Fuzzy Sets, Logic and Systems and Applications
Prof. Nishchal K. Verma
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 35
Extension Principle

So, welcome to lecture number 35 of Fuzzy Sets, Logic and Systems and Applications. In this lecture, we will discuss Extension Principle.

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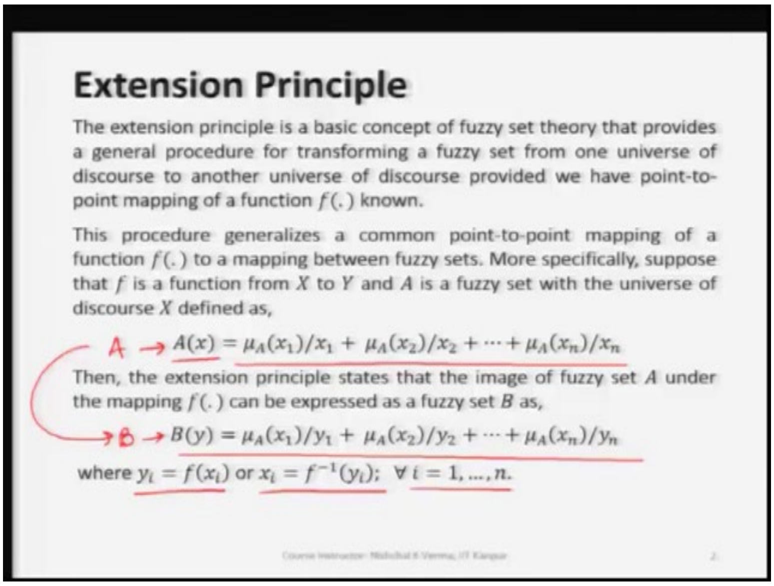

So, the extension principle is very interesting and with the help of the extension principle. We basically convert a transform of fuzzy set, which is defined in one universe of discourse into another which is defined in the other universe of discourse. So, the extension principle is a basic concept in the fuzzy set theory that provides a general procedure for transforming a fuzzy set from one universe of discourse to another universe of discourse provided the mapping function, that means the f which connects one universe of discourse to another universe of discourse is known.

And, this procedure generalizes a common point to point mapping of a function that is $f(.)$ to a mapping between fuzzy sets. More specifically f is a function from X universe of discourse, that means, the universe of discourse X to universe of discourse Y. And, if we take let us say to understand any fuzzy set which is defined in the universe of discourse X.

So, if we take $A(x)$ here. So, normally we write a fuzzy set by simply either A, B or C like that we never write normally $A(x)$, but here since we are putting an emphasis on the generic variable with which it is defined that means, we are indicating here that A is defined in the universe of discourse X. So, A here is $A(x)$. And, this is defined by this equation here, means this fuzzy set here where we have the membership value and then its corresponding generic variable value.

So, $\mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$. So, here if we see A, A is defined in the universe of discourse X with the generic variable x. So, if we are interested in transforming this fuzzy set into the universe of discourse capital Y; that means, if we transform this fuzzy set $A(x)$ into say $B(y)$ where y is the generic variable in the universe of discourse Y.

So, then how we can manage to do that is shown here. So, we see here that $A(x)$ is transform into B, where B is nothing but it is defined by the generic variable y in the universe of discourse Y. And, this is defined by this fuzzy set here $\mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \cdots + \mu_A(x_n)/y_n$. And then these y_i means $y_1, y_2, y_3 \ldots \ldots, y_n$ can be found by substituting x_i values that means, x_1, x_2, x_3, x_4 and so on in $f(x_i)$, that means we use the mapping function to manage to get the conversion of the generic variable let us say x into y.

So, this can also be written like this. So, $x_i = f^{-1}(y_i)$. For every i is equal to 1, 2, 3, 4, 5 and so on up to n. So, this way the extension principle helps us in transforming a fuzzy set which is in a particular universe of discourse into another fuzzy set which is in a different universe of discourse say Y. So, extension principle helps us in managing this conversion

