximum z will be vector plus 18, so this is the solution your given LP problem if you are asked to if you are ask to convert into a what is call primal problem. Then solve it by using simplex method you have to do like this way now next is I told you I mention you if you convert the primal problem that do it. You got it if you convert the primal problem into dual problem that maximization problem, you convert into what is called minimization problems and the way it is converting into dual problem. Now, I explain it if you convert into dual problem and then solve it by what is call simplex method, but that variable dual problem variables are new things that is no way linked with the direct relation with the our primal variables x 1, x 2.

So, there must be some link between the dual variables and our primal variable because our objective is to find out the primal variables of that one the two objectives in mind if you convert the primal problem into dual problem. So, let us first let us now we convert our problem are standard LP problem if you see this standard LP problems this standard LP problems. Now, we will convert into dual problems how you will do now you are doing a first you have to minimize that function how you objective function of that dual problem. You will be considering whatever the coefficients are there that coefficient will come in the objective function of dual problems and that dual problem will be minimization of objective function, so if you do this one dual problem.

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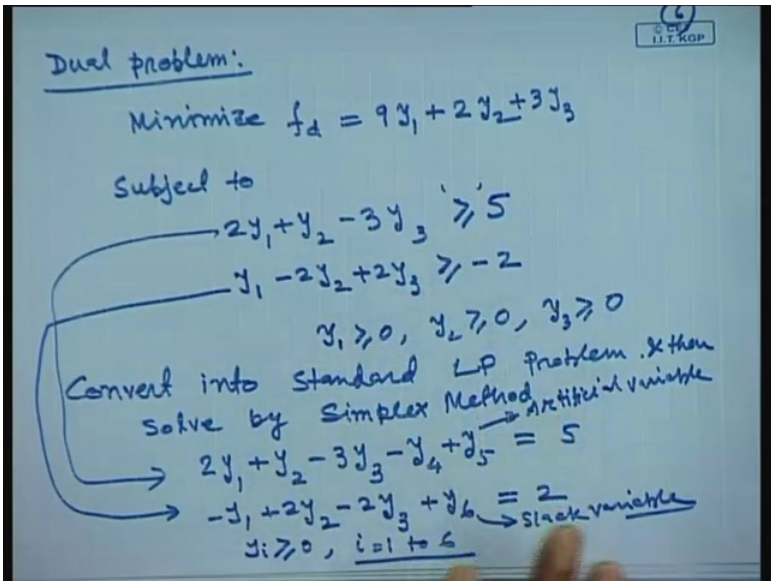

Now, keeping that in mind minimize f of d is say 9 2 x, there are three variables three what is called in standard problem 3 inequality constrains at that 9 into y 1 9 into y 1 a new variables. Then it is a 2 into y 2 that 2 that 2 is y 2, then three into y 3, so this is our objective function dual problem objective function is a subject to. Now, see this one 2 into y 1 this coefficient is 1 plus y 2, then minus 3 into minus 3 into y 3. So, I will write it this first equation, I will write it 2 y 1 plus y 2 minus 3 y 3 is greater than equal to your five.

So, 2 y 1 plus y 2 minus 3 y 3 is greater than equal to 5 that all coefficient of standard LP problem of standard primal problems standard primal problems will come right hand side of the inequality constrains of dual problems. Now, this I can write it second equation that you say this coefficient is 1, so 1 into y 1 plus that is minus sign minus twice y 2 plus twice y 3 is greater than equal to what minus 2. So, I will write it y 1 minus twice y 2 plus twice y 3 is greater than equal to minus 2, see this one agree and we have a new variables.

That dual problem variables are y 1 y 1 greater than equal to 0 y 2 greater than equal to 0 y 3 greater than equal to 0. So, as I mention is say when you will convert into dual problems the right hand side of this constraints does not have any restriction on its sign so we have converted into same this form. Now, if you see this one, then what is this this is as converted into that our standard dual problem. Now, if you want to solve this dual standard we have converted then you want to solve it by using simple method. That means you have to convert, next is convert into convert that dual problem into standard LP problem and then solve and then solve by simplex method.

So, first you convert into standard LP problems this is minimization part is already that we need. Then it will come for each equation it will come one artificial variable agree and result in cost function for the artificial variables artificial variables cost function will be the sum of algebraic some of all artificial variable will be the objective function. So, in this case only one artificial variable y 5 i 2 minimize then y 5 expression, I can write it keeping this variable in right hand side, then y 5 is equal, I can write this expression.

