Fuzzy Sets, Logic and Systems and Applications
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Lecture – 36
Composition of Fuzzy Relations

So, welcome to lecture number 36 of Fuzzy Sets, Logic and Systems and Applications. In this lecture we will discuss the Composition of Fuzzy Relations.

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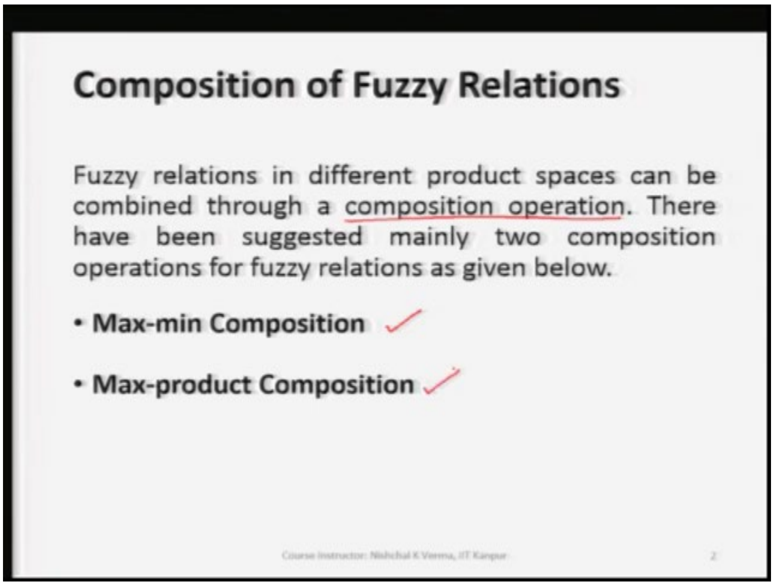

So, fuzzy relations basically in different product spaces can be combined through a composition operation. And there have been suggested mainly two composition strategies. So, what are these two composition operations or strategies for fuzzy relations? So, first one is the max-min composition and the second one is the max-product composition.

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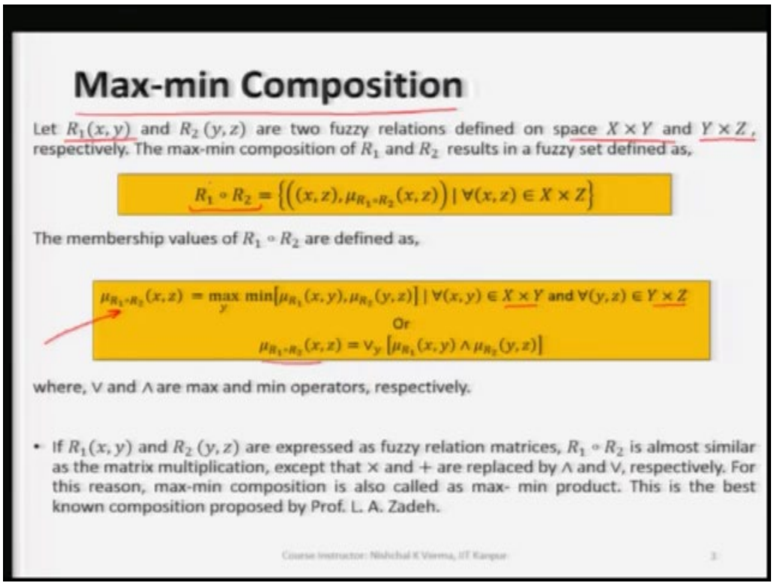

So, let us discuss this one by one. So, max-min composition here, let us have two fuzzy relation sets R_1 and R_2 where, $R_1(x, y)$ and $R_2(y, z)$.

So, you can see here that R_1 is in the universe of discourse X and Y and R_2 is the universe of discourse Y and Z. So, the max-min composition of R_1 and R_2 will result a new fuzzy set which is defined by the $R_1 \circ R_2$. What is this $\circ$? $\circ$ is the composition operation.

So, $R_1 \circ R_2$, so that means the composition operation on R_1 and R_2. So, this is basically this gives us a new fuzzy set which is defined by the elements x, z along with its membership value like $\mu_{R_1 \circ R_2}(x, z)$.