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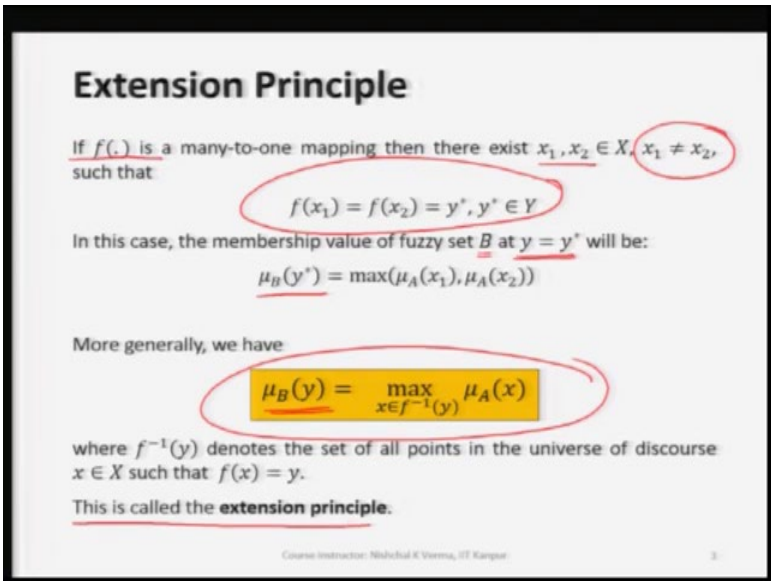

So, let us now understand this further in continuation to this if this $f(x)$ if this mapping function is which is here is many to one mapping then there exist x_1, x_2 that are belonging into X as the universe of discourse, where $x_1 \neq x_2$. So, in that case what will happen, $f(x_1) = f(x_2)$. So, this is a situation where let us say we have two values of generic variable x_1 and x_2. And then their corresponding $f(x_1)$ and $f(x_2)$ both are if equal. So, if both of these are equal means if $f(x_1) = f(x_2)$. And also when we say $f(x_1) = f(x_2)$.

So, it means it is having some values let us say that is y^*, and this y star is belonging into Y universe of discourse. So, what does this mean. This means that if we have any two generic variables if we have any two generic variable values x_1, x_2. And they are not equal to each other, but $f(x_1) = f(x_2)$ then in that case what we do is here. So, what we do here in this case is, the membership value of fuzzy set B at $y = y^*$, y^* is the value which is $f(x_1)$ or $f(x_2)$.

So, this y star will play an interesting role here. We see that how do we find corresponding to this y^*, $\mu_B(y^*)$ you see here that $\mu_B(y^*) = \max(\mu_A(x_1), \mu_A(x_2))$. So, this means that even when the $f(x_1) = f(x_2)$, but $x_1 \neq x_2$. So, in that case what happens $\mu_B(y^*) = \max(\mu_A(x_1), \mu_A(x_2))$. So, this can also be written as $\mu_B(y)$ which is here.

So, the general formula can be like this. So, $\mu_B(y) = \max_{x \in f^{-1}(y)} \mu_A(x)$, which is written here.

So, this is followed when we come across this situation this situation means when I am just repeating the situation where $x_1 \neq x_2$ please understand here $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

So, we will take an example ahead to make it more clear. So, that is how we are able to convert or transform the membership functions from one domain to another, from one generic variable to another, from one universe of discourse to another. So, here very clearly we have transformed A into B. A is defined in the universe of discourse X where B here which is transform fuzzy set is defined in the universe of discourse Y. So, this is called the extension principle.

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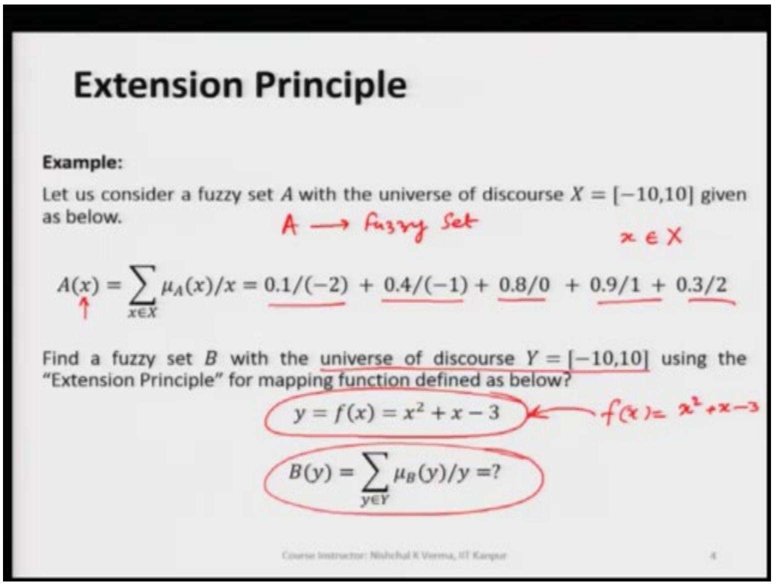

So, let us take again here other examples we are taking this example also to make you understand the extension principle better. So, we have taken an example here and in this example we have taken a fuzzy set A, A fuzzy set which is a discrete fuzzy set of course, you can see and this discrete fuzzy set A is defined in the universe of discourse X. So, that is why x is mentioned here x is the generic variable. And which is nothing, but belonging into the universe of discourse X.

