Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture 47
Linear Dynamical Systems: Definition and Solution via Matrix Exponential

Hello, welcome to another module in this massive open online course. In this module let us look at another yet another application of linear algebra and that is in the context of linear dynamical systems.

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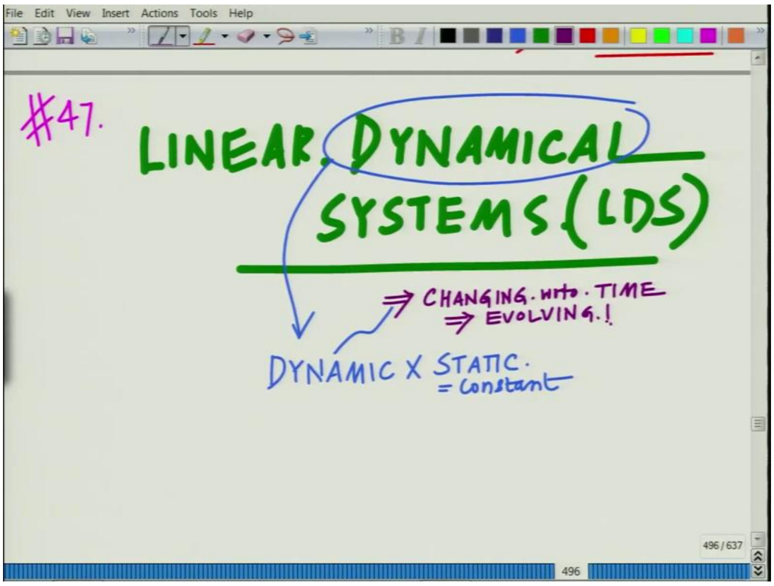
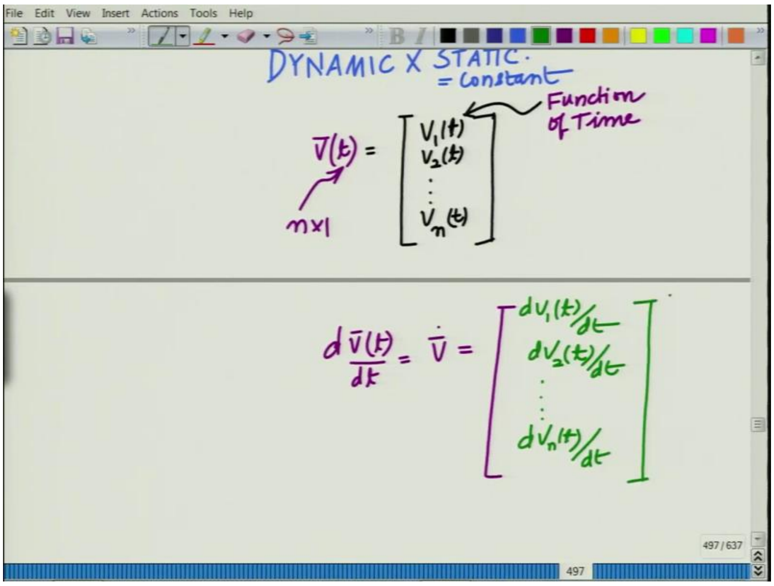
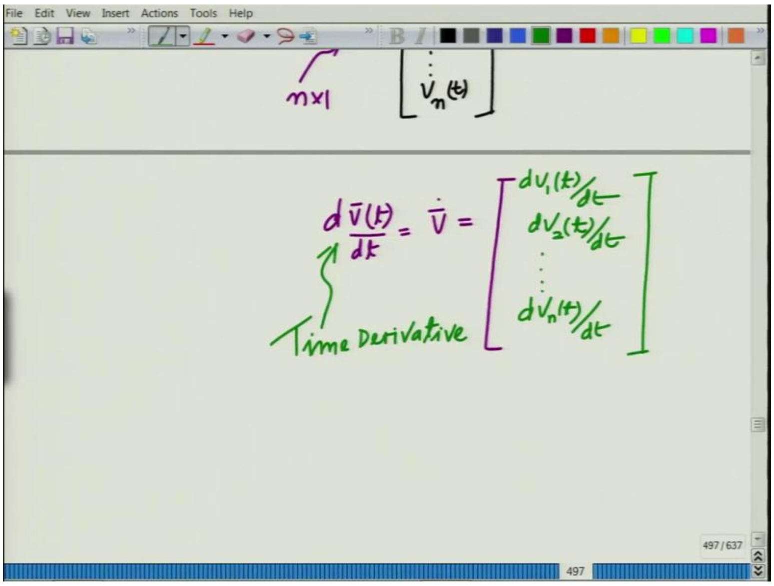




$$\bar{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

$$\frac{d\bar{v}(t)}{dt} = \dot{\bar{v}} = \begin{bmatrix} dv_1(t)/dt \\ dv_2(t)/dt \\ \vdots \\ dv_n(t)/dt \end{bmatrix}$$



These are very important class of systems whose analysis can be conducted using the principles of, these are what are known as Linear Dynamical Systems or LDS, Linear Dynamical Systems and in particular dynamical, the word dynamic essentially means not static but these are dynamic. Dynamic is the opposite of static, static is something that is constant; that is fixed.