Such that for every x, z that is belonging into the universe of discourse X and Z, that means the universe of discourse becomes $X \times Z$. So, that membership values of $R_1 \circ R_2$ can be found by this expression. So,

$$\mu_{R_1 \circ R_2}(x, z) = \max_{y} \min[\mu_{R_1}(x, y), \mu_{R_2}(y, z)] \,|\forall (x, y) \in X \times Y \; and \; \forall (y, z) \in Y \times Z$$

So, this can also be written as,

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y \, [\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z)]$$

So, let us now understand this here that what actually is happening. So, what is happening over here is R_1 is the relation which is defined in the universe of discourse $X \times Y$, R_2 is a relation which is defined in the universe of discourse $Y \times Z$.

So, here we need to understand that R_1 and R_2 are two fuzzy relation sets which are defined in the different universe of discourses. So, from this if we are interested in finding the relation set in between R_1 and R_2 like this like a new fuzzy set which is defined in x and z only.

So, this max-min composition helps us. So, $R_1 \circ R_2$, that means, max-min composition if it is applied on R_1 and R_2 we are going to get the third fuzzy relation which is the outcome of R_1 and R_2 and if you see, the resulting fuzzy relation set will not have the dependency on y. So, that is what is interesting here.

So, if R_1 fuzzy relation set and R_2 fuzzy relation set are expressed as fuzzy relation matrices $R_1 \circ R_2$ which is almost similar as the matrix multiplication except multiplication and plus are replaced by the min and max respectively. So, for this reason max min composition is also called as the max min product. So, this is the best known composition proposed by Prof. Lotfi A Zadeh, who is father of fuzzy logic.

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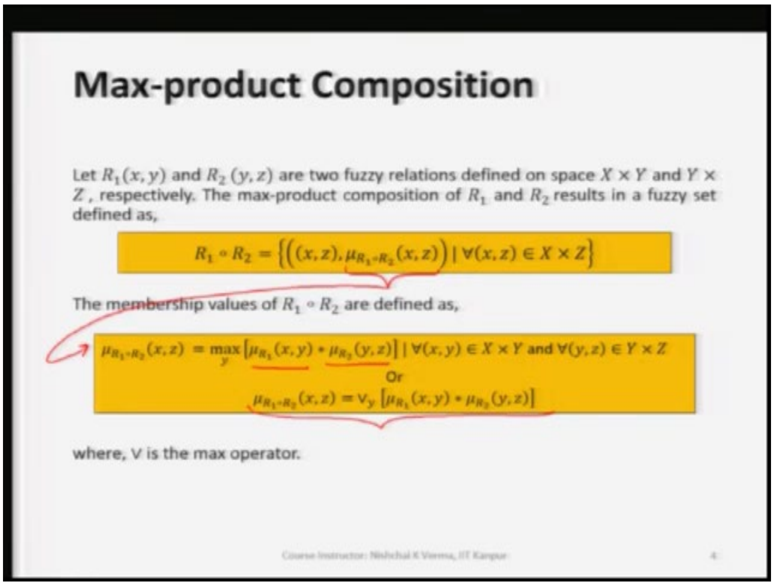

So, when we come to the max-product composition what happens is that the min is changed to product, that means instead of min we multiply.

So, if we take the same R_1 R_2 fuzzy relation sets and they are defined in different universe of discourses like $X \times Y$ and $Y \times Z$ respectively then $R_1 \circ R_2$ is equal to the (x, z), that means the element of the fuzzy relation set along with $\mu_{R_1 \circ R_2}(x, z)$ and this (x, z) is again belonging into the universe of discourse $X \times Z$.

Now, how to get this $\mu_{R_1 \circ R_2}(x, z)$ this? So, this can be computed by this expression. So,

$$\mu_{R_1 \circ R_2}(x, z) = \max_{y}\left[\mu_{R_1}(x, y) * \mu_{R_2}(y, z)\right] | \forall (x, y) \in X \times Y \text{ and } \forall (y, z) \in Y \times Z.$$

This can also be written by this expression here we can use the open inverted triangle for max. So, this can be written simply by this expression also.

