Fuzzy Sets, Logic and Systems and Applications
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Lecture – 37
Properties of Composition of Fuzzy Relations

So, welcome to lecture number 37 of Fuzzy Sets, Logic and Systems and Applications. In this lecture, we will discuss Properties of Composition of Fuzzy Relations.

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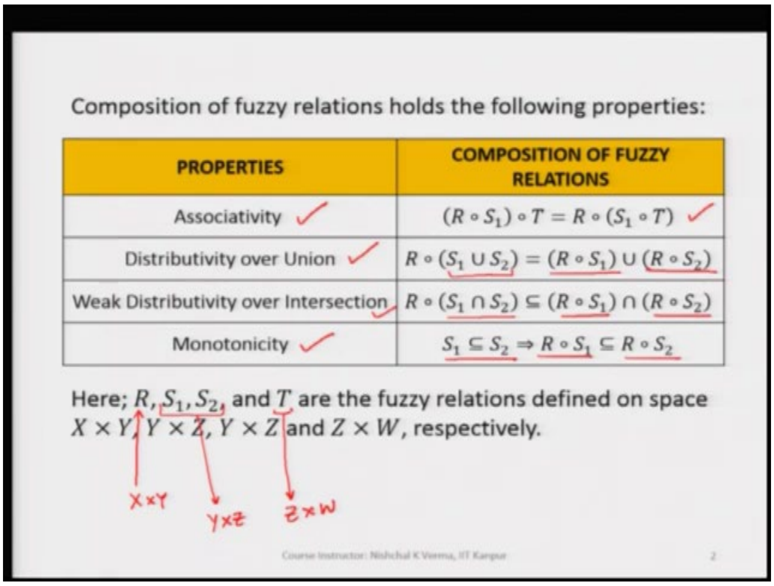

As we already know that the composition of fuzzy relations are based on the either max-min criteria or max-product criteria. And here I would like to mention that the fuzzy relation R fuzzy relation S_1, fuzzy relation S_2 and fuzzy relation T are the fuzzy relations that are defined on the spaces. That means, their universe of discourses $X \times Y$, $Y \times Z$, $Y \times Z$ and $Z \times W$ respectively.

So, this means that relation R which is here; this relation R is defined in the universe of $X \times Y$. Similarly, S_1 and S_2 are defined in the universe of discourses $Y \times Z$, T is defined in the universe of discourse $Z \times W$. So, as I have already mentioned that R, S_1, S_2 and T are fuzzy relations.

And we have already discussed the composition of fuzzy relations in the previous lecture. So, here we will be discussing as to how these composition of fuzzy relations hold these properties. So, first property is the associativity, then the second property is distributivity over union and then we have weak distributivity over intersection. And finally, we have monotonicity. And all these properties are defined here in this column where we have the composition of R and S_1.

So, when we say composition; this composition can be either max-min or max-product. So, the composition of R and S_1 and then whatever we get here out of this $(R \circ S_1) \circ T$. We are going to get $R \circ (S_1 \circ T)$. So, these things must be understood very clearly that the associativity with respect to the composition is satisfied or in other words we can say that the composition of fuzzy relations R, S_1 and T are holding good with respect to the composition.

Similarly, distributivity over union. So, we see here that we have a composition here and please understand this small o is for the composition, this is a symbol for composition. So, we have a $\circ$ so, R is small o and then we have the composition of $R \circ (S_1 \cup S_2)$. So, this can also be written as or in other words we can say $R \circ (S_1 \cup S_2) = (R \circ S_1) \cup (R \circ S_2)$.

Similarly the other property, the third one, the third property is weak distributivity over intersection. So, here we have the $R \circ (S_1 \cap S_2)$. $R \circ (S_1 \cap S_2) \subseteq (R \circ S_1) \cap (R \circ S_2)$. Similarly, here we have the fourth property that is a monotonicity and here if we take S_1 and S_2 fuzzy relations and S_2 is the subset of S_2.

So, then in that case the $R \circ S_1 \subseteq R \circ S_2$. So, this is called the monotonicity here. And as I have already mentioned that this small o represents the composition symbol and this composition can either be the max-min or max-product. So, let us go through these properties one by one.

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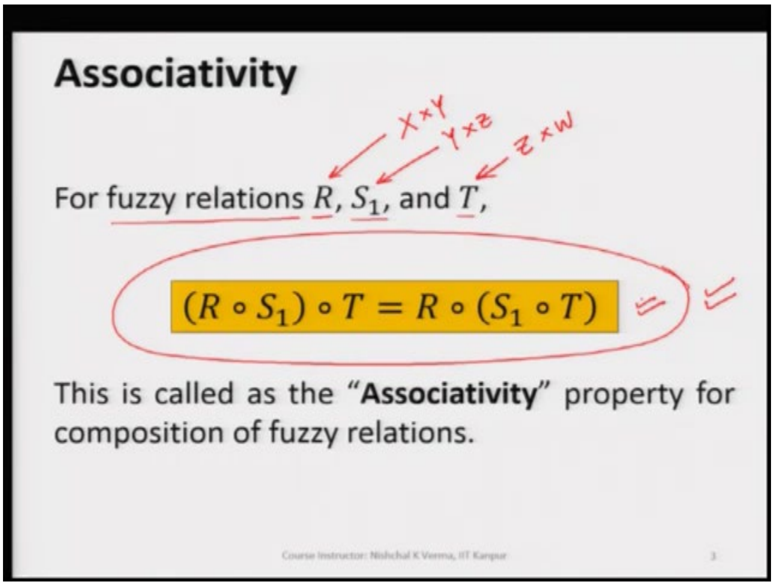

And as I have already mentioned here that for fuzzy relations R, S_1 and T this relation this associativity relation this holds good. And I have already mentioned about R also, R is defined in the universe of discourse $X \times Y$. Similarly, S_1 is defined in the universe of discourse $Y \times Z$ and T here is defined in the universe of discourse $Z \times W$.

So, this needs to be understood very clearly and this is called the associativity property for composition of fuzzy relations. So, this means that we have $(R \circ S_1) \circ T = R \circ (S_1 \circ T)$. So, this is what this means and this is called the associativity property.