Dynamic naturally means something that is not static but something that is changing constantly that is evolving with respect to time in particular, that is essentially what a dynamical system is. Dynamic essentially implies that this is changing with respect to time, implies this is evolving constantly, that is the meaning of this phrase dynamic. So, you have a linear dynamic system; a system that is dynamic.

A dynamic system implies one that is essentially changing with respect to time. Now, this generally has the form if I have for instance define a vector v bar so, naturally it is going to be a function of time, remember we said this is a dynamic system. So, you can clearly see the dynamic nature is reflected by the fact that this is a function of time. So, this is a function of time.

This is an N cross 1 vector, you can write it as v bar t but sometimes you can drop this t just for convenience. This is an N cross 1 vector and then you can define dv bar dt that is essentially you denote by v bar dot. This is d 1 d v 1 t by dt, d v 2 t over dt, the v n t over dt and this is the time derivative.

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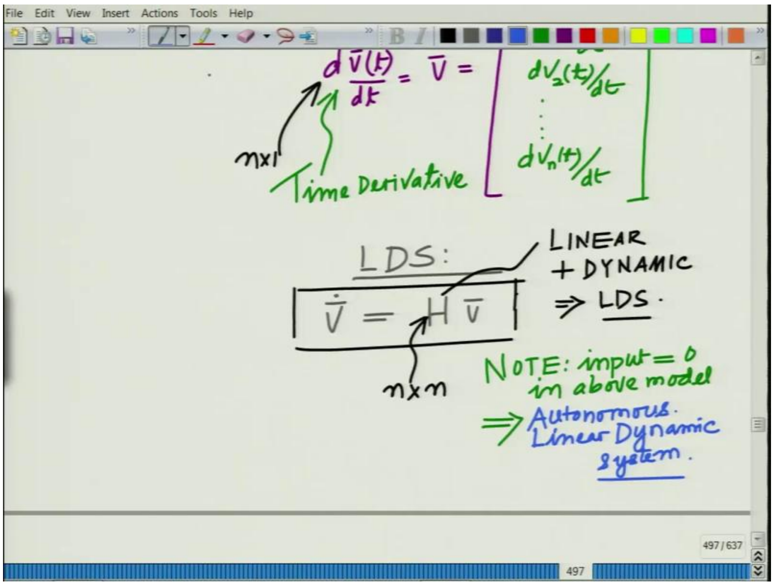

So, this is the time derivative and essentially the linear dynamical system the LDS can be characterized as v bar dot equals H times v bar. This is the fundamental equation of the LDS that is the time derivative v bar dot is a linear transformation of the output is H. This is a matrix H times v bar. So, this matrix H is naturally a square matrix. This is an N cross N because this is also N cross 1 vector.

So, this is essentially linear plus this is a function of time. So, linear plus dynamic is what makes it a linear dynamic system. Now, in particular this is also known as an autonomous linear dynamic system because the input is 0. There is no input. Look at this, v bar the derivative v bar dot is simply H times v bar.

If there were an input H times v bar plus some matrix let us say q times u bar where u is an input then this would this would have been an input driving this linear dynamic system, but because the input is 0, this is essentially an autonomously linear dynamic system. So, note that note: input equal to 0 in our model implies this is an autonomous, implies this is an autonomous linear dynamic system.

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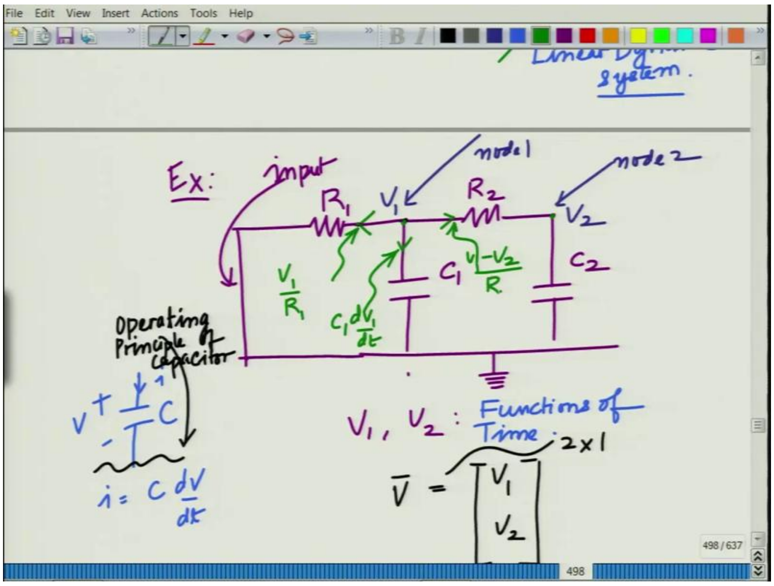