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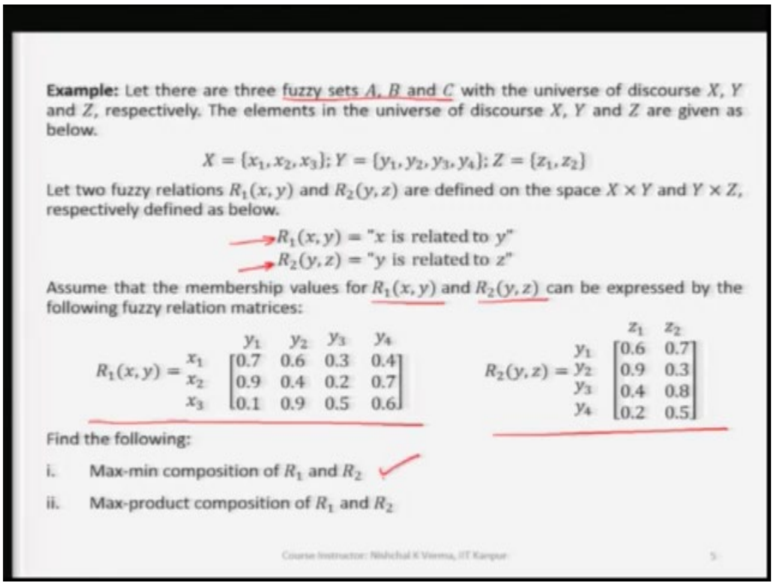

Let us take an example to understand this max-min composition. So, here we have three fuzzy sets and what we will be doing here, from these three fuzzy sets we will be developing the fuzzy relation sets these fuzzy sets A, B and C are defined in different universe of discourses. Like fuzzy set A is defined in the universe of discourse X, B is defined in the universe of discourse Y and C is defined in the universe of discourse Z.

So, if we develop a relation set here R_1 which is in between X and Y. So, we can say that R_1 is a fuzzy relation set which is basically is based on x which is related to y and similarly we have another fuzzy relation set which is R_2 and this R_2 fuzzy relation set in which the y is related to z.

So, let us assume some membership values for R_1 and similarly for R_2 and we write the fuzzy relation matrix R_1 as we have already seen in the previous lectures. So, we have R_1 is a fuzzy relation matrix, R_2 is another fuzzy relation matrix. So, R_1 is in terms of the generic variable x and y, R_2 is in terms of the generic variable y, z.

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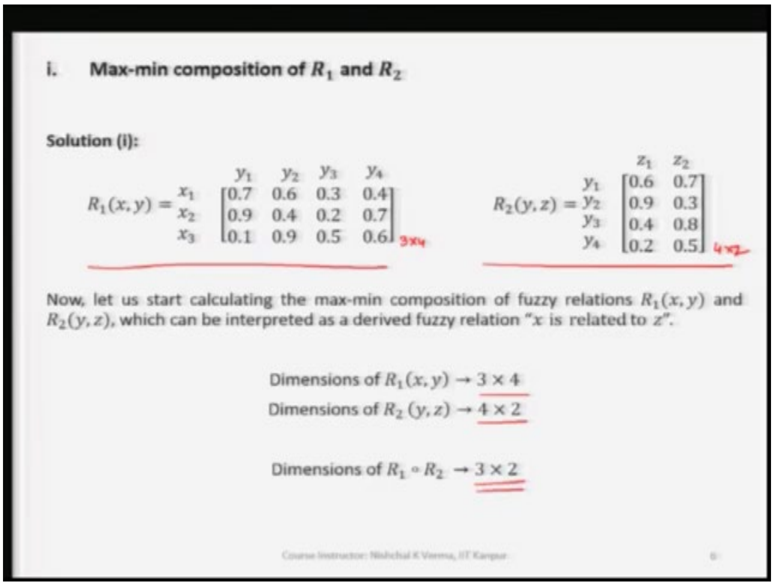

So, let us go for the max-min composition of R_1 and R_2 first and one more thing that is very important here is to note that the dimension of R_1 here is 3×4, the dimension of R_2 is 4×2. So, as we already know that there is a condition that we have to follow that the column of the first matrix should be equal to the row of the second matrix. So, otherwise we cannot multiply.

