Optimal Control

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Lecture - 1
Introduction to Optimization Problem: Some Examples

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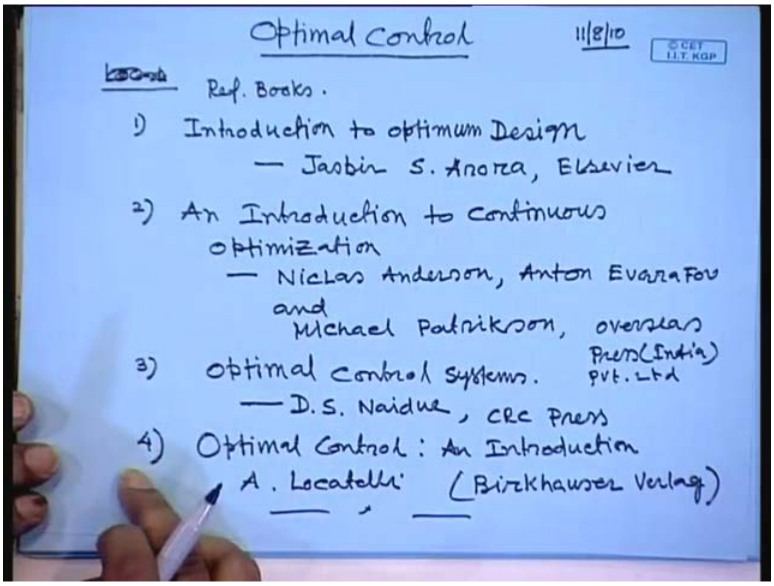

We can start. So, this is the course for optimal control and this course I just split up into 40 lectures and 20 lectures, formally I will cover with the static optimization problem and remaining 20 lecture is a dynamic optimization problems. The books that we will follow as like this, 1, first is introduction to optimum design, the author is Jasbir A Arora, publisher is Elsevier. The second book is the, an introduction to continuous optimization the authors are Niclas Anderson and that Anton Evarafov and Michael Patrikson. This is, this book is overseas press publication, press India private limited.

These two books mainly I will cover the, for what is called the static optimization problems will be covered from these two books. The remaining two books, books three is optimal control systems, author is D S Naidu, it is CRC press. The fourth book is optimal control an introduction, the author is a Locatelli, the publisher is birkhouser, birkhouser verlag. So, I just mentioned is the first two books will be covered the static optimization problems, third and fourth book is for dynamic optimization problem.

Before we start this optimal control problems all they say, what do you mean by the optimization of a function, that will discuss first.

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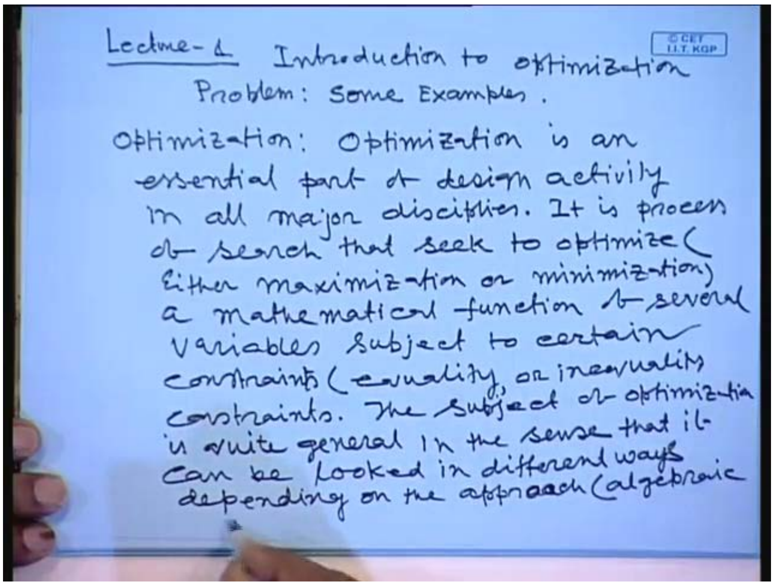

So, our first lecture is introduction to optimization problem and we will explain with some numerical examples or some examples. So, first we will define what is optimization, we understand by optimization, then we will consider an example. Optimization, optimization is an essential part of design activity, again in all engineering field, not only engineering field other fields also, is a design activity in major what is called discipline.