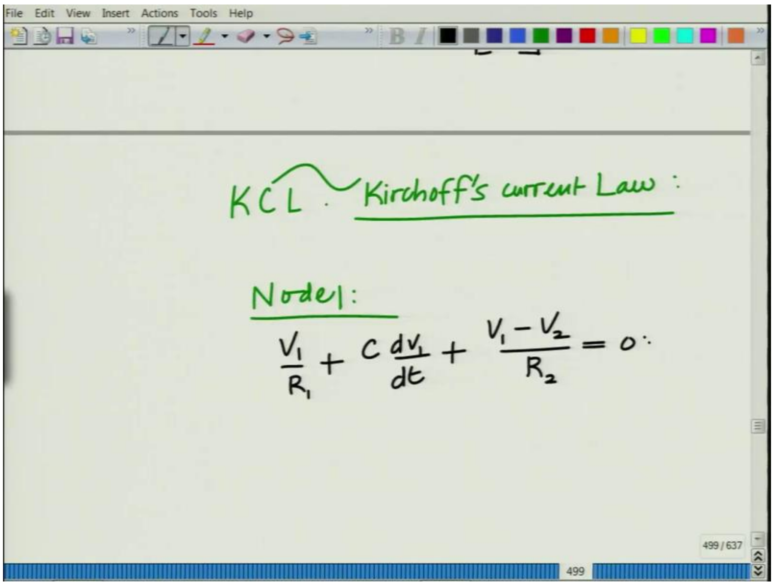

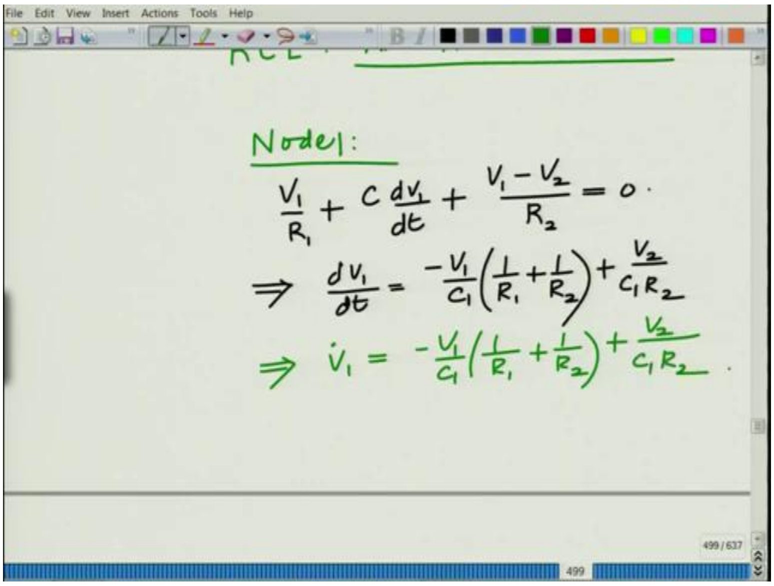

So now, let us look at for instance, an example to understand this. Let us consider a circuit. Circuit is an excellent example for a dynamic system because it contains elements such as capacitors and inductors whose output can be expressed as a function of time. So, I have this circuit. This is a resistance, this is a capacitor, resistance another capacitor and this is let us say, this is the ground and remember, this is autonomous systems.

So, input is 0. So, I do not have any input of course, I can later include an input to make it more general but right now, we are considering autonomous system. So this is R 1, R 2, C 1, C 2, these are the nodes. So, this is your node 1. This is your note 2. Let us call this V 1, let us call this voltage V 2.

Naturally, these are the voltage across the capacitor, capacitor discharges at a certain rate. So, you know that these are essentially going to be this V 1 and V 2 are going to be functions of time. In fact, I should be writing V 1 t, V 2 t as I said for convenience sometimes we can drop these time indices. So V 1, V 2, one should be clear to you, V 1 comma V 2 these are functions of time.

In general, we can use the principle of the circuit that is, if this is a current i and this is the voltage V across a capacitor, then V or I am sorry, i equal to C dV by dt. So, this is the operating principle of capacitor because well, you know as of course, electrical engineers must be familiar with it. Otherwise, you can see basically the charge is the capacitance times q equal to C times V. dq by dt is i which is there for C types dV by dt.

Anyway, point is now let us define the vector V bar as V 1, V 2 this naturally, this is your 2 cross 1 vector, and 1 can write. So this is your node 1, and this is your node 2. So you can write KCL at both nodes. So to develop the system model, one can write KCL at or, this is essentially the Kirchoff's which basically says that the sum of the current flowing into any node or net current flowing out of any node is 0.

Now, therefore, at node 1; what will the KCL be node 1? It is very easy to see at node 1; at node 1 this is going to be current through R 1. So, this is going to be V 1 by R 1. This is going to be C 1 d V 1 by dt and this is going to be essentially the output flowing current is V 1 minus V 2 by R 2 that is from Ohm's law. So, therefore, if you use the principle that outgoing current is 0.

So, we will have V 1 by R 1 plus C d V 1 by dt, d V 1 by dt plus V 1 minus V 2 by R 2 equal to 0 which basically implies that d V 1 by dt equals minus V 1 by C 1 times 1 by R 1 plus R 2 plus V 2 by C 1, R 2 which implies V 1 dot equals minus V 1 by C 1, 1 by R 1 plus R 2 plus V 2 by 1 V 2 by C 1 R 2.

