Optimal Control
Prof. G. D. Ray
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 02
Introduction to Optimization Problem: Some Examples (Contd.)

In last class, we have discussed that what do you mean by optimizations, we broadly classified the optimization in two classes, one is static optimization, another is dynamic optimization. Static optimization is concerned with the designed variables that are not changed with time and we have discussed what are the techniques are available to solve such type of static optimization problems. Another class is dynamic optimization problems and that is concerned with the designed variables that changes with time and time is become a function of these in the problem statement.

So, there also we have discussed what are the techniques are there to solve such type of problems. Then, we have taken what is called that a problem statement is given and for this problem statement we have translated into a mathematical form. Then, we have seen this mathematical form represents an objective function which we are supposed to optimize either minimize or maximize subject to constraints, constraint may be equality constraint or that inequality constraint or both.

In addition to that, there is a side constraints are there; with a suitable example we have explained how to formulate the statement of the problem in to a mathematical form. Then, graphically also we have represented for simple example, graphical cases also you have seen how to represent the optimization problem in graphical form. Then, the next is you have what is called that multiple optimization problems, today we will just talk about the multiple optimization problems.

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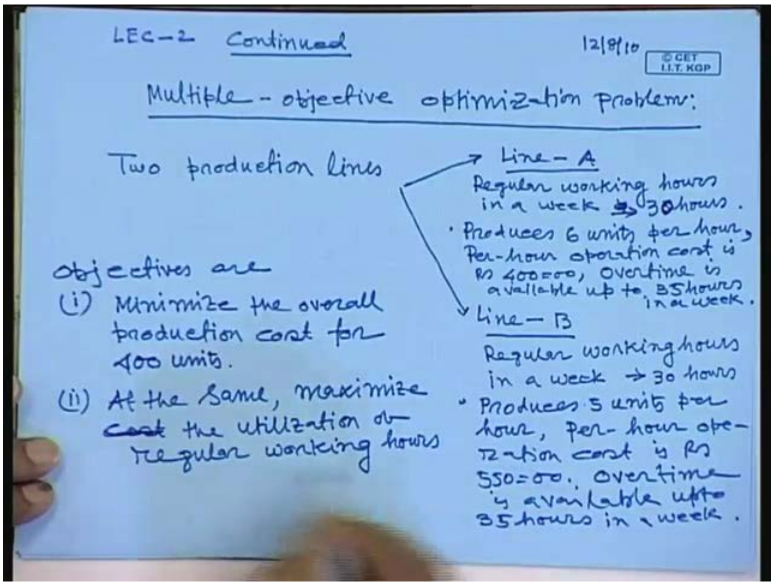

Let us consider a company has receipt what is called a receipt a production of 400 units again and he has to supply within a week. This company has to produce these unit, there are two options are there two production lines are there. One production line is A, another production line is B, in this line A, the regular working hours in a week regular working hours in a week is 35 hours. Similarly, here also line B are let us call 30 hours, this is 30 hours and line B also regular working hours in a week 30 hours. Thus, this unit can be produced through line A or line B, again in addition to this that line A produces that produces line A produces 6 units per hour.

The per hour operation cost is rupees 400, in addition to this that over time is available up to 35 hours in line A, over time is available up to 35 hours in a week through line A. So, similarly in line B, it produces, line B produces 5 units per hour and per hour operation charge per hour operation cost or charge is rupees 550. Similarly, over time is available through line B, over time is available up to 35 hours in a week.

Now, what is our problem, this is a statement of the problem, company has got a production over all of 40 units, and this company has a two lines to produce this unit line A. It is mentioned that in line A the regular working hours in a week is 35, 30 hours line B 30 hours, but in 1 hour, the line A produces 6 units and per hour cost is 400 rupees. It is also mentioned that over time available in line A up to 35 hours in a week, similarly the line B produces 5 units per hour and per hour cost or charge is 550 rupees an over time is also available 35 a hours in a week. Now, our main ob objective of this problem is two folds, first we have to minimize the production cost the total production cost for 40 units.