So, we can write optimization is an essential part of design activity, design activity in all major disciplines, not restricted to engineering problems, agree. So, it is a process of, that may it is a process of search that seeks to optimize. Means in other way to maximize or minimize, to optimize bracket either maximization or minimization of a mathematical function, mathematical function of several variables subject to the constant, the constant may be inequality constant or equality constant. So, a mathematical function of several variables subject to, subject to certain constrain, that constrain may be equality constrain or inequality constrain.

The subject of optimization, subject of optimization, subject of optimization, the subject of optimization is quite general in the sense, in the sense that it can, that it can be looked, it can be looked in different ways depending on the approach, depending on the approach. This approach may be a, what is called the algebraic approach or geometrical approach.

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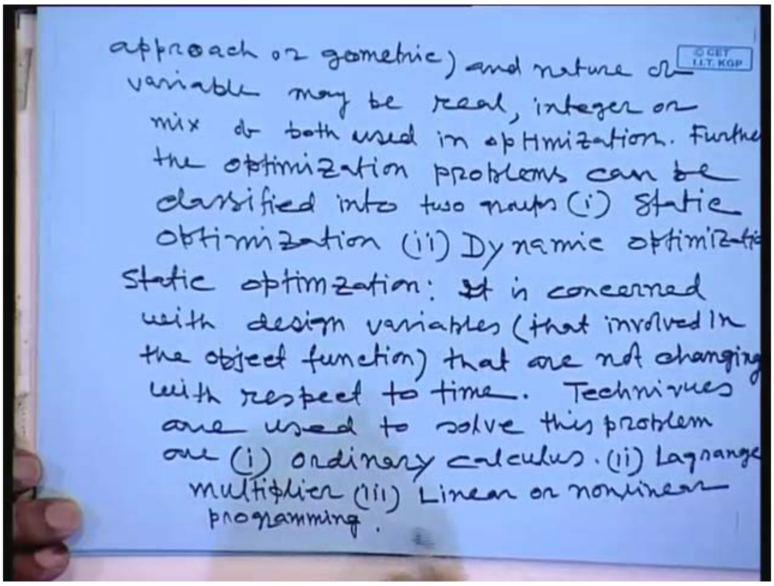

Algebraic approach or geometric and the nature of variables, the nature of variable associate in optimization problems may be a real, integer or the mix of both. And the nature of variables and nature of variables may be real integer or the mix of both. When it a real variable, this is a continuous variable, when it is a integer variable it is a discontinuous variable. And it can be a combination of discrete and continuous variable both, used in optimization problems. Further this, optimization problem can be classified into two groups, one is called static optimization, another is called dynamic optimization.

So, further the optimization problem, problems can be classified, can be classified into two groups. One is static optimization problem and second is dynamic optimization problems. So, static optimization problem basically is a variables, is a variables which does not change with time, the static optimization problems are the variables, that are associated in the optimization problem and it does not change with time and that is called static optimization problems. And static optimization problem can be solved by, what is called organic calculus, second the techniques used for solving the optimization, static optimization problem is the organic calculus, then you LaGrange multiplier method, then third is linear and non-linear programming methods.

So, our just, I will write static optimizations, what is this it is concerned with, it is, it is concerned with design variables that involve in the objective function, that bracket you can write it that, involved in the objective function. That are not changing, the desired variables that are not changing and not changing with time, with respect to time, that means this is a constant variables, designed variables. The techniques are employed, techniques are used to solve this problem, the static optimization problem, to solve this problem are, I just mentioned earlier also, it a ordinary calculus method, ordinary calculus. Second is LaGrange method, LaGrange method or LaGrange multiplier, third is linear or non-linear methods, programming methods, linear or non-linear programming technique.

So, this is the static optimization. So, our, you can say that static optimization is the, is concerned with the designed variables, that associated in the objective functions. And that variables does not change with time, that is called static optimization problem. And that optimization problem can be solved based on what is called techniques, the techniques are ordinary differential calculus. If you know ordinary calculus, you can solve it or by using LaGrange, what is called multiplier or linear or non-linear programming technique.