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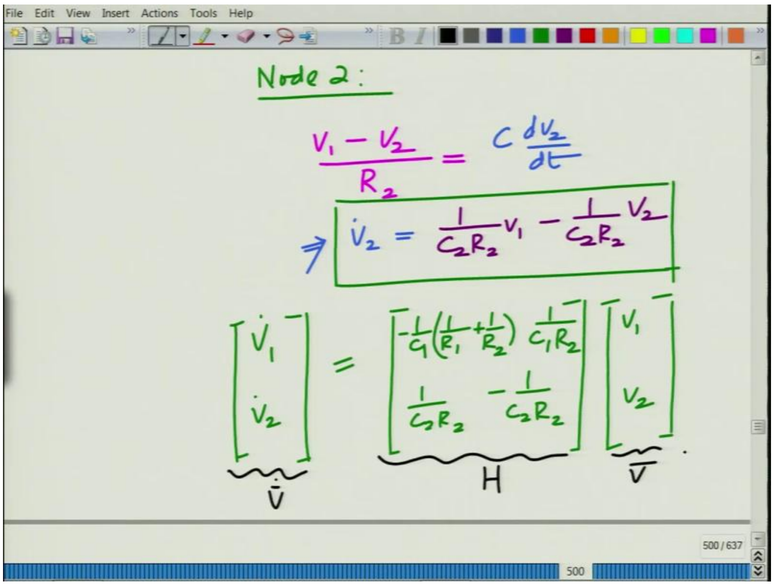

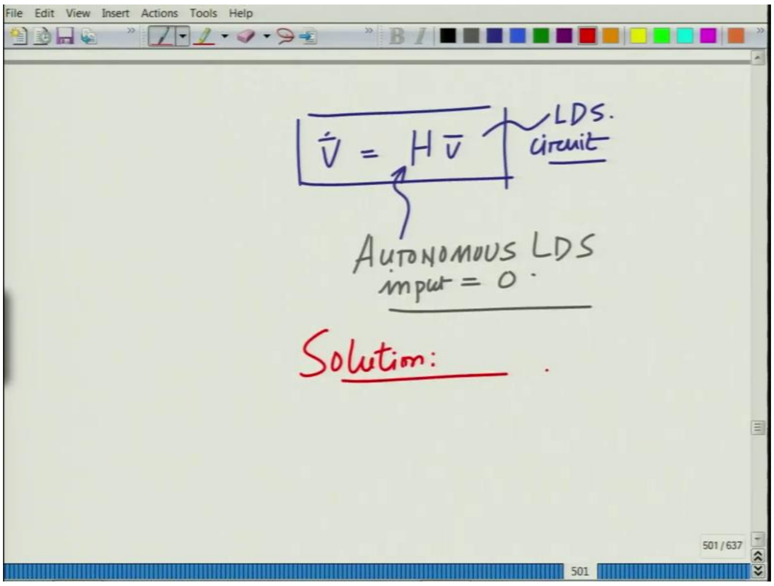

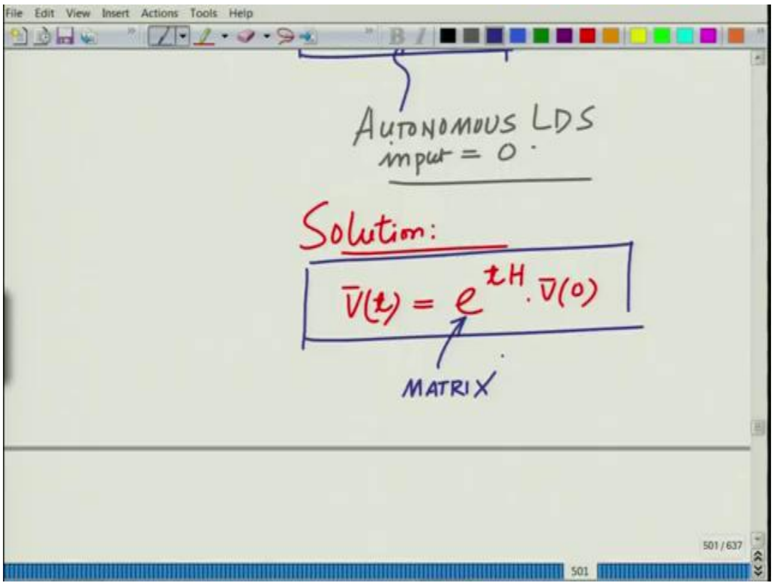

Similarly, if you write it at node 2, node 2 if you look at this node 2, this is V 1 minus V 2 by R 2 and this here is C 2 d V 2 by dt. So, current flowing into node 2 is current flowing out of node 2 This implies that we have V 1 minus V 2 by R 2 equals C d V 2 by dt. This implies that V 2 dot equals minus 1 by C 2 R 2 that is C 2 R 1 by C 2 R 2 V 1 minus 1 by C 2 R 2 V 2. So, this is essentially this and therefore, now if I look at these two so, I can write V 1 dot, V 2 dot equals a matrix times V 1, V 2 of course, this will be 1 by C 2 R 2 minus 1 by C 2 R 2 and these entries here will be minus 1 by C 1, 1 by R 1 plus 1 by R 2 times 1 by C 1 R 2 and this is essentially your matrix H.

This is your V bar dot and this is your v bar. So, essentially what you have is you have V bar dot equals H v bar and this is essentially now, you can see our circuit can be modelled, circuit with elements such as capacitors and so on that can be essentially modelled as a linear dynamical system. So, we have the equation V 1 bar v bar equal to that is V bar dot equals H times v bar where h is this matrix.

And now, so and remember this is not just any LDS but it is an autonomous LDS because input is 0. This is an autonomous LDS because input is 0. So, the solution is given as solution to an LDS V bar equal to is very simple interesting solution let me describe this. V bar t equals e raised to the power of t H times v bar 0. So, this is a very interesting solution. So, and the most interesting aspect of this quantity, this solution is this e raised to the power of t H.