We have to minimize first this is the objective function another objective function is at the same time we have to maximize the what is called the regular working hours in a week. So, that is the two objectives we have in our hand, so it is called multi objective more than one objective is a multi objective optimization problems. We have A from the statement of the problem. We have different constraints there, now our job is to convert this problem into mathematical form and then solve it.

So, our main objective if you say the objective are production of the objectives, minimize the overall cost, overall production cost for 400 units that we received order from. Then, we have to maximize at the same time, maximize cost of, sorry maximize by utilization of regular working hours in week hours in a week. So, this is our objective that there are two objectives are there minimization something and maximization. Also, simultaneously you have to maximize another functions, so let us take the statement of the problem, let us write in mathematical form, so this problem is we can tell more specifically it has two objective optimization problems.

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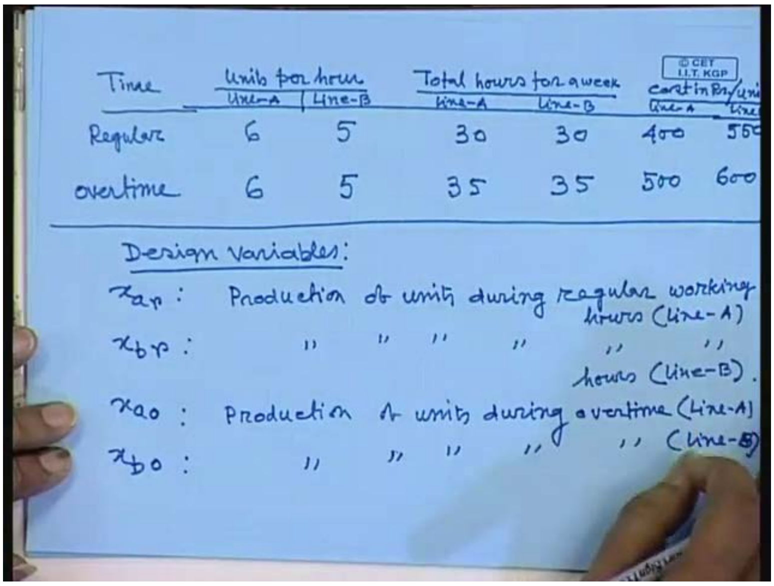

So, let us write all the information in tabular form, so we can finally write in mathematical expression form. So, we have a units per hour another is total hours for a week total hours for a week that cost in rupees per unit and units per hour can manufacture through 2 lines. So, line A we have defined the two lines, line A and line B and total hours for a week line A and line B cost or the charge operation charge for each unit or per unit line A and line B. So, here is the hour time whether is that regular time or over time, so regular time the units of per hour through line A produces 6 units through line B, 5 units naturally over time.

Also, same units will be produced over time the per hour through line A is 6 per hour through line in B is 5 minutes total hours for a week, line A is 30 hours line B also same time or hours an overtime. In both the lines is 35 hours cost per unit through line A is 400 rupees per unit and through line B is 550, when it will go for over time. Then, it is a 500 rupees instead of 400, 500 and that is a 600 rupees. So, this table we can we have written from the statement of the problems, so let us consider our design variables.

The objective function that is what we will define that is the function of our design variables first x a r, this indicates production of units production of units during regular working hours per week through line A that is the unit. That means x a r unit is produced during the week through regular hours working hours, then x b, b stand for line B. Similarly, production of units during the regular working hours through line B per unit, now x a suffix 0 line through line A during the over time working hours that is the production of units during over time through line A.

Similarly, through line B over time production of units during over time through line B in a week. So, these are variables we have defined, now immediately we can say that what is the cost that is overall cost is for the for manufacturing 40, 400 units what will be the total cost will be for the 40 units.