So, we see that we have R_1, 3×4, R_2, 4×2; so that means, we can multiply these two matrices. So, let us do that and the outcome of this multiplication will generate another fuzzy relation matrix and the order of this matrix will be 3×2.

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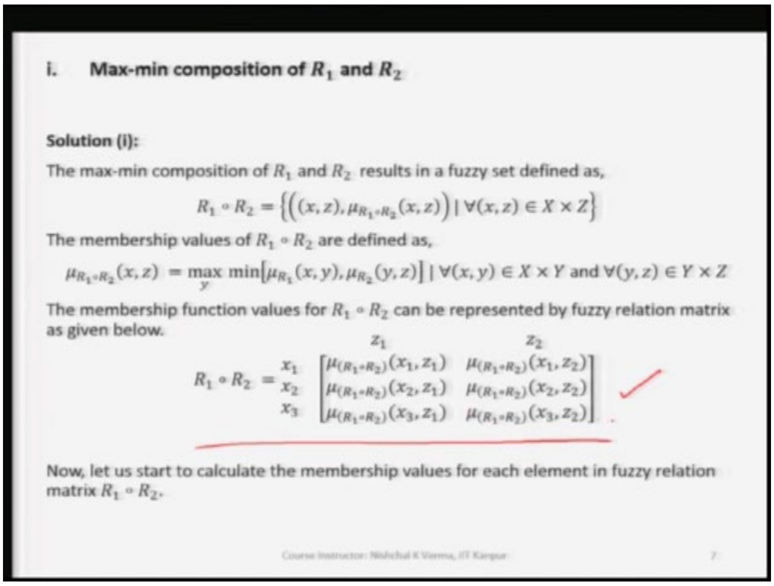

So, let us represent this R_1 composition R_2, and this R_1 composition R_2 will have the x and z generic variables and this can be written by this matrix here and we already know that this is going to be a 3×2 matrix because the R_1 is 3×4, R_2 is 4×2.

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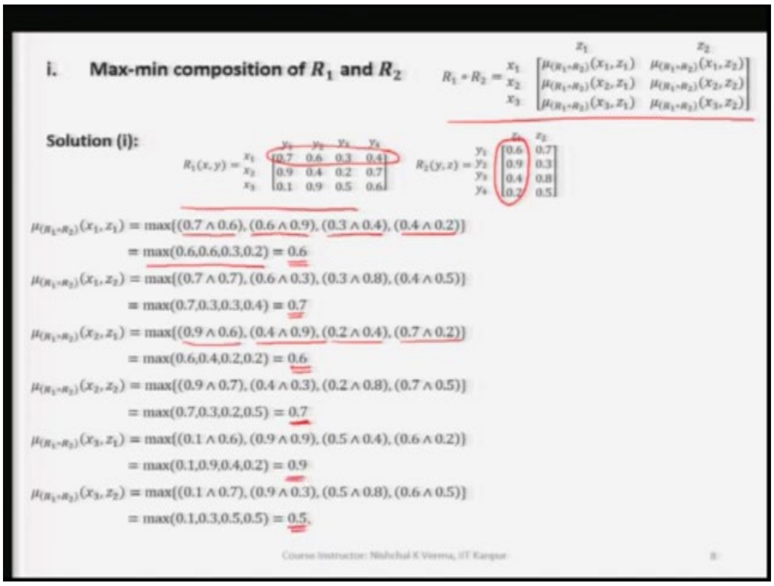

So, let us know multiply this. So, R_1 composition R_2 we have R_1 here, we have R_2 here and let us now multiply we take the first row of R_1 and the first column of R_2. So, when we take first row of R_1 here and then first column of R_2 like this.