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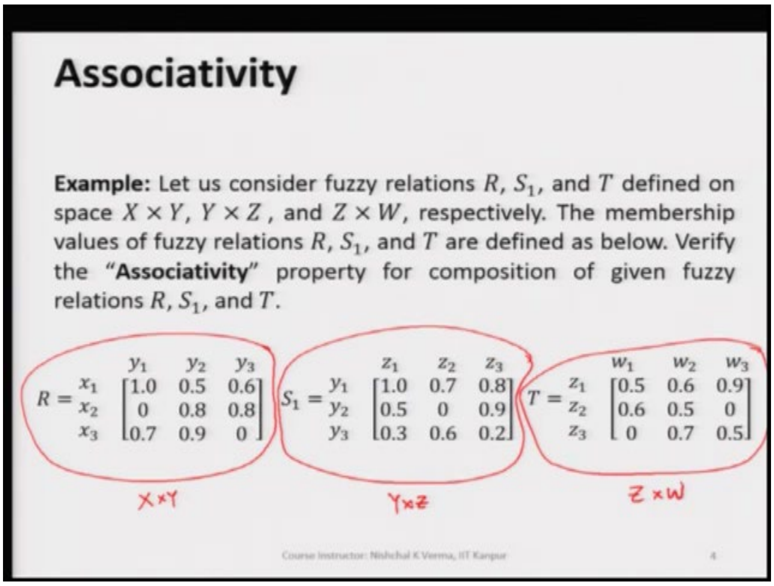

So, let us take an example to understand this better with respect to the fuzzy relations. So, let us understand this associativity for composition of fuzzy relations. So, here we have taken the fuzzy relation R first. We see here this is R fuzzy relation and then we have another fuzzy relation here as S_1 and then we have the third one is the fuzzy relation T. And we clearly see that R is defined in the universe of discourse $X \times Y$. Similarly, S_1 is defined in the universe of discourse $Y \times Z$ and here T is defined in the universe of discourse $Z \times W$.

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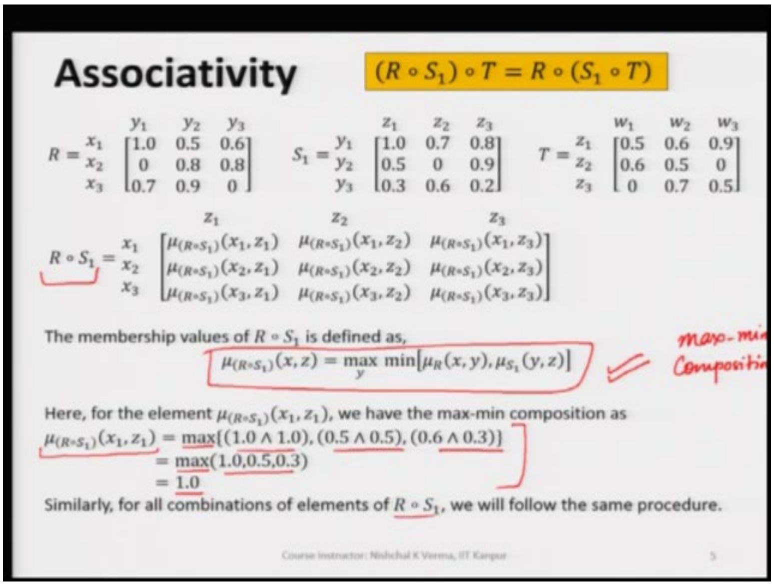

So, let us now compose R and S_1 first, we already have done this exercise in the previous lecture. So, based on that we'll try to generate the composition matrix. So, we have the $R \circ S_1$ which is this. And this is going to be equal to all these elements, means we have 3×3 elements here in this composition of fuzzy relation matrix.

And here, please understand that when we compose a fuzzy relations. So, all these exercises must be in conversant with the order of the matrix. This means that we need to have the suitable order of the fuzzy relation matrices that is needless to say. So, here now as I have already mentioned that we have followed this criteria for max-min composition, max-min composition.

Here we could also follow max product if it is set. So, but in this case we will only be interested in max-min. So, we have followed max-min criteria and based on this when we apply this criteria, we get $\mu_{R \circ S_1}(x_1, z_1)$ like this. We have already done this exercise in the previous lecture. So, I am not going to explain this again. If you have any doubt, you can go through the previous lecture to understand this better.

So, the max of all the mins here and then when we take a max of these the outcome of the mins then we are going to get 1 here in this case. So, similarly all combinations of elements of $R \circ S_1$.

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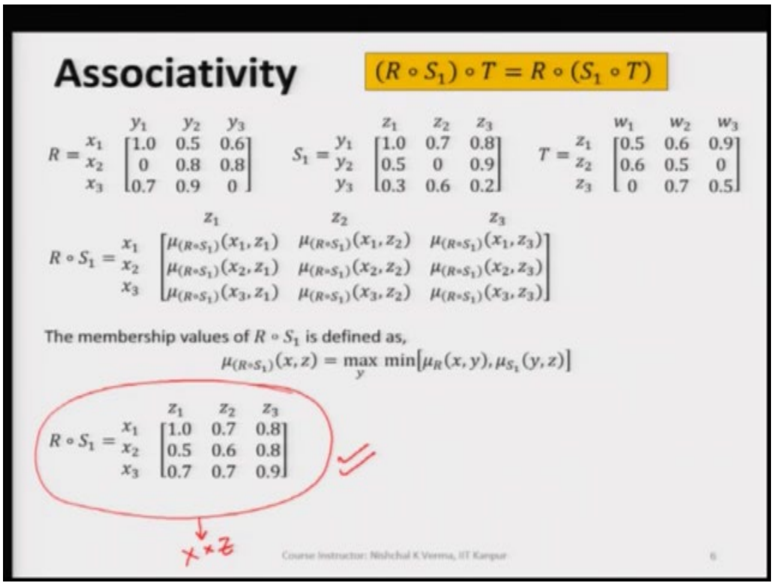

We are going to get here this matrix, $R \circ S_1$. So, you can just try this and you are going to get the composition of fuzzy relation matrix R and S_1 here, which is 1, 0.7, 0.8, 0.5, 0.6, 0.8, 0.7, 0.7, 0.9. And this outcome is again you can clearly see that is defined in the universe of discourse $X \times Z$.

So, we have now at this moment, the composition of fuzzy relation R and fuzzy relation S_1. Now, let us go further. So, this part is done here, so now let us go further and whatever is the outcome here is we compose with T.

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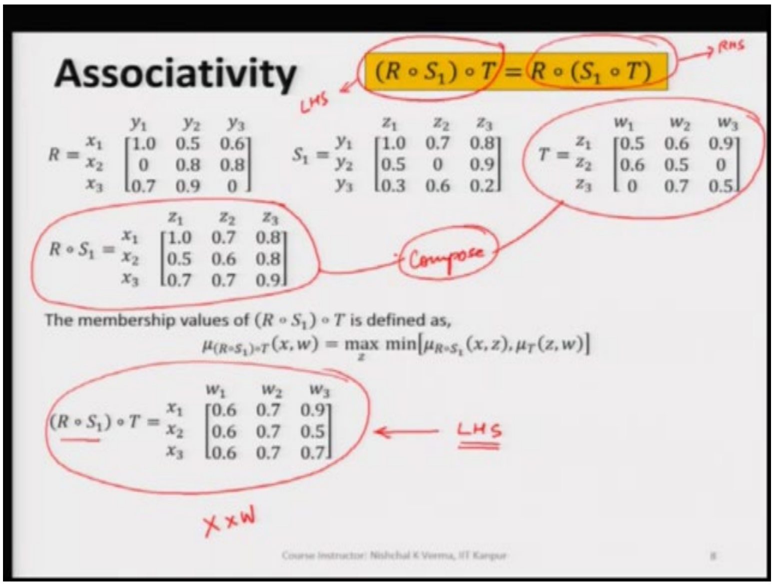

So, let us take T here and when we compose this T with $R \circ S_1$. So, means we take first $(R \circ S_1) \circ T$. So, both $R \circ S_1$ and T, both are fuzzy relations. Because, $R \circ S_1$ is going to give us another fuzzy relation. So, here we compose these two here.