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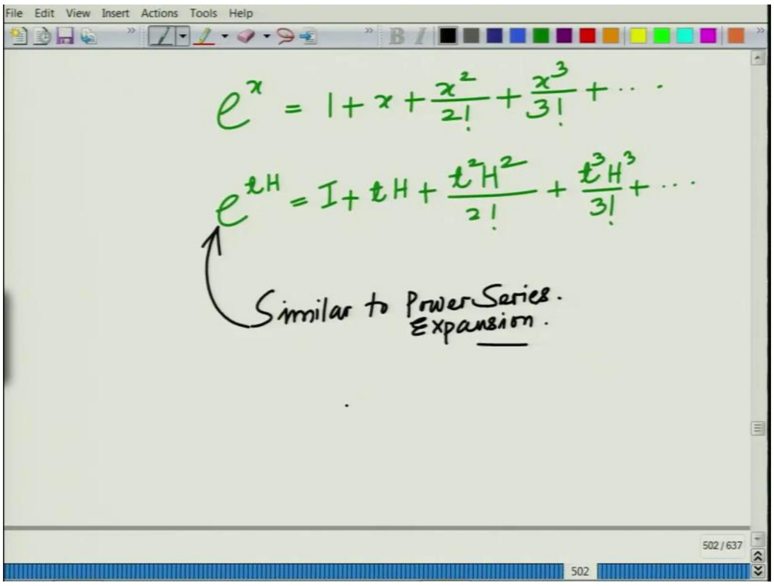

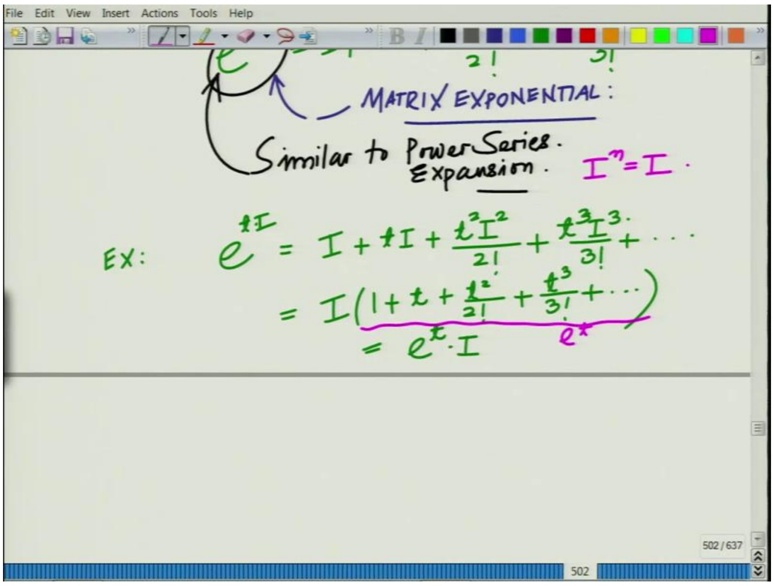

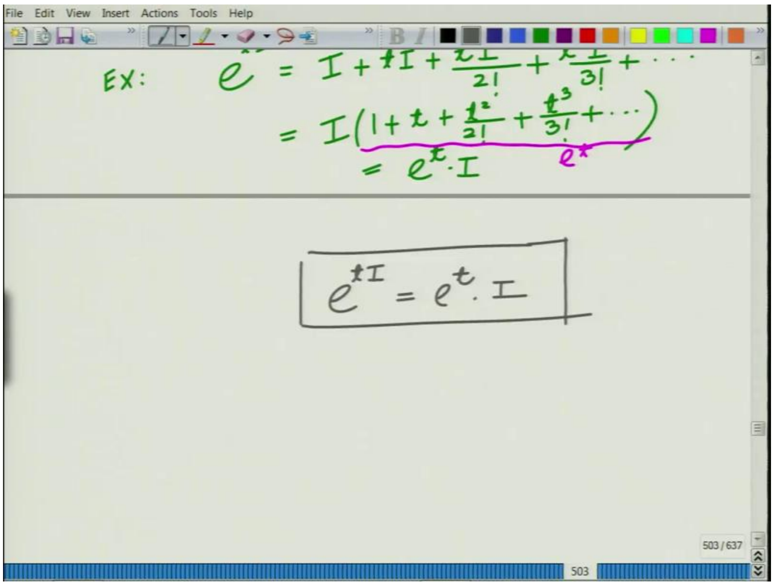

This is termed as recognize this is e raised to the power of a matrix, e raised to the power of t times H. We have not seen this before. This is known as a matrix exponential. This is a very interesting definition. This is a matrix exponential and the way it is defined is, you know the power series e x equals 1 over 1 plus x plus x squared over 2 factorial plus x cubed over 3 factorial plus so on and so forth.

e raised to t H is similarly defined as 1, I am sorry identity plus t times H plus t square H square by 2 factorial. Remember it matrix H square so, I can compute H square t cube H cube by 3 factorial plus so on. So, that is the interesting thing and therefore, this is similar to a power series expansion, that is the interesting thing; similar in nature to a power series expansion.

For instance let us take a simple example. So, this is a very interesting thing. So, this is the matrix exponential which has several powerful implications. So, this is what is known as, I already told you this is what is known as matrix exponential. This is e raised to the power of a matrix. So, that is the interesting thing. Let us look at some simple computations of this. For instance e raised to the power of t times the identity matrix.