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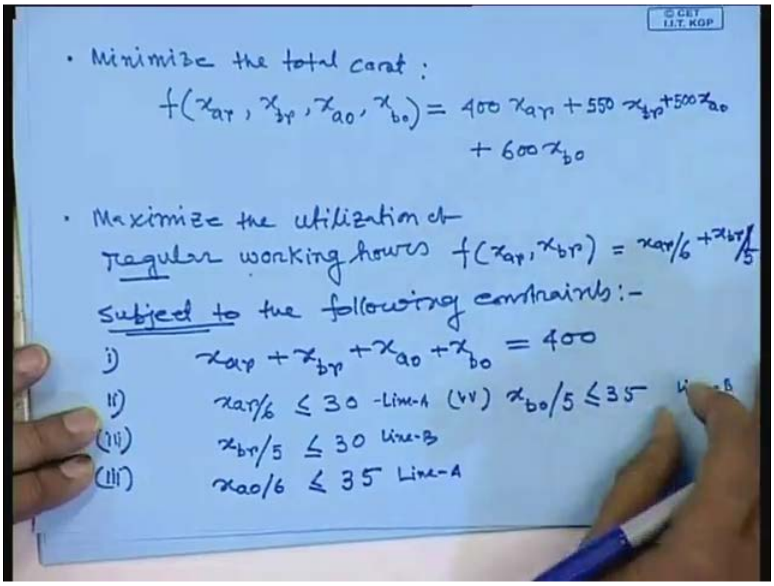

So, our problem is minimize the total cost, so we know if you see that from the table, we know the production unit during the regular working hours through line is a r. This number of units through line A is produced or manufactured, then per unit cost is 400 rupees, so 400 multiplied by a r is the total cost through line A during the working hours regular working hours. So, it will be this total cost minimization total cost is a function of design variables x a r, x b r, x a 0 over time. Number of units in x b 0 and which equal to 400 into x a r plus 400 rupees. When you will manufacture the unit through line A during, what is called working hours, the cost per unit is through line A is 400 rupees, we have x a r number of units produced to the through line A.

Similarly, through line B is x b number of units produced through line B during the working hours regular working hours. The cost of this one is if you see from the table that is given 550, so 550 into this 1 plus over time through line A is number of unit produced in through line A during the over time is x a 0 and the cost is your 500 rupees over time through line A. So, it is multiplied by 500 plus number of units produced during a number of units produced through line B. It is during the overtime hours is x b 0 and each unit cost, so 600 rupees each unit cost is 600 rupees.

So, it is 600, so this is the total cost we have to minimize that objective function, I told you last class is an scalar function and it is a function of design variables our design variables are four a r b, x b r, x a 0, x b 0. So, you have to minimize this cost total cost, so in addition to do this, we have another objective function that is the maximization of regular working hours. So, another objective function is the maximize the utilization of regular working hours, so that is let us call we write that is the function of is a function of what x a r, x b r regular working hours, in the regular working hours, we have produced x a r units and through line A how many we have produced x b r through line B.

Then, what is the time required to produce that x a r units through line A during the regular hours. So, that is we have seen from the table or from the statement of the popular problem that 6 units is prepared what is called 6 units is manufactured in 1 hour. So, it will take how many hours is there x a r by 6 plus x b r by 5, so this is the regular hours through both the lines. So, that time we have to maximize, so there are two objective functions as their and this objective function. Simultaneously, one is mini maximization this maximization another is minimization. So, subject to our constraints, then what are the constraints the subject to the following constraints and those constraints, we are obtained from the statement of the problem constraints subject.

So, what is this that number of units we have to produce through regular working hours through line A and B as well as number of units we have to produce through over time in line A and line B. So, what is the total units, we can write it 1 that is one conditions that x a r plus x b r plus x a 0 over time number of units produced through line A during the overtime hours is x a 0 plus x b 0 is equal to 400 units. That is as far from the statement of the problem, second condition is given constraints here, given that if you see the total hours through line A maximum hours in a week is 30 line B and the line over time through line A 35, line B is 35, so we have this constraint.

So, we have x a r is the number of units produced through line A during the regular working hours, so how many hours is required, then 6 units is produced in 1 hour, then this much of unit will produced in this much appear hours. So, that must be less than equal to 30 hours as far statement of the problems, so another things are there that is x b r by 5 also less than equal to 30, x b r is the number of units produced through line B during the working regular working hours. The production through line B 5 units that time required per hour, it can produce 5 units, so total hours is this much, it must be less than 30 as per specify a statement of the problem and also through over time.