And when we compose this what we are going to get upon applying the max-min criteria is this. So, $R \circ S_1$ and whatever is the outcome here is composed with T is going to give us a new fuzzy relation matrix which is defined in the universe of discourse $X \times W$. So, this is what is the outcome out of this exercise. So, this is the LHS, I can say this is the LHS because left hand side is coming like this. Now, in the property let us now try to find the RHS.

So, we see that this is the RHS part and this is the LHS part. So, we have done LHS, now let us find the RHS part using the same set of fuzzy relation matrices.

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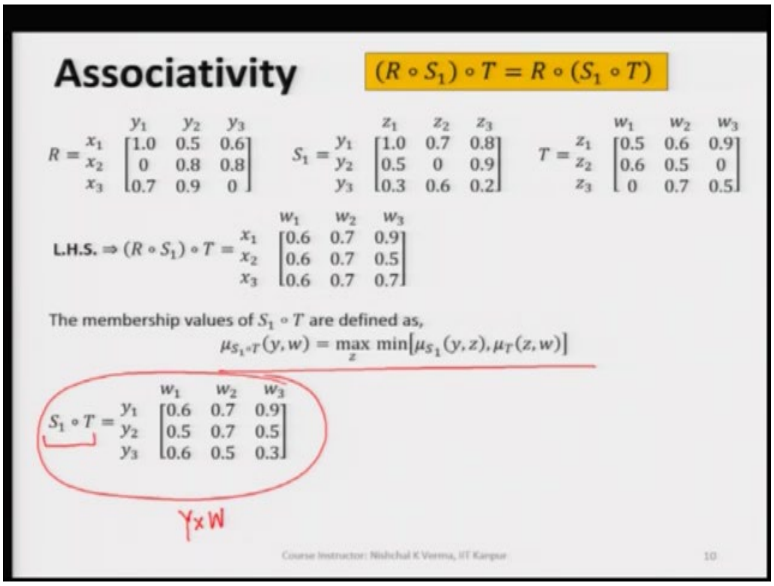

So, let us now find the $S_1 \circ T$. So, when we apply the max-min criteria as we see here with the same max-min criteria, if we compose S_1 and T which is here we get this matrix.

So, we get a new fuzzy relation matrix which is defined in the universe of discourse of $Y \times W$. Now, in the RHS, if we see we have another fuzzy relation R with which this outcome needs to be composed. So, when we compose, when we take the composition of $R \circ (S_1 \circ T)$. So, $S_1 \circ T$ we already have now, let us take R and make a composition of this and let us see what we are getting.

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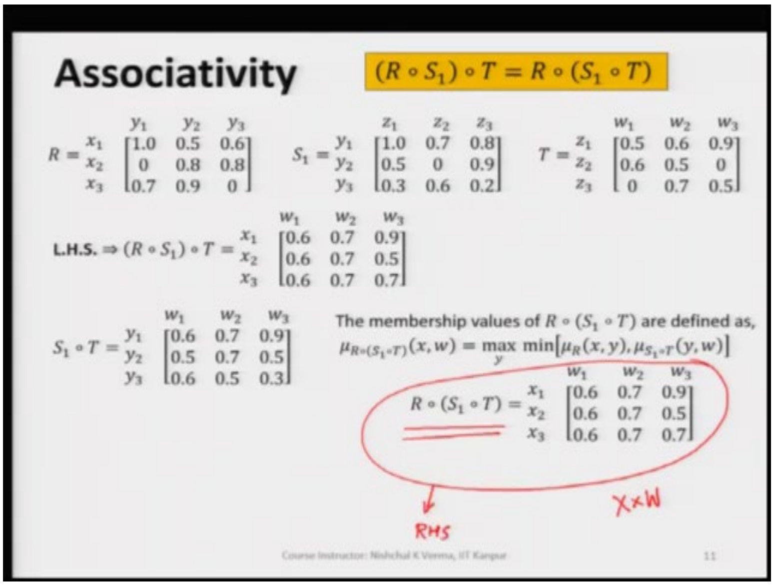

So, here when we do that we are getting this outcome. And we clearly see that we are getting a new fuzzy relation matrix out of these compositions which is defined in the universe of discourse $X \times W$. So, this is nothing, but the RHS part.

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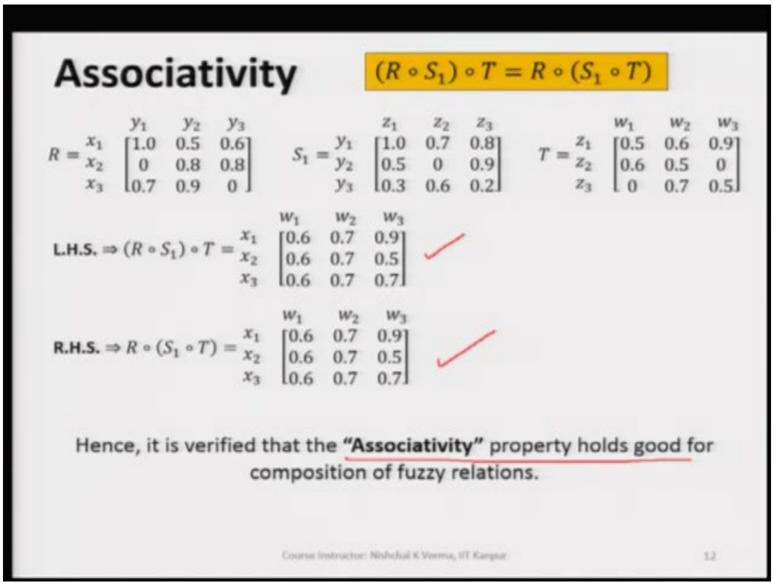

So, when we write LHS here and RHS here means both LHS and RHS here. We see that both the fuzzy relation matrices are same, are equal. So, we can say that the associativity property holds good for the composition of fuzzy relations and with this example we are able to understand that if we take any 3 fuzzy relations and then when we apply the associativity property.

So, with either the max-min or max-product, this property is well satisfied or in other words we can say that the associativity property for composition of fuzzy relations hold good.