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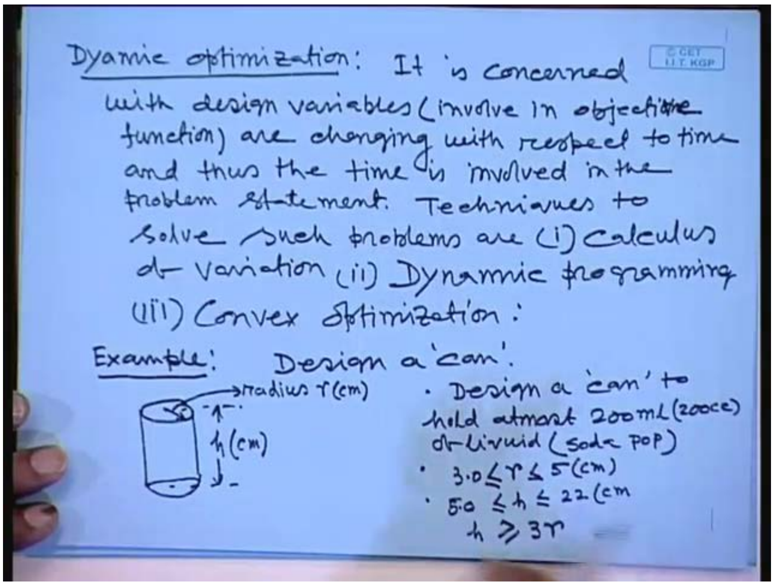

Similarly, the static optimization ((Refer time: 16:33)) similarly, the dynamic optimization problems, optimization is it, this is dynamic optimization is concerned with the design variables, that involved in the objective function. Again that design variable are a function of time and the time may be involved in the statement of the problem. So, that type of problem is called, what is called dynamic optimization problems and dynamic optimization problems are solved, the techniques available for solving the dynamic optimization are first is your calculus of variations, people can apply calculus of variations. Second is your, you can apply this dynamic programming, third is convex optimization problems. So, first we will see now that work, how we will formulate the optimization, for static optimization problem based on the statement of the problem, then how we will you formulate.

So, I will just write it, the definition it is concerned with design variables involved in the optimization function, involved in the, in objective function, objective function design with design variables are changing with respect to time. And thus the time is involved, the time is, time is involved in the problem statement, just I mentioned is the techniques are available to solve such type of problems, are three techniques are. You can say, techniques to solve such problems are one, that calculus of variation, of variations two, the dynamic programming, third is your convex optimization problems, problem convex optimization.

So, basically in optimization problem we have a two categories, static optimization problem and dynamic optimization problem. Now I will, already explained what is static optimization problem and dynamic optimization problems. Next I will, tell that given the statement of the problem, how we will form the mathematical model of this optimization problems.

So, let us take one simple example and see how one can formulate, a one can get the mathematical model of that optimization problem, example. Suppose you have to design a can, container so can is I am just showing it here, it is this type of can, whose height is, the height of the can is h, which is expressed in terms of centimeter, unit is centimeter. And the radius of this can, this radius r in centimeter, this is the can we have to design, but we have a what is our requirement and what is the constants involved while we will design the can.

So, first statement of the problem, this can volume should be not more than 200 milliliter, that means can contained should not contained more than 200 milliliter of liquid or you can say a soda pop. Because, it is restricted that people should not consume or should not drink more than 200 milliliter of soda pop at a time, on that this is, this can is needs to be designed. In addition to that there are some constants also involved, keeping all these things in mind our job is, what is the manufacturing cost is for the can, minimum manufacturing cost. So that, we obtain these, these constants so I will write one by one, what is we needed it to design, before we design this container we manufacture this container.

So, design a can to hold at most 200, milliliters or 200 cc of liquid or you can say, the soda pop. That, this is restricted because people should not drink more than 200 milliliter of soda pop at a time, that is why this, the volume of this one is restricted to this one. Second things, in additional to this they are some constants imposed on the physical parameters of the, what is called the can, that physical parameters is r radius of the can is less than equal to, less than or equal to 5 centimeter and more, greater than equal to what is called 3 centimeter.

This is the constant on the radius of this pile, radius of the can is imposed when you will manufacture the can, the radius will be in this range. So, next is impose that height, the height of the can is than equal to 20 centimeter and greater than equal to 5 centimeter, 5 centimeter. This is the height constants is given, but for aesthetic reason, aesthetic reasons means, when you we will manufacture the can so that, we can comfortably hold the can. For aesthetic reason, that is another constant is imposed in the design, that constant is the height of the, the height of the cylinder is greater than equal to 3 r.