The third x a 0 divided by 6 time is must be the over time must be less than 35 and fourth constraints is x b 0 number of units produced through line B during the over time working hours is x b. Since, it is required 5 b in 1 hour through line B, 5 units can produced, then this units how many times or how many hours is required that we can find out that must be less than equal to 35. So, this is line A, through line A, line B you can say line A and line B, these are the constraints is given. In addition to that, we have a side constraints of that what is the side constraints all the design variables, this design variables x a r, x b r, x a 0, x b 0 is greater than equal to 0, it cannot be negative quantity.

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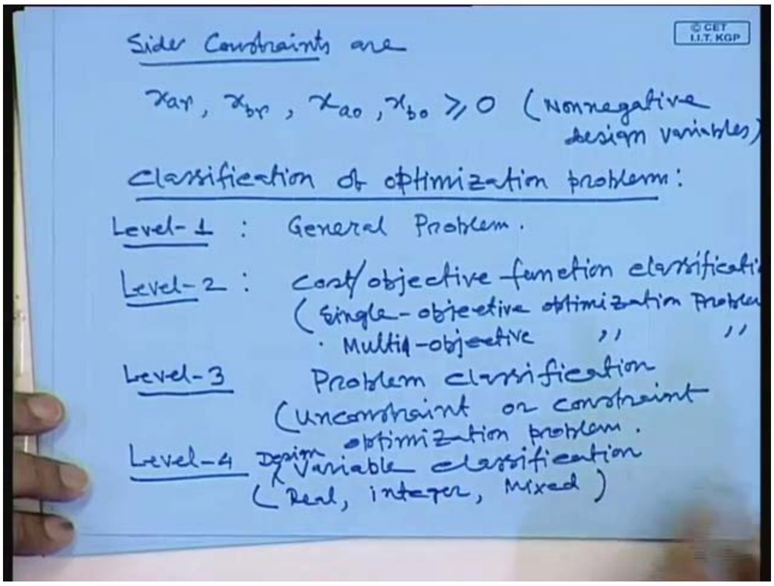

So, another side constraints are the side constraints are x a r, x b r, x a 0, x b 0 is all is greater than equal to 0, non negative. This is called non negative design variables and this is called side constraints, so this is the mathematical form, these are starting from let us call this is equation number one this is equation number two and this equation number two is a equality constraints constraint. Then, you can say equation number three, four, five, six all are inequality constraints, these things are inequality constraints and these are the side constraints.

So, this is a mathematical form of mathematical model with two objective functions, so it is called multi objective optimization problems. So, we will discuss later how to solve such type of problems, so we have now come across what is called single objective function and multiple objective functions in this case. So, next is your classification of optimization problems, so if you see this thus two example, we have considered one for single optimization problems. From the statement of the problem, we have we have formed the mathematical formulation of optimization problems.

Then, multiple objective optimization problems we have seen and we now only discuss the classification of optimization problems. So, first is level one, level one is the statement of the problem is given that means general problem that next problem, next level is here cost or objective function classification. So, next level is cost or objective function classification, what is this means whether is a single objective function optimization problem or it is a multiple optimization problems. It is either this is a single objective optimization problem or multiple multi objective optimization problems, so this is classification in second level, third level is problems classification.

So, problem classification means whether it is an unconstraint optimization or it is constraint optimizations that you have to classify. So far we have discussed the two problems, both the problem says constraint optimization; it involves inequality constant and equality constants some problems may be there is no constraints involved in the problem. So, we have to maximize or minimize or optimize the function, with which is a function of disagree variables only agree there is no constraint. So, we have a problem, we have to say whether these optimization problem is unconstraint or constraint optimization problem level three, level four is variable classification.

Variable classification means that that design variables what we are considering here that design variables are real or integer or mix of both. So, depending upon it will be called that classification of variable, we are dealing with the continuous variables or discrete variables, now not discrete, instead what is called integer variables or mix of both real and integer variables. So, next is variable classification variable design variable, you write design variable design variable classifications that is a real variables are real or integer or mix both real and when you have to call real variable. It is a continuous integer means this form, so this last level is your level five, function classification.