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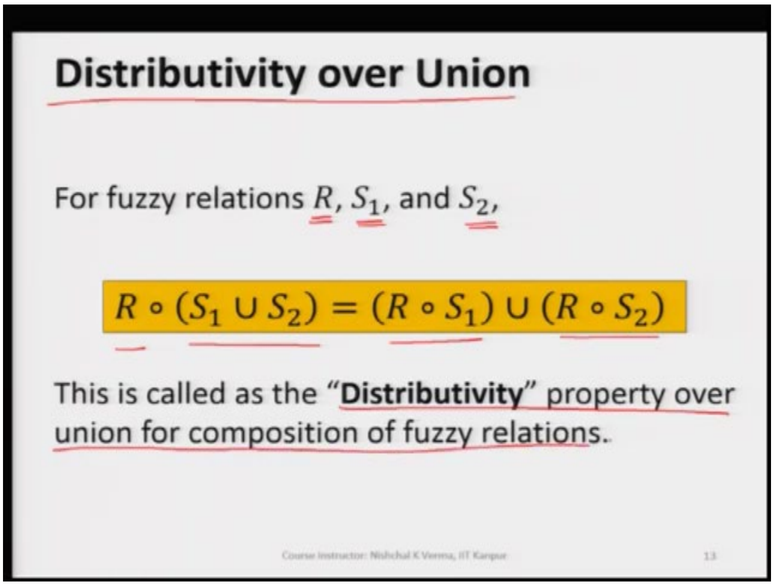

Now, the second property is the distributivity over union. And this is also for the composition of fuzzy relation. So, when we say composition of fuzzy relations. So, this distributivity property over union is also holding good. And what is this? This is nothing but if we have fuzzy relations R, S_1 and S_2. And this R, S_1, S_2 are defined in the universe of discourses as we have already discussed.

So, in that case this distributivity over union we have the $R \circ (S_1 \cup S_2) = (R \circ S_1) \cup (R \circ S_2)$. And if since these 2 are equal. So, we can say the distributivity property over union for composition of fuzzy relations hold good.

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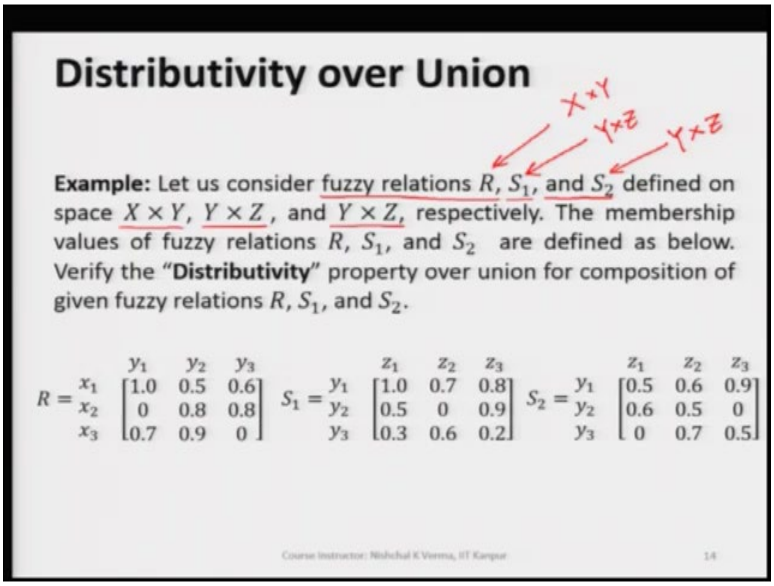

So this also let us understand properly by taking an example. So, here also we are taking fuzzy relations R, S_1 and S_2. Again, these are defined on spaces $X \times Y$, $Y \times Z$, $Y \times Z$ respectively. So, means R is fuzzy relation set is defined in the universe of discourse $X \times Y$. And S_1 is defined in the universe of discourse $Y \times Z$.

Similarly, S_2 is defined in the universe of discourse $Y \times Z$. So, this way we see that we have R, S_1 and S_2 as 3 fuzzy relation matrices.

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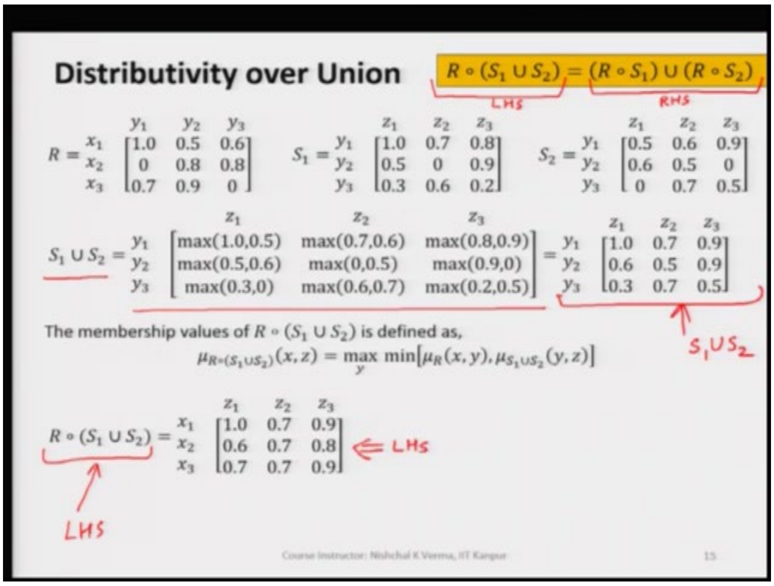

Now, let us try to see whether the distributivity over union property holds good for these fuzzy relations R, S_1 and S_2. So, in this process we have first the LHS and then we have RHS.

And let us move ahead, first take the LHS part. So, LHS is here this is our LHS. So, for computing LHS we first need to find the union of $S_1 \cup S_2$. So, we already know how to find the union of $S_1 \cup S_2$. So, when we take the union of S_1 and S_2, we simply take the max of the corresponding membership values. So, when we do this exercise we get here as union of S_1 and S_2.

Now, I am writing just this is the matrix which is $S_1 \cup S_2$. So, when we take the composition of R and the union of S_1 and S_2 here, we get this as the outcome. So, this is nothing but the LHS part. So, this is my LHS part.

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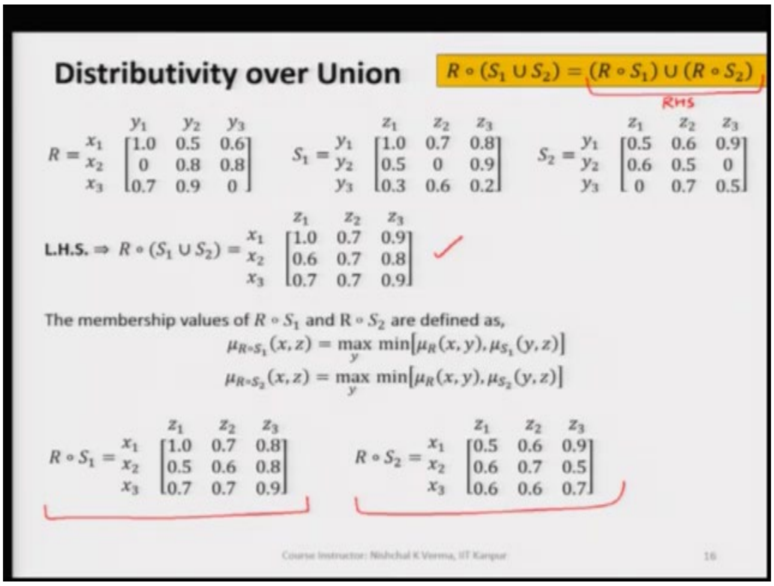

Now, let us move ahead and we see that in the right hand side, the RHS we have the union of two compositions. So, let us first find both the compositions, the first composition here is the composition of R and S_1 which is this. This by applying you know the max-min composition as I said before that it depends if you have been asked to find the $R \circ S_1$ by max-product, then you can use the max-product criteria. But here we are taking we are using the max-min criteria, so the outcome of $R \circ S_1$ is this and then $R \circ S_2$ is this.