So, like a normal matrix multiplication we will multiply this, but instead of using the multiplication sign we use min and instead of plus sign we use max. So, here 0.7×0.6. So, when we use multiplication, we simply take the product of 0.7 and 0.6. So, here we will not take the product rather than we will take min of it. So, we used the min sign here the open triangle and then again what we do here is that we again go to the second element of the first row of the R_1 and then second element of the first column of R_2.

We have 0.6 and 0.9, so instead of taking the product we will take min again. Similarly, here instead of taking product of 0.3 and 0.4 we take the min, similarly here the 0.4 and 0.2.

So, this way we see that we have multiplied the first row and first column of R_1 and R_2 respectively. Now, when we have taken the respective minimum values, then whatever values that we have now we take the max. So, when we are multiplying we were just adding, but instead of adding here we take the max. So, when we take max we are getting 0.6.

So, likewise here as we have taken the first row of R_1 and first column of R_2, now we will take first row of R_1 and second column of R_2. So, when we do that we find 0.7 after taking max-min and then similarly we pick the second row of R_1.

So, when we take second row of R_1 and first column of R_2. So, we see that here we have 0.9 min of 0.9, 0.6 and then min of 0.4, 0.9 and then we have min of 0.2, 0.4, then min of 0.7, 0.2. So, like that and then we have 4 values coming after taking mins. So, then we take max of these 4 values and then when we take max of these 4, we get 0.6. So, that is how we take the composition of the second row and first column.

Now, similarly the we use the second row of R_1 and second column of R_2 and when we do that we again get the max single value and then similarly we take the third row of R_1 and take first column of R_2 and then third row of R_1 and second column of R_2. So, this way we are getting all these values.

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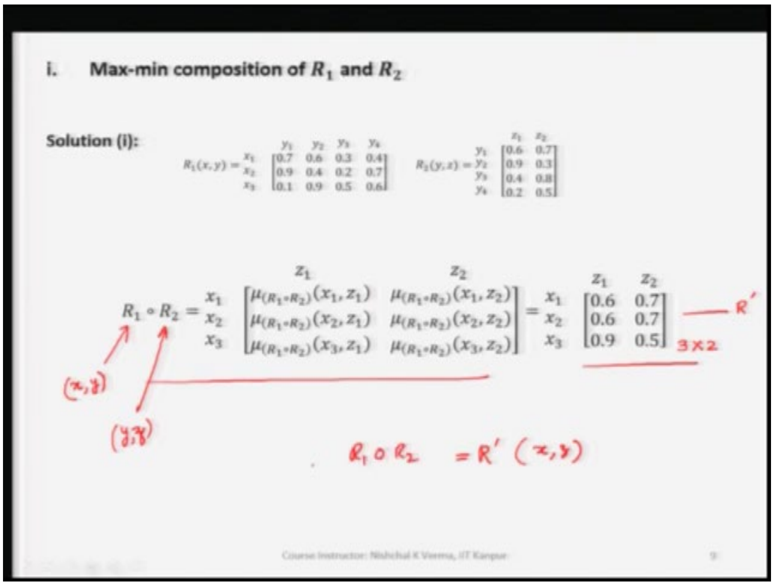

Now, when we have found all these max min values, then we let us arrange and when we arrange this we are getting a matrix as a result which is of 3 cross 2. So, this gives us the max-min composition of R_1 and R_2 and let us recall that this R_1 is defined in terms of x and y means the fuzzy relation set R_1 is in the space in the universe of discourse $X \times Y$ and here R_2 is in the universe of discourse $Y \times Z$.

But the max min composition is giving us a new fuzzy relation set which is let us say R dash. So, this R' is going to be in X, Z and this is nothing, but the max min composition of R_1 and R_2.