What is this going to be? Example: this is going to be I plus t times I plus t square times I square divided by 2 factorial, recall I square equal to I t cube, I cube divided by 3 factorial, I cube is also equal to I. So, this is basically I times 1 plus t plus t square by 2 factorial plus t cube by 3 factorial plus so on and this is e raised to the power of t times I, because if you look at this quantity here, if you look at this quantity here.

This is nothing but e raised to the power of t. So, this is e raised to the power of t I and we use the property that I raised to the power of n equal to I. So, e raised to, so the simple naturally e raised to the t I is simply e raised to the t times I very simple; something that you would probably expect. e raised to the t I, you take each element that is basically it is it is a diagonal matrix entries t t and therefore, it is nothing but e raised to the power of t times I.

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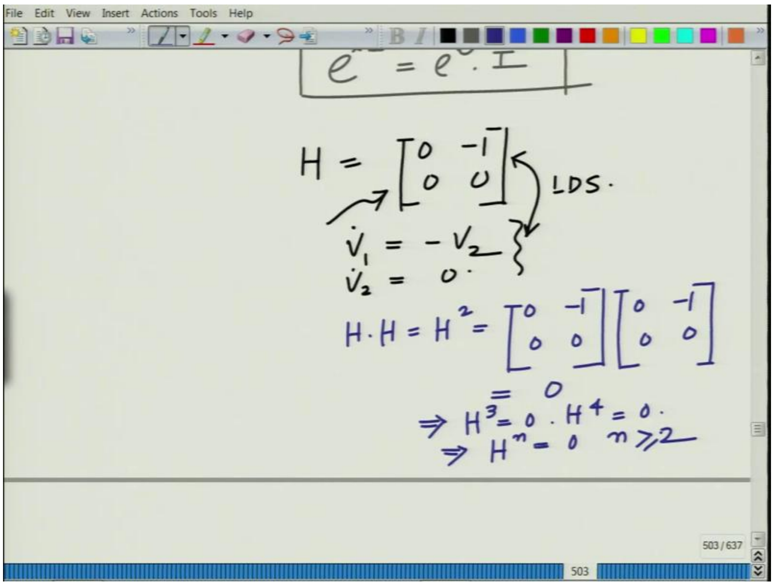

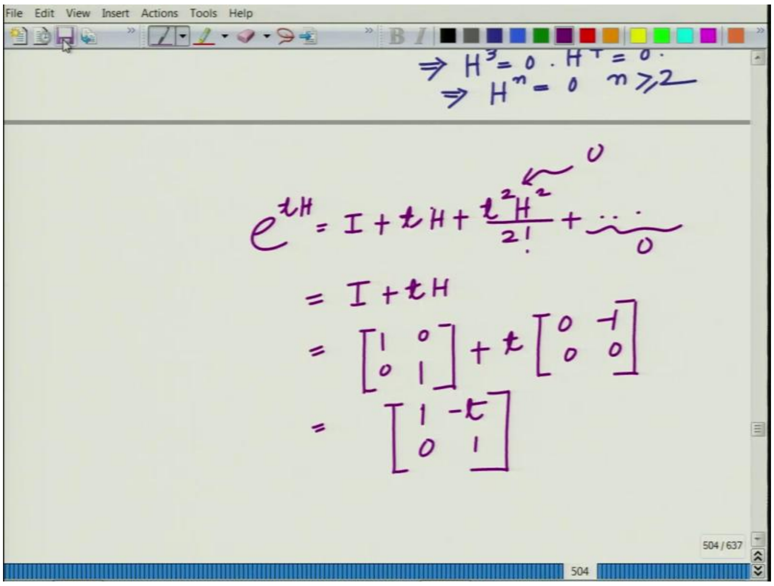

Let us take another interesting matrix for instance, let us say H equals, this matrix 0, minus 1, 0, 0, which implies that your linear dynamical system is basically V 1 dot equals minus V 2 that is and V 2 dot equals 0. This is the corresponding LDS, corresponds to this LDS. Now, if you look at this, this satisfies the property H times H equal to H square. This is basically equal to 0, minus 1, 0, 0, times 0, minus 1, 0, 0 equals 0.

So, this implies H cube equal to 0, H 4 equal to 0 so on. This implies H n equal to 0 for n greater than or equal to 2. Now, you can ask the question; what is e raised to the power of t H? For this H that is I plus t H plus t square H square by 2 factorial equals I plus t H remember, all these terms are 0. So, this entire thing will be 0. So, I plus t H which is basically 1, 1, 0, 0 plus t times 0, minus 1, 0, 0. So, this is simply 1 minus t that is your e raise to the power of t H, where H is equal to the matrix 0, minus 1, 0, 0 that is your matrix H.