Basically, you see r and h radius of the can and the height of the can is a side constants means, side constants means r cannot be more than this or less then this. Similarly, height cannot be more than this or less than this and this constraints is given due to the aesthetic reason. In the sense that we can hold the can comfortably, with this constraints if the height is more, the diameter is radius is small then something is not convenient or the diameter is, radius is more, height is small then it is not convenient to hold the can comfortably. So, this is the constraints, now what is our problem, we have to a design a can in such a way so that, manufacturing cost is minimum.

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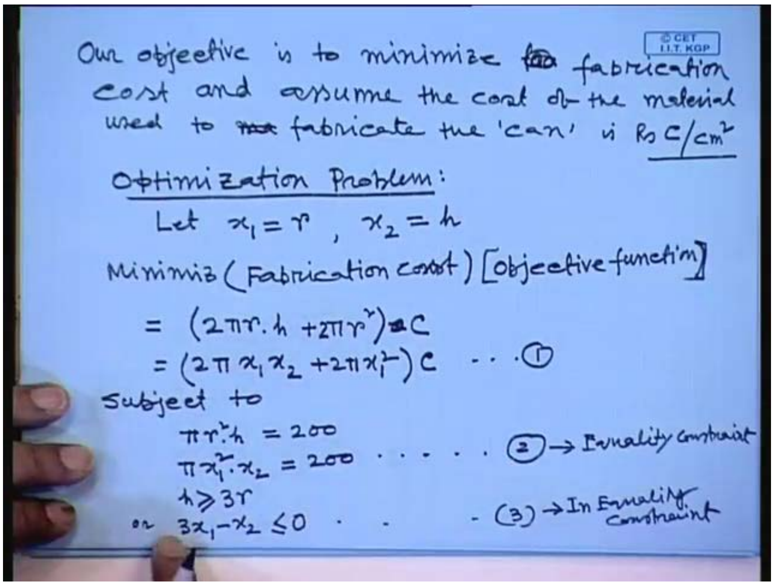

So, our objective is now, our objective is to minimize fabrication, fabrication cost and assume the cost of the material, the cost of the material used to manufacture the can, cost of the material used to manufacture or to fabricate the can is rupees C, rupees in C bar centimeters square. This is cost per unit centimeter square of the metals sheet or whatever metal you used to manufacture the can. So, this is our constraint everything is given, now I can easily formulate the problems.

So, what is our optimization problems? If you look carefully the optimization problem, I am just writing, given the statement of the problem I am now formulating in mathematical model, from. So, our first problem is optimization problem is, let us before that I just considering x 1 is the variable which is denoted by radius of the circle, the x 2 is another variable, which is the height of the cylinder in centimeter. So, our problem is minimize bracket you can say, the fabrication cost.

So, our manufacture fabrication or manufacture cost of can should be minimized, can and that is generally called objective function, that we call generally call so objective function or cost function, in optimization problems, objective function. So, what is our objective function, that manufacturing cost is involved, the circumference of the cans can which we have assumed circular circumference of can, we are assuming that thickness, that cost of the can, the cost of material used for fabrication for a fixed thickness, it is given that one and with this shape we are fabricated the can.

So, what is our cost involved, first is twice pi r circumference of this one is multiplied by height. So, this is the material used for covering the circumference of this can, plus the bottom and top. Since, it is a circular it is a pi r square, but they are twice. So, two pi r squares so if you use this, what is called the variables now, that this 2 pi r is x 1, I have defined that r, h is x 2, then 2 phi x 1 square. You see this is the centimeter square, we know the cost is C rupees, C per centimeter square. So, you have multiple by C similarly, here in terms of variable x 1, x 2, we have multiply by c.

So, our problem minimize this manufacturing cost of the can which is expressed by this expression and that expression, if you see is a non-linear function and function of x 1 and x 2 radius and height. And C is the cost of this material is fixed for centimeter square. So, what is constants involved here, next our constraint are here we mentioned it that, the can contain at most the 200 milliliter of soda pop or any liquid. So, we can write the minimization the subject to so what is this one, the volume pi r square, what is the volume of this, this, this one pi r square into h must be equal to 200 milliliter or 200 CC. So, which we can write it pi x 1 square into x 2 is equal to 200 cc.

So, let us call this is, this is equation number 1, this is equation number 2. So, this equation if you see, this is nothing but a equality constraints so I will call this equation is equality constraints. And another constraint we have imposed for aesthetic reason so that you can hold the can comfortably, that constraints is given like this, that each h is greater than equal to 3 r. In other words we can write it this one, that r 3 x 1, r is equal to x 1 minus x 2 is less than equal to 0, this is. So, this is equation number 3 so this is equal to, you see this is the inequality constraint, this is the inequality constraint.