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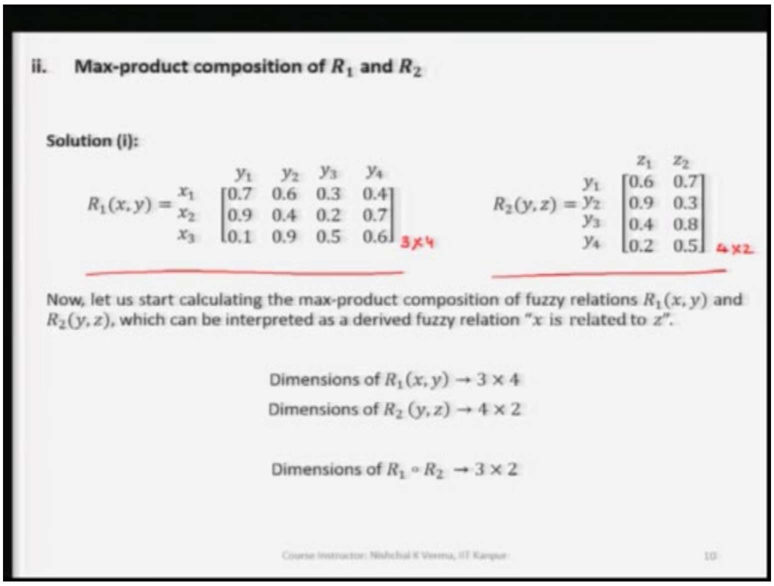

So, now, on the same lines we can get max-product composition here the difference is nothing, but only instead of taking min we take the product at rest other things remain the same.

So, if we take the same example here means R_1 fuzzy reason set we have here as it is mentioned as it is written here and then similarly the R_2 also and here we see that again we have 3×4 matrix and then we have 4×2 matrix. So, we see that we can multiply these two matrices and the resulting matrix will again be 3×2.

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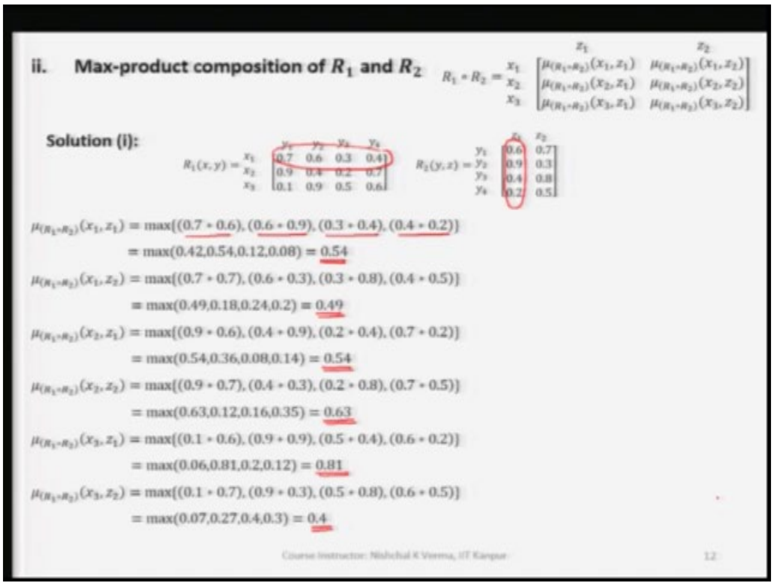

So, what we do here is we first take the first row of R_1 and then first column of R_2 and here we multiply we take the product of the corresponding elements like 0.7 and 0.6 we take the product of it. In earlier case where we discussed max-min we were taking only the minimum of 0.7 and 0.6, here we will be multiplying 0.7 and 0.6.

So, that is what is the difference in between the max-product and max-min. So, since we are dealing with max-product composition. So, here we will multiply 0.7 and 0.6, then we will multiply 0.6 and 0.9, we will multiply 0.3 and 0.4, we will multiply 0.4, 0.2.

Now, these multiplications will result 4 values and when we take max of it we are going to get 0.54. So, similarly this first row of R_1 and then second column of R_2 will result 0.49 and then the second row of R_1 and first column of R_2 will result 0.54, second row of R_1 and second column of R_2 will result 0.63.

The third row of R_1 and first column of R_2 will result 0.81 and finally, the third row of R 1 and second column of R_2 will give us 0.4. So, now, we have got all these values of the membership and these values that we have found these are the membership values which are with the resulting fuzzy relations set and this is resulting out of the max-product composition of R_1 and R_2.