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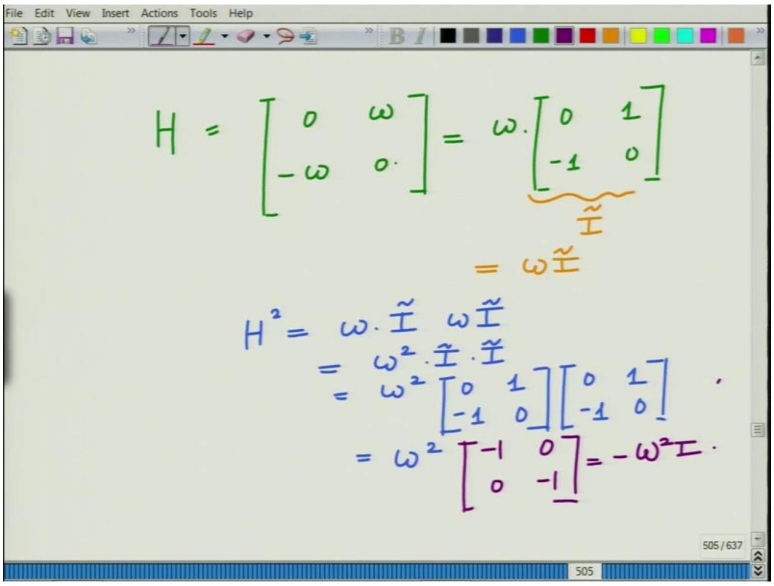

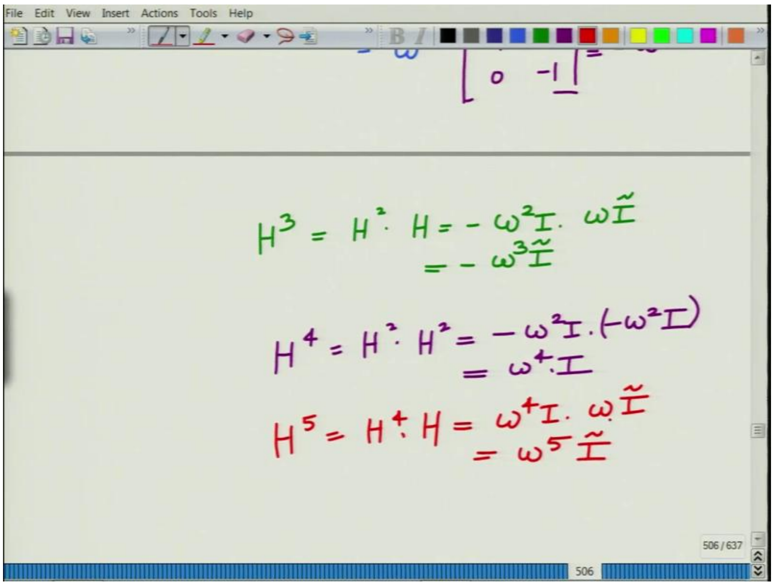

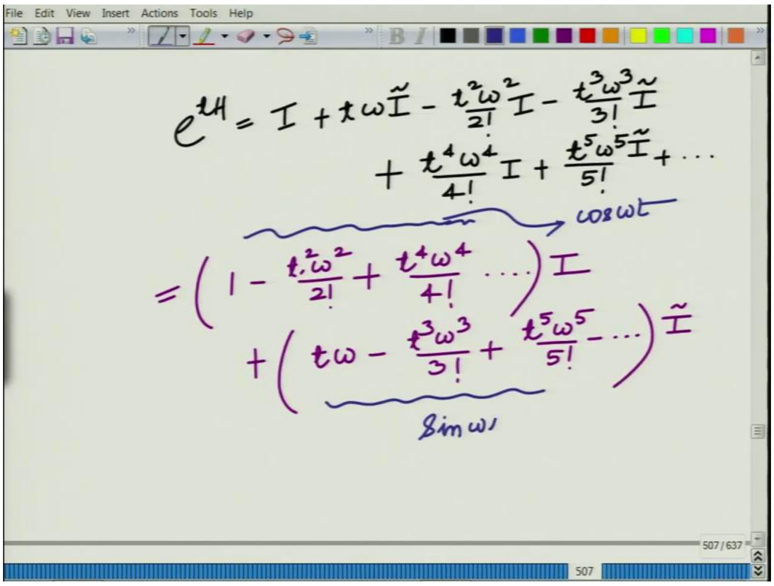

Let us now look at another last example, to compute understand how this matrix exponential is computed, that is your H equal to let us say 0 omega minus omega 0. I can write this as omega times or I can write this omega into the matrix 0, 1, minus 1, 0 for lack of anything better I will call this simply as I Tilda.

So, this is omega times I Tilda. Now, look at this H square will be equal to omega times I Tilda into omega times I Tilda equal to omega squared times I Tilda into I Tilda which is omega square 0, 1, minus 1, 0 times 0, 1, minus 1, 0 which is omega square times minus 1, 0, 0, minus 1 which is minus omega square times I.

And similarly H cube this will be equal to H squared times H. This is minus omega squared times I times omega into I Tilda. So, this will be minus omega cube times I Tilda. H to the power of 4 equals H square into H square. This is minus omega squared times I into minus omega squared times I. So, you get back omega raised to the power of 4 I, H to the power of 5 equals H 4 into H equals omega raised to the 4 I times omega I Tilda. This is omega raised to the five times I Tilda and so on and so forth.

I can compute any arbitrary H raise to the power of n and now, substituting this you get e raised to the power of t H equals I plus t H which is omega which is t omega I Tilda minus plus t square H square which is minus t square omega square by 2 factorial into I plus t cube H raised to the power of 3 which is minus t cube omega cube by 3 factorial into I Tilda plus t 4 omega 4 by 4 factorial into I plus t 5 omega raised to the power of 5 divided by 5 factorial into I Tilda and so on.