Now let us take the union of these two. So, when we comes to the union, then we apply the max criteria again and we take max of the corresponding membership values of the both fuzzy relations matrices.

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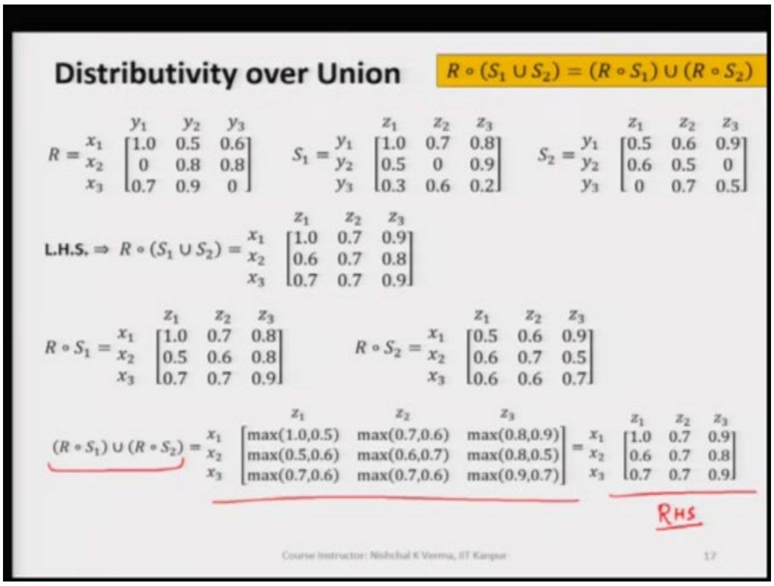

So, we see that here when we take the max criteria which is this, then the outcome is this. So, this is nothing but the RHS part. So this is the outcome of the $(R \circ S_1) \cup (R \circ S_2)$.

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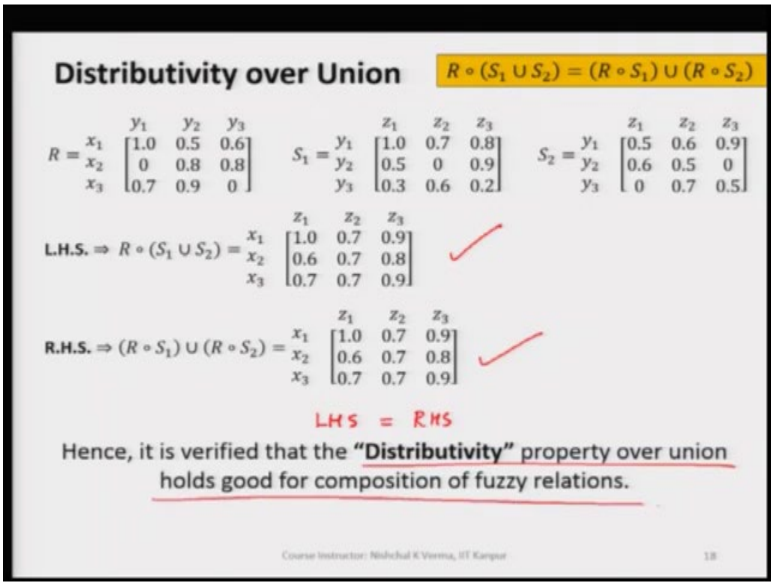

So then when we write the outcome of LHS and the outcome of RHS we see that both of these the fuzzy relation matrices that are the outcome of the $R \circ (S_1 \cup S_2)$ and the

$(R \circ S_1) \cup (R \circ S_2)$. So, both are same. So, in this way we can say the LHS is equal to RHS.

So, this means that the matrices that we took. The fuzzy relations sets that we took holds good for the distributivity over union property. So, in other words we can say that the distributivity property over union holds good for the composition of fuzzy relations.

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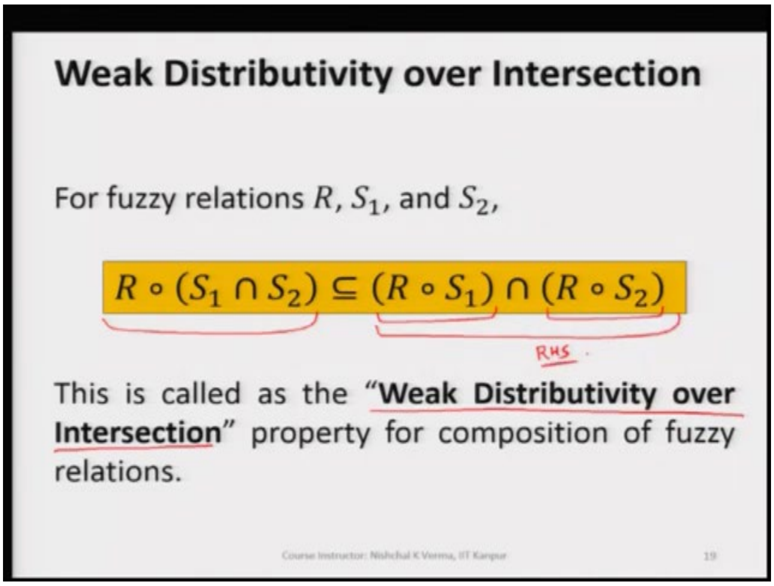

And now the next is the weak distributivity over intersection. So, when we have fuzzy relations R, S_1 and S_2 and if we have the $R \circ (S_1 \cap S_2)$. So if we have this thing, this is going to be the $(R \circ S_1) \cap (R \circ S_2)$. This is called the weak distribution over intersection.

So what essentially we are doing here is that we are taking the intersection of the 2 composition in the right hand side. So, we first have the composition of 2 fuzzy relations, means the $R \circ S_1$ and $R \circ S_2$ we are taking the intersection of it here and we have then the $R \circ (S_1 \cap S_2)$. So, if we have this $R \circ (S_1 \cap S_2)$, this is going to be the subset of the outcome of the RHS.