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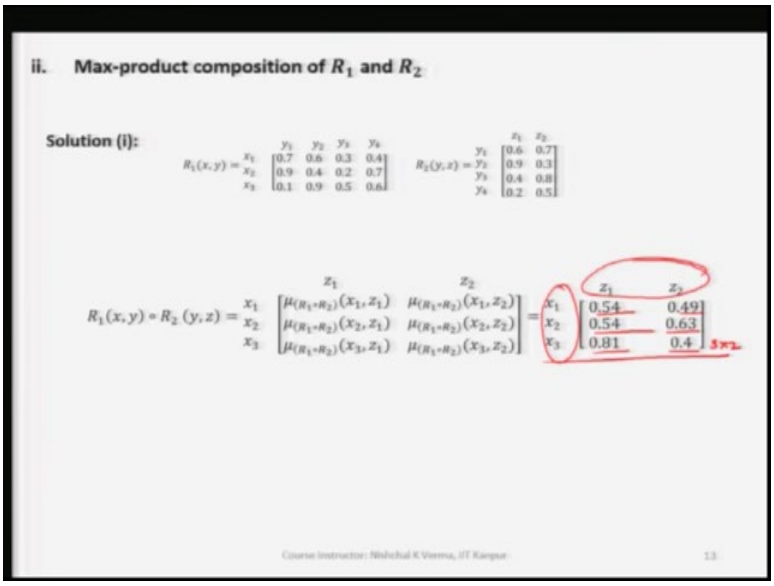

So, let us know substitute these values and we get the max-product composition of R_1 and R_2 here like this where we have the 0.54, 0.49, 0.54, 0.63, 0.81, 0.4 as the values of the resulting fuzzy set. And of course, as I already mentioned that we have the resulting fuzzy relation set the order of this matrix is 3×2.

And it is clearly visible here that the matrix that we have got here is out of the relation between x and z. So, this means that the max product composition of R_1 and R_2 is giving us a new fuzzy relation set and this new fuzzy relation set is defined in terms of x and z.

So; that means, here the new fuzzy relation set is defined in the universe of discourse $X \times Z$. So, this is very important point that has to be noted that if we have any fuzzy relation set which is in the universe of discourse of $X \times Y$ and another fuzzy relation set which is defined in another universe of discourse $Y \times Z$. Then by doing this exercise either max-product composition or max-min composition we can find the fuzzy relation set in the universe of discourse $X \times Z$. So, this is a very important point that has to be noted.

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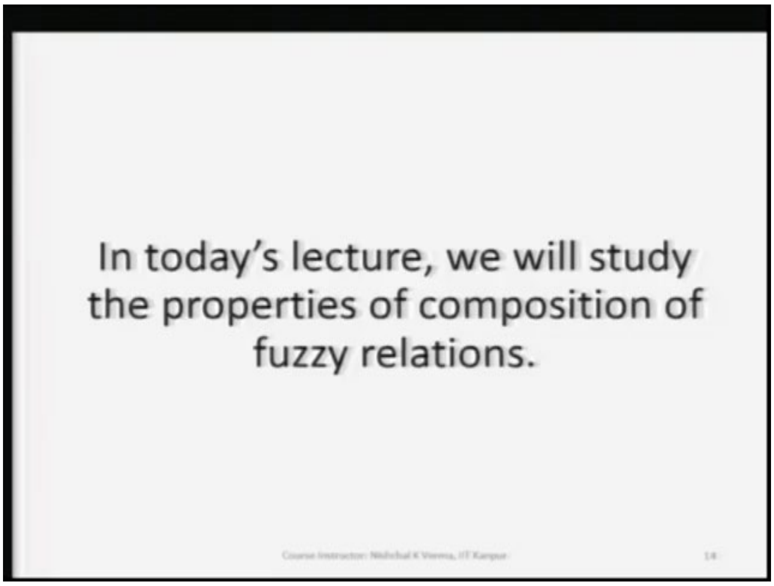

So, this way we have understood the max-min composition of R_1 and R_2 and max-product composition of R_1 and R_2 and we have seen this with a couple of examples and we have understood this very clearly. So, I will stop here for this lecture and then in the next lecture we will study the properties of composition of fuzzy relations.

Thank you.