So, we have a objective function we can, from the statement of the problem we can write it the, what is the objective function. For this particular example the, our objective function is the minimization of manufacturing cost of a can, then we have a equality constraints. From the statement of the problem, we can write it equality constraints another, another constraints is there, which is inequality constraints from the statement of the problem we have written. There are some other constraints are there, we call what is called, side constraints.

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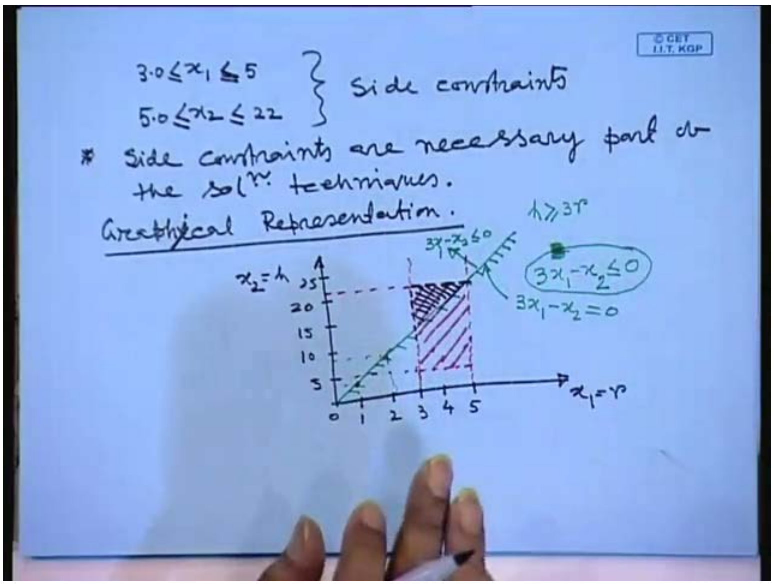

So, this side constraints are this one, that we have written, that radius of these can, cannot be more than, is less than equal to x less than equal to pi and greater than equal to 3. Another constraint we have given the height of the can, will be less than equal to 22 centimeter and greater than equal to 5 centimeter. So, these constraints are called the side constraints and x 1, x 2 you will see is the only design variables, which will play role for minimization of the cost manufacturing cost.

So, this variables, side constraints is given to you. Now one can solve this problem by keeping in mind, we have the objective function, any optimization problem we an objective function, then we have a constraints, equality constraints and equality, inequality constraints may be there or both is there equality and inequality constraints. Then we have a side constraints so next is this, the side constraints are necessary, to just you can say, the side constraints are necessary part of the solution, techniques. This will tell you about the acceptable region of design variables, acceptable region of design variables.

So, let us see graphically this problem, we can represent this problem graphical representation. Now see this one, x is I have considered x 1 the radius of the can at yx is x 2, which is a height of the can, this is you can say radius of the can. Then with, based on the side constraints I am drawing this one, you see 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5 this negative cannot be because radius cannot be negative that dimension. So, this part is not there in the, another is your height varies from what is called, lower limit is 5 and upper limit is 22. Let us call this is 5, 10, 15, 20 and this is 25 so I know with this constraints, side constraints where our design variables may lie.

So, this is 3 x 1 varies from 3 to 5 and x 2 varies from 5 to 22, this is 20. Let us call this is 22 somewhere here so this is our region, where our design variables x 1, x 2 must lie to obtain the minimum fabrication cost, somewhere here. Not only this, there is the another equality constraints are there further, the search region we can just compress it. What is this, constraints are there another constraints if you see this one, that it is given that, h is greater than 3 r, which in turn we can write it, which in turn we can write it that 3 x 1 minus x 2 less than equal to 3, because our constraints if you re recollect our constraints h is greater than is equal to 3 r so 3 r, r is a, r is a we have defined as a x 1 variable. So, 3 x 1 minus x 2 is equal to these constraints, so this inequality constraint I can show in this graph, where it is.

So, first I will draw 3 x minus x 2 is equal to 0 so 3 x, 3 x 1 minus x 2 is equal to 0, it is a equation of a straight line. And in this case, this equation of straight line whose slope is 3 that means, this angle will be you can just draw it from this one. This is x is equal to 1, that will be x 2 will be 3 somewhere here, x is equals to 2, that will be 8 somewhere here.