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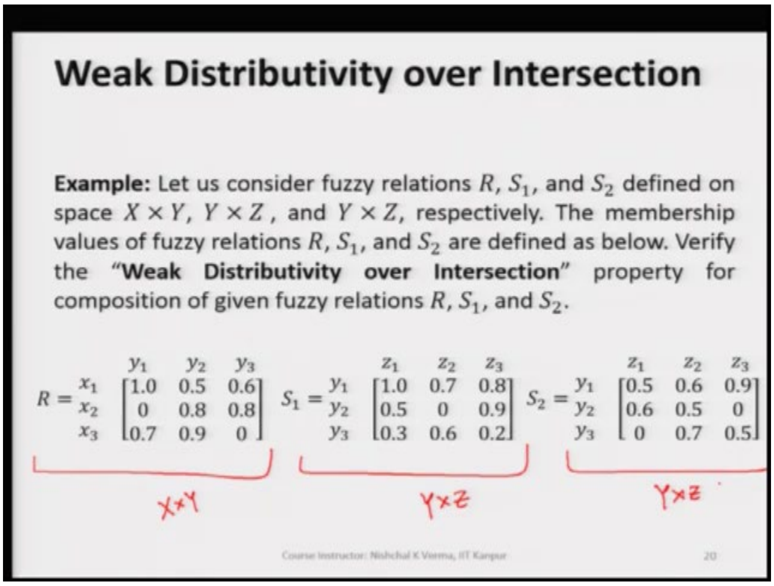

So, let us take an example again to understand the week distributivity over intersection again. So, here we have 3 fuzzy relation matrices R, S_1 and S_2 and again I would like to mention here that R is defined in the universe of discourse, $X \times Y$ and S_1 is defined in the universe of discourse $Y \times Z$ and S_2 is also defined in the universe of discourse $Y \times Z$.

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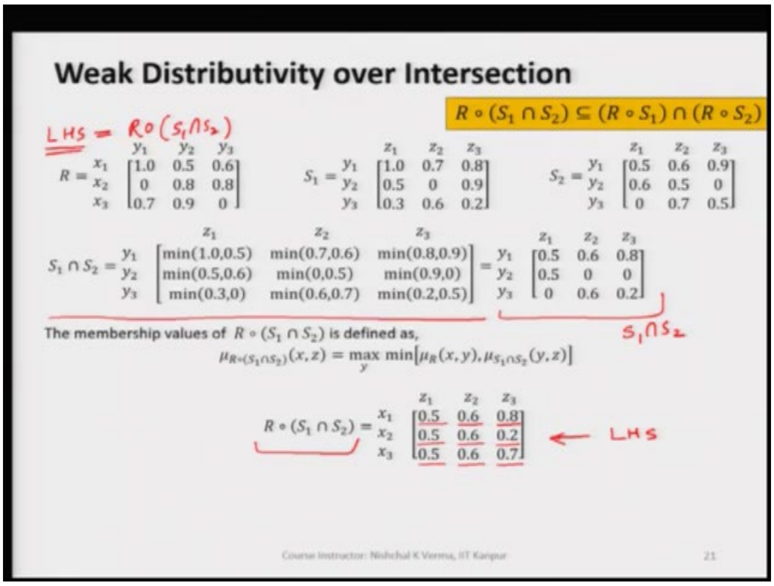

So, let us now first take LHS here and for LHS we have to have the $R \circ (S_1 \cap S_2)$. So, let us first find this. So, let us first find the $S_1 \cap S_2$ and when we take intersection we know that we use min criteria we take min of the corresponding membership values from S_1 and

S_2. So, when we do that we are getting here this as the outcome. This is nothing but, the $S_1 \cap S_2$.

So, when we have this fuzzy relation matrix as $S_1 \cap S_2$. Now, let us take the max-min composition of R and this matrix which is here. So, when we do that. So, since we are using max-min criteria, we are getting the final outcome as this where the elements are 0.5, 0.6, 0.8, 0.5, 0.6, 0.2, 0.5, 0.6, 0.7. So, this way we get the LHS computed for our example using R, S_1 and S_2. Now, when we have LHS computed.

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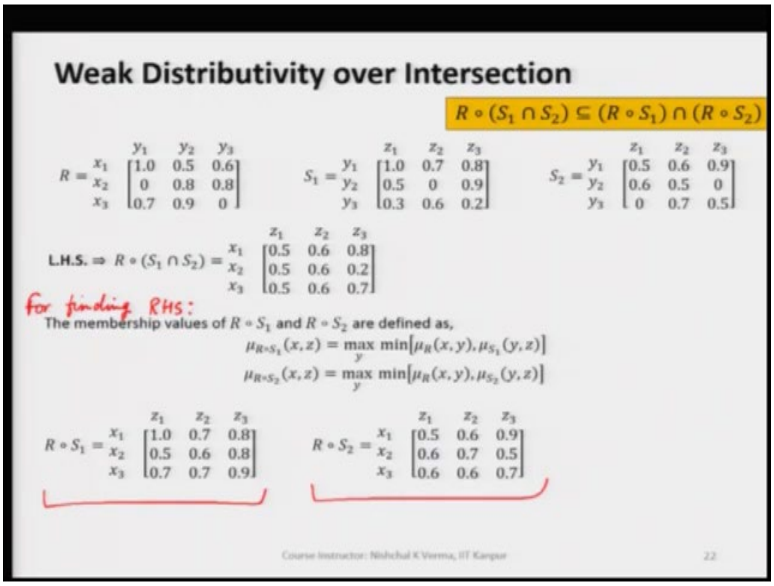

Let us now find the intersection of the $(R \circ S_1)$ and $(R \circ S_2)$. So, here we have the $(R \circ S_1)$ and $(R \circ S_2)$. So, this is in the process of computing the LHS for finding RHS.

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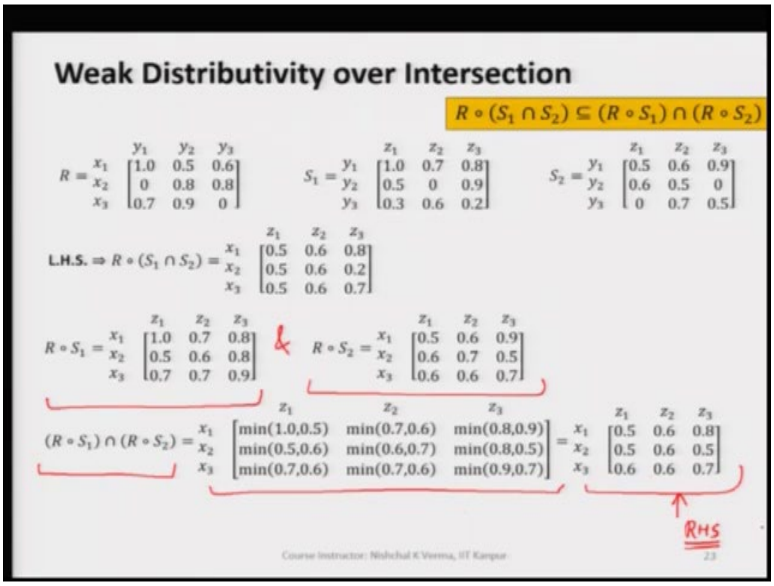

So, let's now move ahead and see what we are getting. So, we see that $(R \circ S_1)$ is this and this composition of course is again a max-min, here also the composition is max-min. So, we can quickly get the $(R \circ S_1)$ and $(R \circ S_2)$. So, when we do that then, let us now take the intersection of these two.

So, when we take the intersection of these 2 fuzzy relation matrices. We since we are taking intersection. So, we use the min criteria, the basic intersection. So, this can be found by taking the min of the respective membership values from both the fuzzy relation sets $R \circ S_1$ and $R \circ S_2$ and final outcome is here. And this is nothing but the RHS which is the intersection of both the fuzzy relation sets.