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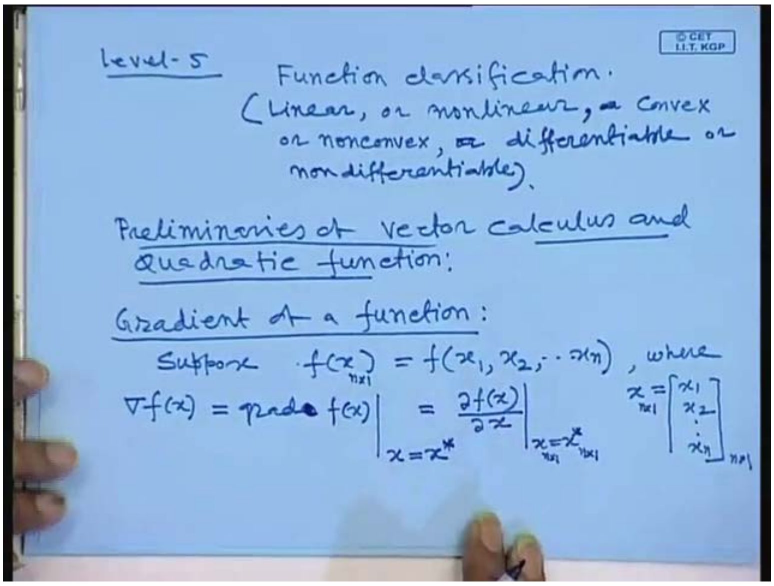

That means function classification is this optimization problem, whether it is a linear or non linear optimization problems or converse optimization problems or non converse optimization problem. It is this problem that optimization problem is differentiable or non differentiable, so we can do this one, if you see the example one, we have considered that example one for designing. So, that is a non linear optimization problems because in objective function was non linear and other constants you can see it that mainly objective function is a non linear functions is there.

If you see that carefully, the equations see this equation objective function is a linear equation that constraint this is the linear equation. Then, this equalities all are linear equation, so this multi objective optimization problem is a linear optimization problems. So, function classification means linear the optimization problems is linear, non linear or non linear, non linear convex or non convex or differentiable or non differentiable. So, basically the general problem the problem of optimization can be leveled into five, one is general problem statement.

From second level are your cost and optimization cost and objective function classification; third is problem statement problem classification whether it is un constraint or constraint optimization problem. Then, fourth very well classification means whether the designed variables are real integer or mix of both and last level, five is a function classification whether it is optimization problem is a linear non linear convex or non convex or the f. In this objective function, all this are differentiable or non differentiable classification classify, so this is that what the about the details about the different types of classification, what is optimization problems before that we get the conditions for optimal to get the optimal of the function all these things.

We must know something about what is called minimal is of vector calculus and the quadratic functions what is quadratic function all this things. So, the preliminaries of vector calculus and quadratic function most of the optimization problems involve the function of several variables, if it is the most of the optimization problems, the function of several variables. So, it is necessary to introduce vector calculus, so what is vector calculus, we will see so gradient of a function, suppose gradient of A, suppose we have a function of A, suppose we have a function f of x, now this x is I will define as a vector of dimension n cross 1.

That means x is a vector this elements of this vectors I denote it by x 1, x 2, x 3, x 4 dot x n that is why I have given the dimension is n cross 1. So, this is more details, one can write is a function of x 1 x 1 dot x n, which in compact form I can write function of x whose dimension is n cross 1 or you can see where x is equal to x 1 x 2 dot x n. So, I have written here the dimension of this n cross of 1, here also you can write it n cross 1, so we have a function that function is a that f is a function of several variables x 1, x 2 dot x n. So, we know already how to find out the gradient of that function, so gradient of function is denoted by symbol f of x. This symbol will read as a grad of f x and this grad of f of x we are finding out.