So, if you just draw a this, this, this one, this is our line so this line represents so like 3 x 1 minus x 2 is equal to 0, but our constraint is given 3 x 1 minus x 2 less than equal to 0. Then which side of this straight line, which side of this straight line satisfy this constraints? One can easily say the upper portion of this line, upper portion of this line satisfy this equation. So, I am just writing in this side arrow so this is the, this part is 3 x 1 minus x 2 is less than equal to 0, you can check by any value in this one, in the region it, that this inequality satisfied.

Now see, previously we have shown it this whole shaded area, now this has become the such area now, becoming a smaller with this constraints. So, our if you see this one, this is our the searched region, with the black one, this. So, in order to optimize the objective function that our design variable lies in this zone, in this region. Now question is I know this, in this region we a have a design variables of this one, if it is in this region, some value of x 1 and x 2, the our what is called the manufacturing cost will be minimum. But we do not know what should be the value of x 1 and x 2, but we know that region, it is in this region.

So, this is the graphical representation to understand some optimization problems, again. If you look in this pair, this optimization problem may be two category I am talking about all r static optimization problem, because x 1 and x 2 does not vary with time. This is a fixed this one, now you see this, this optimization problem just I have written it here.

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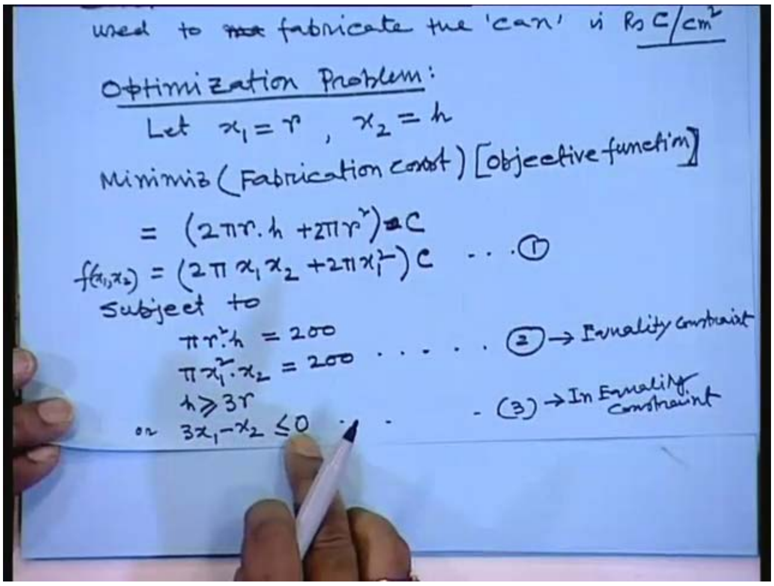

So, this you see f the minimization of fabrication cost, if you consider this f, f 1 is function of x 1. So, it is a non-linear function, this is a non-linear objective function whereas, see equation number 2, this is also non-linear inequality constraints. Because of it is a product of x 1 square and x 2 and here is product of x 1 square and also x 1 and x 2. And the third equation which is inequality constraint, it is linear equation so this problem is called non-linear static optimization problems, we have in our hand.

So, there are two types of optimization, static optimization may be linear optimization problems is called non-linear optimization problem. We will call the linear optimization problem, when all the, when the objective function and all the constraints are linear, then we will call it a linear optimization problems. And non-linear optimization problem you will call, if anyone of this either objective function or any inequality constraint may be more than one equality constraints is there, in the problem statement or more than one inequality constraints is there, in the problem statement, it a. If any one of the constraint is non-linear either objective function or constraints, either equality constraints or inequality constraint, then we will call it is a non-linear static optimization problems. Any one either objective function or constraints, any constraint or more than one constraints are non-linear function, in variables then we will call, non-linear optimization problems.