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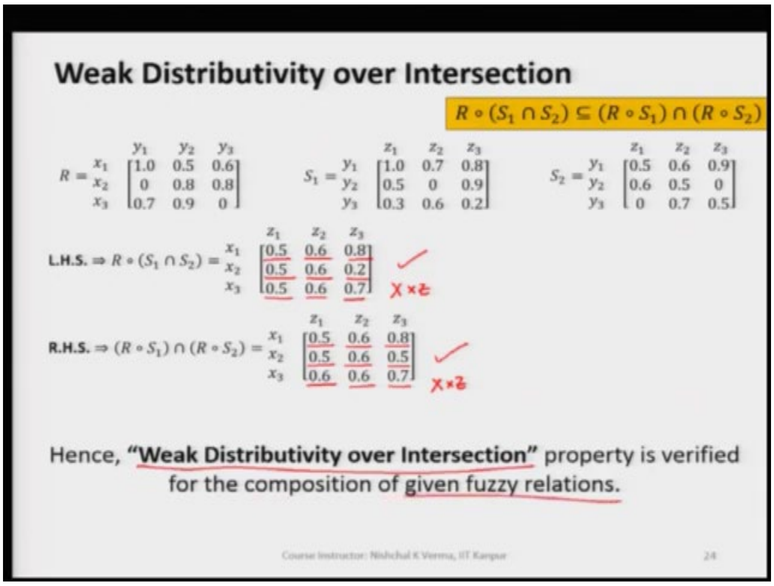

So, this way we have here the LHS and RHS. Now, if we see LHS and RHS both, we see that both of these fuzzy relation matrices are not same. We can clearly see that the LHS which is the $R \circ (S_1 \cap S_2)$ has the elements 0.5, 0.6, 0.8, 0.5, 0.6, 0.2, 0.5, 0.6, 0.7. And similarly, when we see the corresponding elements in the case of RHS where we have the $(R \circ S_1) \cap (R \circ S_2)$. We see that we have 0.5, 0.6, 0.8, 0.5, 0.6, 0.5, 0.6, 0.6, 0.7.

So, first of all let me make it clear here that, both of the fuzzy relation matrices of LHS and RHS they are defined in the same universe of discourses. So, here the universe of discourse is $X \times Z$ and here also the $X \times Z$. The universe of discourse is $X \times Z$. And when we see the corresponding elements in the fuzzy relation matrices. So, we see that the LHS the each elements are either equal or lesser than that of the elements in the RHS fuzzy relation matrix.

So that means, here the we can say that the $R \circ (S_1 \cap S_2) \subseteq (R \circ S_1) \cap (R \circ S_2)$. So, we see that each elements or every elements of LHS is either less or equal to the corresponding RHS fuzzy relation matrix. So, this way we can say that the fuzzy relation sets R, S_1, S_2 holds good here or we can say satisfy the weak distributivity over intersection criteria.

So, in general, we can say weak distributivity over intersection property is holding good for the composition of given fuzzy relations.

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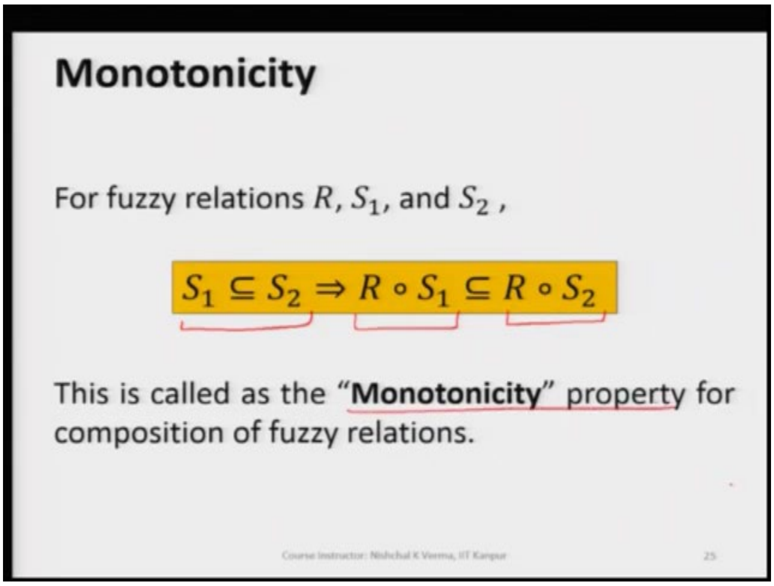

Now, let us discuss the monotonicity as the other property, the fourth property here for fuzzy relations R, S_1 and S_2. So, it is very simple in the sense that if we have S_1 and S_2, two fuzzy relations and $S_1 \subseteq S_2$. So if this is the case, then the $R \circ S_1 \subseteq R \circ S_2$. And this is called the monotonicity property. So, let us take an example similarly here and let us see how the monotonicity property also is holding good for fuzzy relations.

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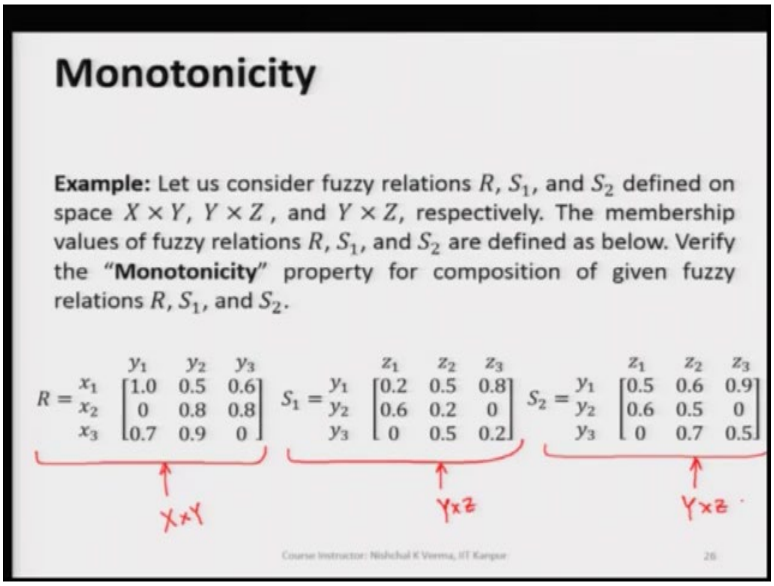

So, let us take the fuzzy relation set R, S_1 and S_2. And here also we see that the universe of discourse for R is $X \times Y$. For S_1, the universe of discourse is $Y \times Z$ and for S_2 also the universe of discourse is $Y \times Z$.