The value of this gradient of the function at x is equal to x star, some value is given that vector x 1, x 2 value numerical values are given here this point, what is the gradient of this function. We find out gradient of the function is nothing but a partial derivative of the function with respect to different variables x 1, x 2 dot x n. So, this we can write it symbol del f, del x divide by del x, same thing by retain methodical is like this way, x is equal to x star, where x star dimension is n cross 1.

Similarly, x n dimension is f, so you complete this one and then that will give you the information of the gradient of this function. So, how you will complete that 1, so I told you just is a partial differentiation of function with respect to x 1, then partial differentiation of the function with respect to x 2 keeping all other variables are fixed.

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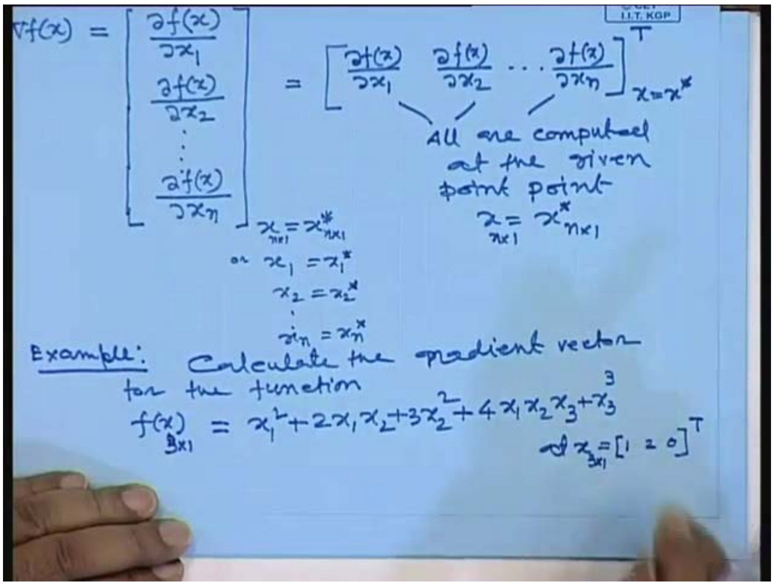

So, this equal to you can say this equal to or gradient of f of x equal to partial differentiation of the function x which is a function x means x 1, x 2, dot x n differentiate these with respect to x 1. So, this function you differentiate with respect to x 1 keeping other variables x 2, x 3 dot x n constraint. So, similarly again you differentiate f of x with respect to second variable x 2 keeping x 1 x 3 dot x n remain constant in the function and then differentiate. This way you precede and f find up to x n, so once you got it this one, you put the value of x is equal to x star. So, this indicates gradient of the function, which a function that half is a function of x 1, x 2, dot x n and compute the gradient of the function at x is equal to x star.

So, this you can complete like this way or one can write it this is nothing but a, this if you see x, x is a vector. So, other way you can say or you can write x 1 is equal to x, so one star put it in this x 1 and x 2 is equal to x 2 star and dot x n is equal to x n star. This one can write it like this way also the transpose form del f of del x, del x 1, del f of del x del x 2, dot del f of del x, del x and this whole transpose of a matrix transpose of a vector this. So, gradient of a function which is a scalar if you differentiate with respect to a vector, the results is a column data again. So, this I can write into this row factor, take the transpose also in this form.

So, these are all compete at the given point, all partial derivatives all are compete can write it here x is equal to x star at the given point x is equal to x star whose dimension is n cross 1. So, gradient of a function that functions is nothing but a, you do the partial derivative of f with respect to different variables into a vector form. So, that is gradient of this one, so next is if you difference if you do the derivative of a gradient of a function once again, then you will get a matrix what it is. If I differentiate vector with respect to a vector, then you will get a matrix, so before that I will take one example and tell you how to compute this one example.

The gradient vector for the function f of x is equal to x, x is a vector in your case is dimension is 3 cross 1. That means it has a x 1 x 2 and x 3, so this function is written x 1 square plus twice x 1 x 2 plus 3 x 2 square plus four x 1 x 2 x 3 plus x 3 cube, calculate the gradient of this gradient vector for the function this at x is equal to x dimension is 3 x 1, x 2, x 3. That is why I am writing the suffix this and that value is 1, 2, 0 and I have written is transpose. That means the row vector transpose is becoming column vector, so this you can find out this gradient of this vector like this way.