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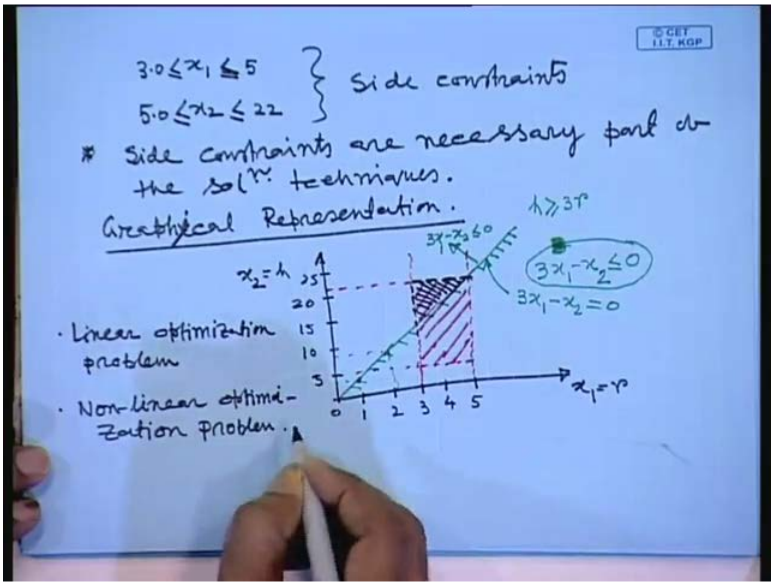

So, we have a two, now if I write it in general statement of this, this one you will see our problem is like this way. Now we have a, we can say it is a linear optimization problem we have attend linear optimization problem one, then another is non-linear optimization. I mentioned you the linear optimization problem we will call, if the objective function and including all the constraints, either equality constraints or non-inequality constraints all are linear, then it is a linear optimization problem, linear programming problem. In any one of this objective function or inequality function either inequality, inequality constraints is non-linear one or more than one is non-linear, then it called as non-linear optimization problems.

So, in general now I will write the, suppose we have an, an, any problems optimization problem is given statement of the problem is given to you. Next with the, with the, with the statement of the problem I can formulate the mathematical model of the optimization problems. In general, if you have a design variables more than two, we have consider two design variables, one is for this example, one is radius of this can and height of the can. In general, it may be a n variables.

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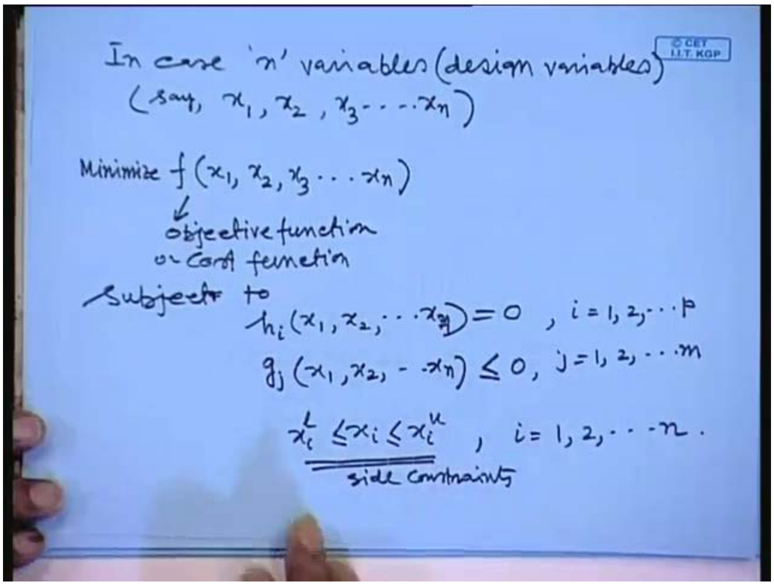

So, we can write in case of, in case of n variables, n variable means design variable and design variable what are they, those are involved in the optimization problems or statement of the optimization problems as related design variables. So, we have a n variables of their and let us say this n variables are, in case of n variables bracket you can write say, we have a variable x 1, x 2, x 3 dot dot x n. These are n variables of that, all these variables are not changing with time, that is called static optimization problem, these design variables.

So, we have a n variables are there, this n variables are there so we can now in general, we can write the problem, optimization problems, when you will call optimization problems, the problem may be a maximization of a function, objective function or minimization of a objective function. Previous example it was a minimization of a function, cost minimization, fabrication of the can should be minimized. So, in general now I am writing f is a function of x 1, x 2, x 3, x dot dot x n.

So, this the objective function or it is called the cost function, our problem is minimize this function, which is called the objective function or this is called objective function or it is called cost function minimize. Minimize this objective function, then subject to condition are there, subject to in general, we have shown it here that, in general we can write it h I, x 1, x 2 dot dot x n equal to 0. So, this is a our equality constraint h i, how many equality constraint are there I am wiring, i is equal to now 1, 2 dot dot p.