And now, if you divide this into two parts; one will be I or one will be 1 minus t square omega square by 2 factorial minus t cube omega 2 by 3, I am sorry, collecting all the terms with I plus t 4 omega 4 by 4 factorial times I plus t omega minus t cube omega cube by 3 factorial plus t 5 omega 5 by 5 factorial minus so on times I Tilda. Now, if you look at this, you can readily recognize this, this is nothing but your cosine omega t that is 1 minus t square because cosine x is 1 minus x squared by 2 factorial plus x 4 by 4 factorial, so on and so forth.

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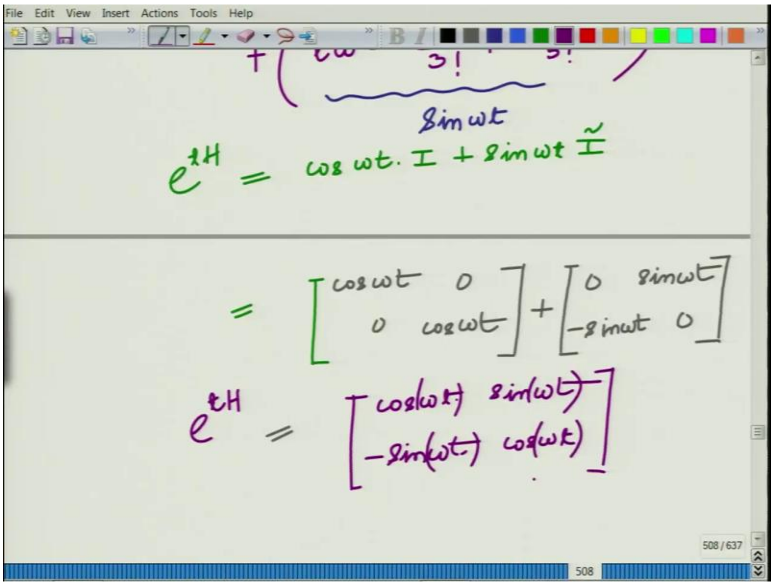

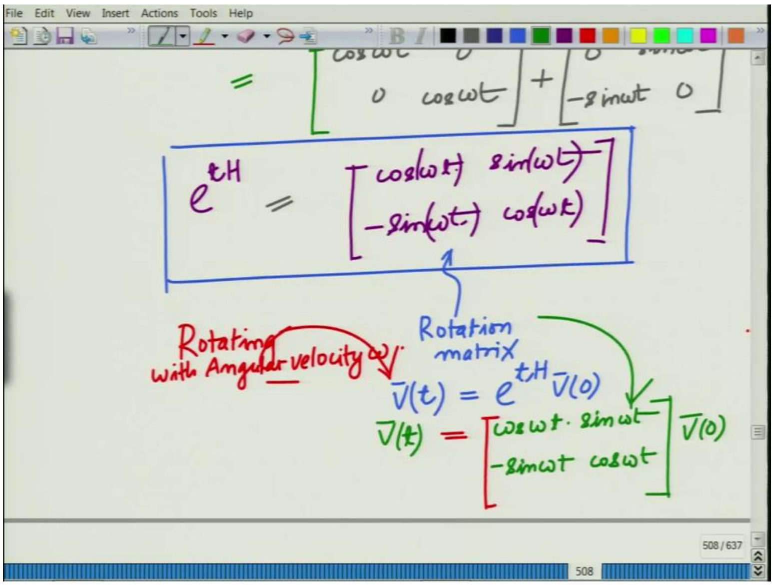

And this one this other term is nothing but sin omega t. So, this is essentially what you will get here is interestingly you will get cosine omega t plus times I times sin omega t into I Tilda which is essentially you get cosine omega t, cosine omega t plus 0, I think sin omega t, I am writing sin omega t times I Tilda, minus sin omega t is 0. So, essentially if you expand if you write this now, this will be cosine omega t, sin omega t, minus sin omega t, cosine omega t.

So, this is a very interesting expression for this and you can readily recognize this is nothing but the rotation matrix. So, this is nothing but rotation matrix which is rotating with angular velocity omega. So, what you will have is V bar dot or you will have V bar t equals e to the power of t H remember times V bar 0 and this is a rotation matrix which implies this is basically rotating.

If you look at this, what is happening to this rotating with velocity omega or essentially angular velocity omega? Let me just clarify this this is rotating with angular rotating with angular velocity. So, if you look at this what is this doing and ask the question what is this doing? This is rotating with angular velocity angular velocity omega that is essentially what this is doing.

So, this is essentially going to be your cos omega t, sin omega t, minus sin omega t cos omega t times V bar 0 and this we are what is what we are seeing is a rotation matrix. So, V bar t is basically a rotated version of V bar 0. So, V bar 0 is continuously rotating to obtain V bar t and the angular velocity is basically omega and you can also see this is essentially going to be periodic because you have cosine omega t, sin omega t and so on and the period.

So, this is going to be a, essentially this is going to be periodic this thing and so, this is basically the output is a rotated version of the V bar at 0. So, that is a very interesting application of the analysis that is a matrix exponential. So, we have seen a very interesting concept.

This is which is known as a matrix exponential and the solution to an autonomous linear dynamical (line-up) linear dynamic system which arises very frequently for instance, we have seen a call a simple application in the context of circuits many other systems in which it arises very frequently and solution can be described very interestingly in a very compact fashion using the matrix explanation. So, let us stop here and analyse further properties of the matrix explanation in the subsequent lectures. Thank you very much.