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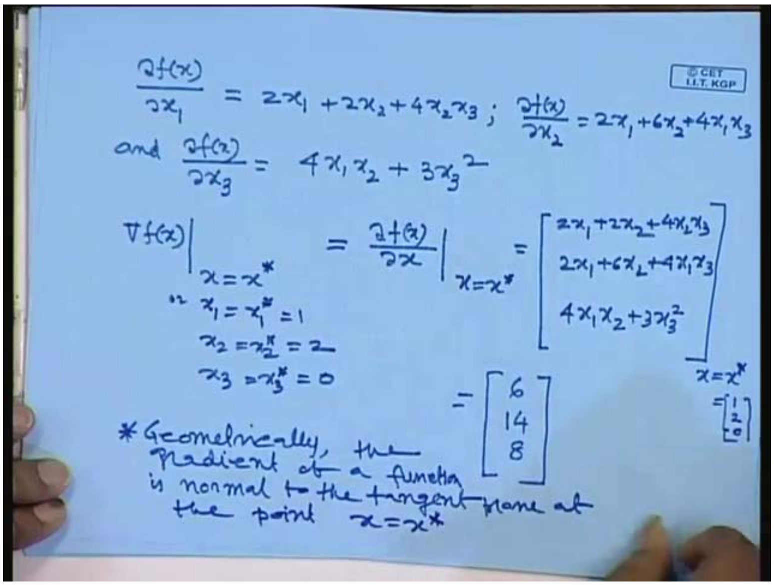

So, first you find out f of x with respect to x 1, we calculate this expression is twice x 1 if you see the expression that one, you have to differentiate this one with respect to x 1 keeping other variables constant. So, if you do this one, it will be 2 x 1, 2 x 2 plus 4 x 2, x 3, similarly del f of x del x 2 will come 2 x 1 plus 6 x 2 plus 4 x 1 x 3, I told you I repeat once again. Then, when you differentiate partial differential row with respect to x 2 keeping x 1 and x 3 constant and del f x, then del x 3 is equal to 4 x 1, x 2 plus 3 x 3 x

3 square. So, these things I just retain from this functions partial differentiation of the function with respect x 1, x 2, x 3.

Now, the gradient of that vectors is nothing but a grad of this function gradients of the function, these have to compete x is a grad 2 x star and this star means at for all value x is given. So, this is you can say x 1 is given what value of this one is given x 1 is equal to x 1 star whose value is 1 x 2 is equal to x 2 star, whose value is 2 x 3 is equal to x 3 star whose values is 0. So, you put this value, this is nothing but mathematically you can write at del f of del x of del x, x is a vector of dimension 3.

So, this port x is equal to x star and this values are like this way if you put this values are here 2 x 1 plus 2 x 2, I am writing from this x del f del x 1 plus 4 x 2, x 3. Then, del f del x 2 is 2 x 1 plus 6 x 2 plus 4 x 1 x 3, then from this one del f del x 3 is equal to 4 x 1 x 2 plus 3 x 3 whole square and put this value, x is equal to x star and this value is x 1 is x 2 is equal to x 2 star is 2 and 3. So, these values are nothing but if you see this one is nothing but 1, 2, 0, if you put this values this in this shows the value is 16, 6, 14, 8. So, this is mathematically, geometrically, what does it mean, geometrically it indicates the gradient of a function the gradient of a function is normal is normal to the tangent plane at the point x is equal to x star.