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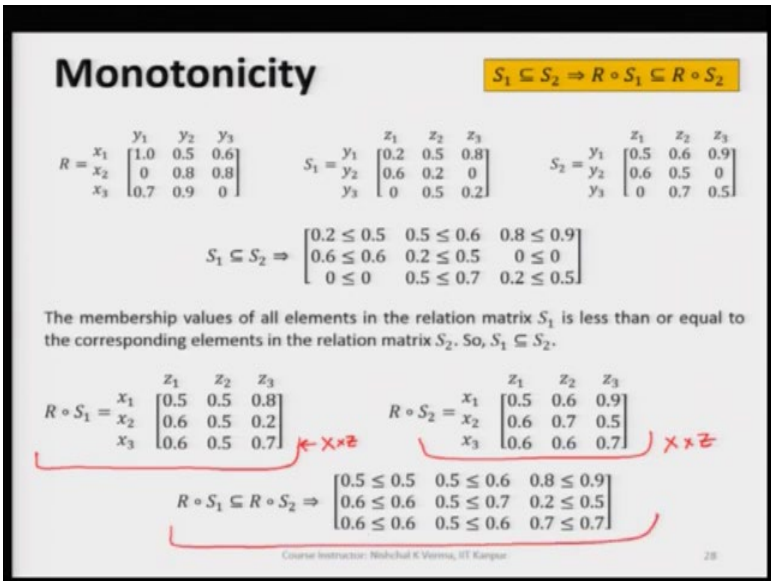

Now, let us choose S_1 and S_2 in such a way that our $S_1 \subset S_2$. So, if we choose S_1 and S_2 suitably, so that $S_1 \subseteq S_2$. We can check here that each and every element of the $S_1 \leq S_2$. So, when we choose the S_1, S_2 fuzzy relation set like this. Then let us now take the composition of S_1 and S_2 separately with R. So, here when we take the $R \circ S_1$, we find here a new fuzzy relation matrix which is defined in the universe of discourse capital $X \times Z$.

Similarly, when the fuzzy composition of R is taken with S_2 again we have the outcome and which is fuzzy relation in the universe of discourse $X \times Z$. Now, let us compare the $R \circ S_1$ and $R \circ S_2$. So, if we compare this with element wise we see that all the elements of $R \circ S_1$ or either less or equal to the $R \circ S_2$. Or in other words we can say that the elements of $R \circ S_1$ are either less or equal to the corresponding elements of the $R \circ S_2$ which is clearly shown here.

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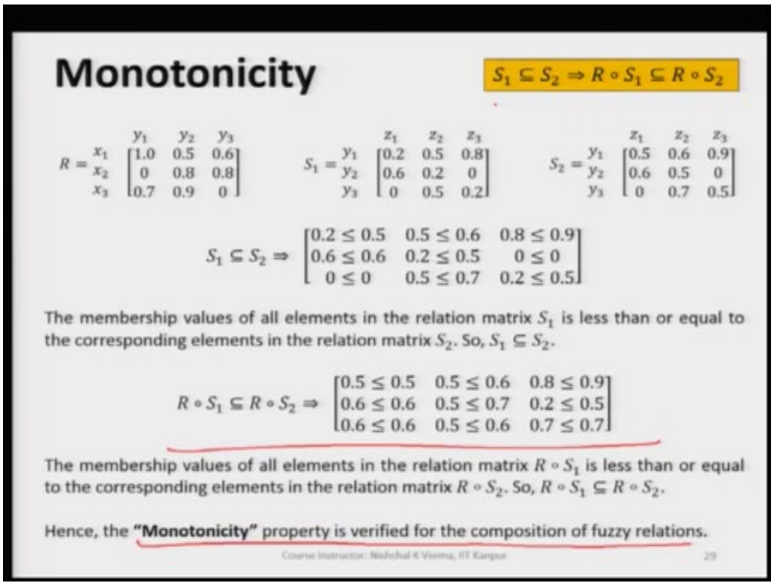

So, this way we see that when we see that all this criteria when we apply all the elements of $R \circ S_1 \leq R \circ S_2$, then we can say that the monotonicity property is verified for the composition of fuzzy relations.

In other words we can say the monotonicity property holds good for fuzzy relations and especially with this condition that when $S_1 \subset S_2$ and when we take a composition either max-min or max-product like the composition of R and S_1 and composition of R and S_2, then the $R \circ S_1 \subseteq R \circ S_2$.

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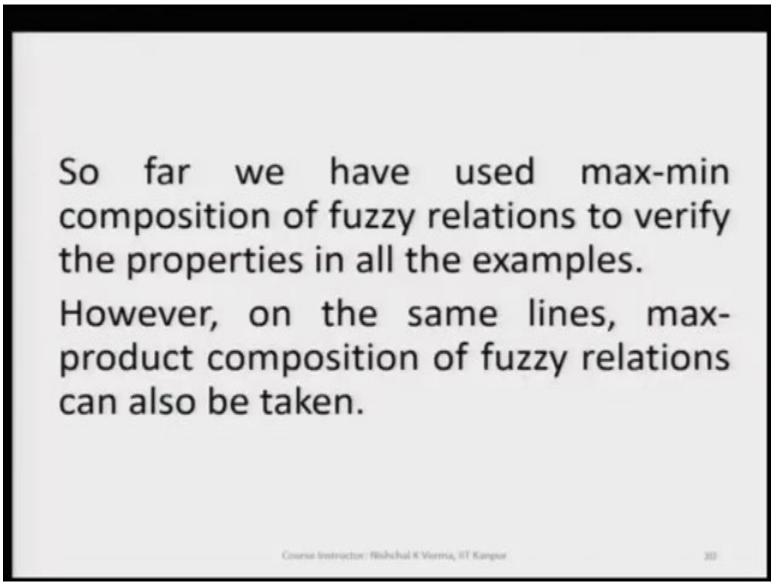

So, so far we used max-min compositions of fuzzy relations to verify the properties in all the examples. However, on the same lines max-product composition of fuzzy relations can also be taken.

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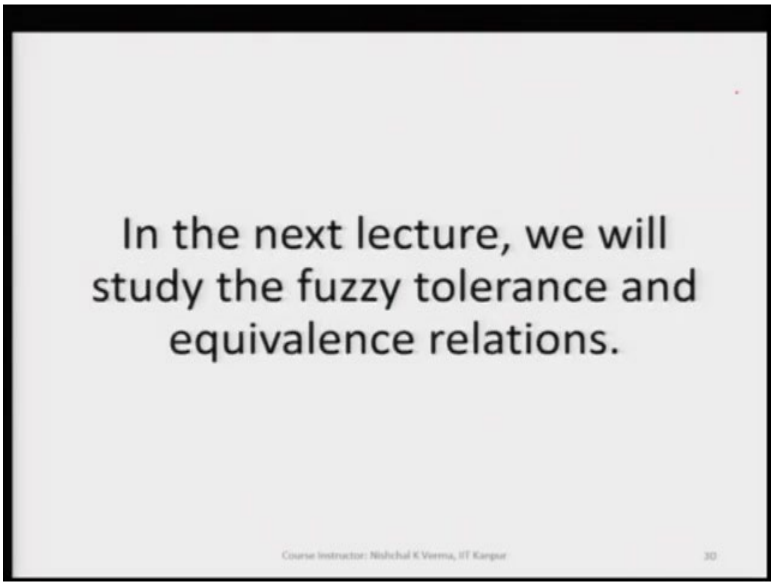

So, this way we have seen all the four properties with respect to the composition of fuzzy relations in today's class and now in the next lecture we will study the fuzzy tolerance and equivalence relations.

Thank you.