So, we have any objective function is a scalar function, but in this case only we have considered a single objective function, we will show with an example that we may have an, in optimization problem more than one objective functions, simultaneously we have to optimize more than one objective functions. At this movement I am just considering one objective function and subject to, there are p equality constraint and g j, which is a function of design variable x 1, x 2 x dot dot x n, this x n, x n is less than equal to 0 for j is equal to 1, 2 dot dot m. So, we have a m equality constraint there is a, p equality constraint and there is a m inequality constraint are there.

In addition to that, I told you there is a some, what is called side constraint are there, what is the side constraint x i is less equal to x i u, less than equal to x i l, this superscript u indicates the upper valve of this design variable x i. And superscript l means the lower range or valve of x I, design variables and i varies from what? Which I we have considered there are n variables are there. So, you can write i is equal 1, 2 dot dot n variables.

So, this is our more general statement of our optimization problems, we have objective function which is a scalar and which is a function of n design variable x 1, x 2 x dot dot x n. Subject to equality constraint, how many equality constraint are there? p constraints are there and how many inequality constraints there are? There are n inequality constraints and these constraints are called side constraints, these are called side constraints.

So, this our statement of the problem in general so if this function this, this optimization part function will be a linear optimization problem or non-linear optimization problems, if all the objective function, equality constraint, inequality constraint are linear, then it will be called linear optimization problem or linear programming problem. If any one of this either objective function or any one this function is a non-linear, then it is non-linear optimization problem. So, one can write this equation more complex form.

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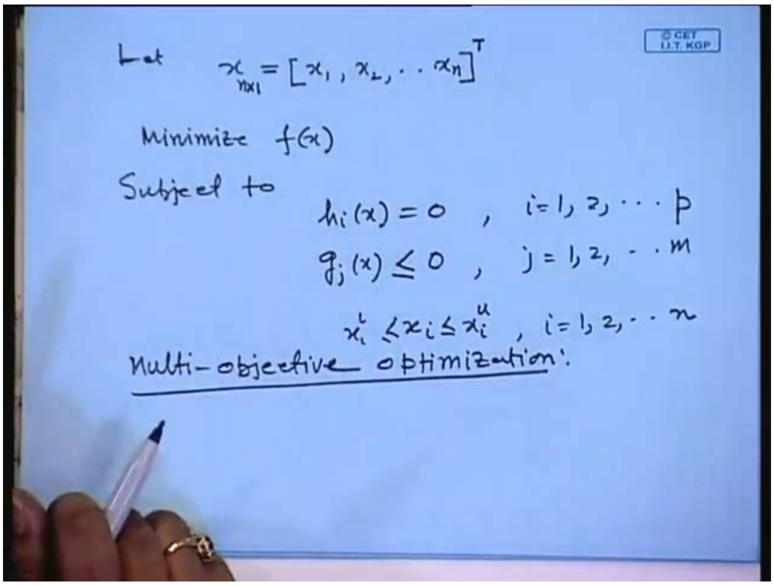

By assigning, by assigning that say let, let x is vector whose dimension is n cross 1, I am writing this one x 1, x 2 dot dot x n is the dimension x 1, x 2, the elements of vector x. How many elements? n element so dimension is n plus 1, is column vector from I denoted. So, the, our problem statement now I can write it, minimize f of x, subject to h i of x equal to 0, i is equal to 1, 2, dot dot p and g j of x is less than equal to 0 and j is equal to 1, 2 dot dot m. And our side constraints are, we can write it our side constraint is x i, x i u and this is x i l and i is, in this variables, how many variable are n variables are there. So, it is a general what is called the statement of the optimization problems so this is only one optimization problems are there, only one single objective function is there, single objective function is there, we are minimizing or maximizing.

So, next is you, what is called multi objective optimization, that means in this case we have a more than one optimization problems, more than one objective function is there. Same as this one, only our objective function is more than 1 and we have simultaneously, we have a subject to equality constraint and inequality constraint. And we have also, side constraint so this type of problem are called multi optimization problem, more than one optimization, more than one objective function to be optimized.

That means may be one objective function is maximization, another objection may be minimizations. Such type of problem we have and how to handle that type of problems, we will discuss, first we will, how to formulate that type of problem from the statement of the problems. So, I will stop here today. So, next class I will discuss what is multi objective optimization problem.

Thank you.