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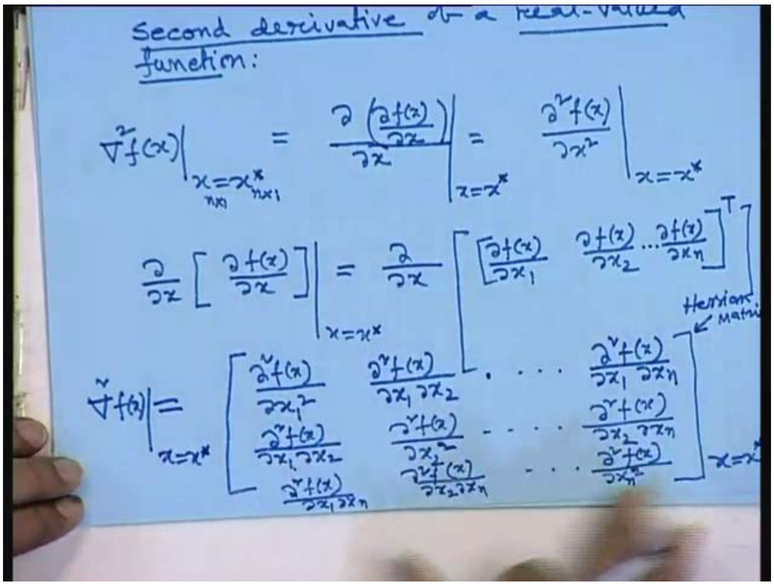

This is now the second derivative of a real value function, how to compute real value function, so same derivative of a real valued function means the function is given first, you derivative that function with respect to x. So, that is called gradient of that function, then gradient of that function once again you differentiate with respect to a x, an x is a vector.

So, firstly while we are finding out the gradient of a vector a gradient of a function that function is a quantity, if it is then we will get a vector again, what we are doing, we are differentiating that vector with respect to vector, then we will get a matrix. So, let us see what we are doing it here, so the second derivative of real function is function is denoted by f of x that is and finding out the value x is equal to x star double star. By definition, I told you first you take the gradient of this function f of x, then del of x and then once again you differentiate this with respect to this one, with respect to del x, this is nothing but a.

Then, you put the value of x is equal to x star, where the variables are n cross 1, let us take this and this you can write it del square of x del x square. The second partial derivative, we have to do it and put the value x is equal to x term, so let us see what is done, this one del f of del x. So, first we have done the differentiation of f gradient of f with respect to x that we know what this expressions is. So, I will write this expression here what is this expression is put the value x is equal to x star del x as it is, I have written then value, I have written this value is what del f del x, del x 1, del f, del x, del x 2, del f, del x dot x n.

Then, whole transpose at this should be a column factor, so I can write it row factor transpose, it does not matter this one, so this thing gradient of the function is this one, then you can you will differentiate once again. So, how you will do it this is a scalar quantity, this you see when you differentiate f with respect to r, you will get again, you are differentiating the scalar quantity with respect to that term. Again, you are differentiating this scalar quantity with respect to that, so ultimately you will get a matrix. So, this will become now del square f of x x is equal to x star, the matrix will look like this way del square f of x del 1 square.

Then, del square f of x del x 1 del x 2 and dot, you will get del square f of x is equal to del x 1 del x n. So, again just you differentiate these things, you will get these results del square f of x, del x 1, del x 2. Actually, it will be del x 2 del x 1, but these are two values are same, so next is del square f, del x 1, x 2 square, x 2 square dot del square f, del x 2 into del x n differentiate. In this way, you continue the last row will be del square x, del x, this is the del x and x 1 del x n, then del square f del x del x 2 del x n and so on last element of this matrix will be del square f del x and square.

So, this is a matrix and you have to find out the value of this matrix elements at x is equal to x 1. So, this is called that second derivative of real functions, how to find out and if you look these matrix it is a symmetric matrix and that matrix name is call Hessian matrix, in short it is called Hessian. So, this is a symmetric matrix, so in short I can write tell you if you differentiate second if you take the days second derivative of a real value function of f of x, which is a scalar quantity, then the result is a matrix. That matrix is a symmetric matrix and each element of this matrix is nothing but the partial derivative of x second partial derivative of x with respect to x 1 x 1 and x 2 all these things.

So, one can complete this one and that matrix is called Hessian matrix, simply it is hessian which is a symmetric matrix this is a symmetric matrix and you know the definition of symmetric matrix. In general, if a is a matrix and elements is a i j, a i j is equal to a j is not equal to j. So, this today I will stop it here. I will continue next class with an example that how to find out the second derivative of a function, considering the same example.

Thank you.