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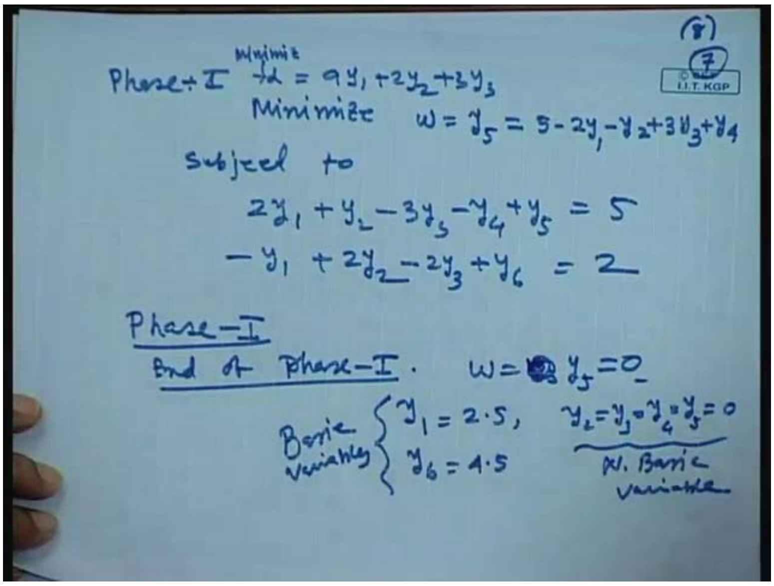

So, our phase one if you see our phase one problem is like this way minimize w is equal to y 5 and what is y 5. If you see this one, 5 minus twice y minus twice y 1 minus y 1 plus three y 2 if you take their set plus y 4. So, I am writing 5 minus twice y 1 minus y 2 agree plus 3 y 3 plus y 4, so this is our objective subject to subject to what this constrains subject to 2 y 1 plus y 2 minus 3 y 3 minus y 4 plus y 5 is equal to 5 and minus y 1 plus y 2 y 2. These two equation what we have written same equation I am writing here minus two y 3 plus y 6 is equal to 2.

We have our original objective function is what if you see our objective original objective function is minimized if you see this one minimize f d is equal to 9 y, 1 agree plus 2 y 2 plus 3 y 3. So, if in phase one in tabular form, we have to minimize first this one then in phase 2 we have minimize that one. So, when you will minimize in first one tabular form both the function we have to write it simultaneously. So, this we have to write simultaneously this one, so I live it an as in exercise to work out and you will get it after 5 phase 1 after completion of phase 1 the solution of this one you will get that after completion of phase 1, you will get the value of this.

That phase 1 end of phase one end of phase 1 you will get it, the value of w is equal to 0. Then one in phase you have a two iteration that I am telling you, so after the end of the phase 1, you will get w 1 is 0 means w is 0 means y 5, y 5 is equal to 6. This 0, then you will get it y 1 value 2.5 y 2 y 3 y 4 y 5, these values are 0 and y 6 values is 4.5, these are the basic variables you will get it and these are non basic variable non basic variables.

So, this is the end of phase 1 that means we have minimize the artificial variable which we are expecting that y 5 values should be 0, it is coming 0. Then at the similarly, next you ignore y 5 in the tabular form we remove it from the tabular form this y 5 and then proceed for phase 2.

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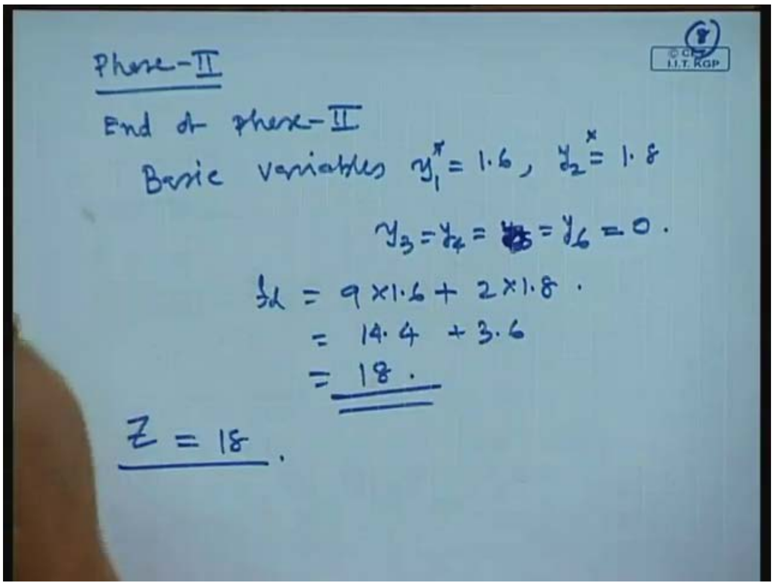

If you see that phase 2 at the end of phase 2, in that phase 2, you are minimizing the objective function the changed objective function. During this process, when you did it phase 1 in this process the objective function is change because you have done to lot of elementary low operations there. So, it is changed, so phase 2 the end phase 2 end of phase 2 you will get basic variables y 1 star is equal to 1.6 y 2 star, you will get 1.8. Then y 3 y 4 y 4 y 5 then y 5 then y 6 no that y 5 is not there y 3 y 4.

Then, y 6 these values will be 0, now you see what is the objective function value y 1 is 1.6 y 2 value is that 1.6 y 3 value is 0. So, nine point if you see the f of d this is 9 into 1.6 plus 2 into 1.8 if you do this one, this will be a 3.6, 14. So, it will be a this will be a 18, so f d value is would be a 18, then what is our z value z value how you find z value we do not know the value of x. So, there must be some transformation from, what is called dual variables to primal variables. That we will discuss next class, but we can say this value objective function in the primal must be also same agree that 18. We will discuss next class in details how to convert the dual variables into a primal variable design variables, now I will stop it here.