Now, this is very simple this is very clear very easy to understand that we have a simple discrete fuzzy set A, which has five elements. Now, our job is to convert or transform this fuzzy set into $B(y)$. And here the new fuzzy set the transformed fuzzy set you see it is defined in the universe of discourse Y and this is possible when the $y = f(x)$ that is the mapping function when this is known this is possible. So, let us do that.

And here this problem will include the universe of discourse which is mentioned here that is from -10 to 10. So, let us now use the extension principle for mapping function and let us move ahead. Mapping function is given in this problem mapping function is $f(x)$ and $f(x) = x^2 + x - 3$ which is here. So, the transform fuzzy set that is $B(y)$, B of y will look like this of course, now we have to find its membership values corresponding to its generic variable values small y's.

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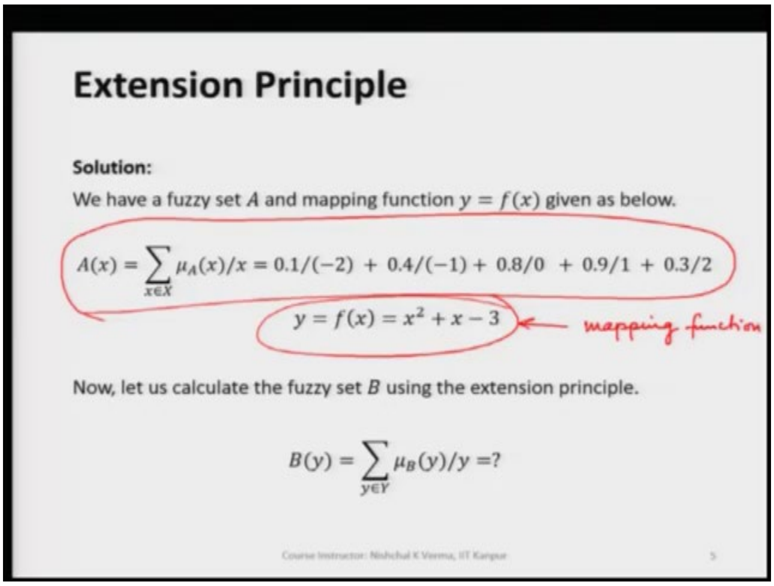

So, let us now quickly move ahead and let us see what we do to get this. So, we simply write here the fuzzy set A which is in the universe of discourse X. And then we write the mapping function this is the mapping function. So, it is very easy to convert $A(x)$ into $B(y)$ and we simply write $B(y)$ in terms of $\mu_B(y)/y$'s. And then we find the values of $\mu_B(y)$'s and y's means the membership values corresponding its generic variable values.

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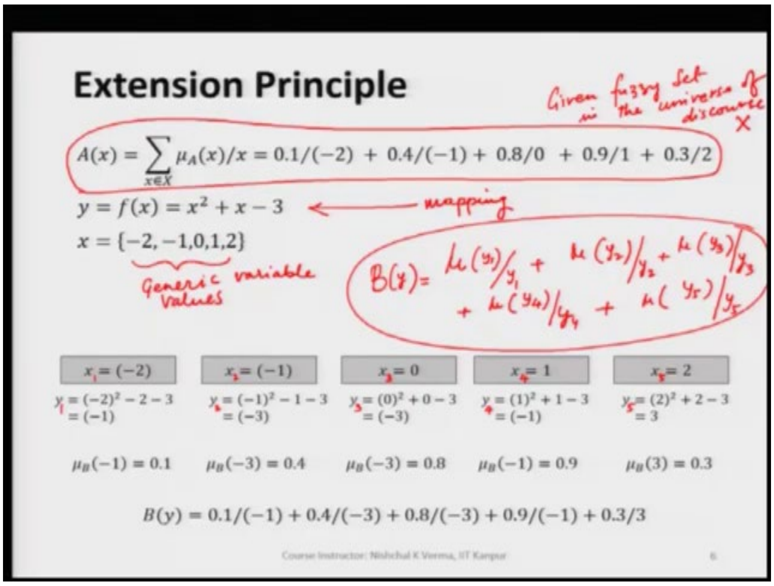

So, the mapping function is here this is given and here we have the fuzzy set this is also given. So, I am just writing that given fuzzy set in the universe of discourse X. So, since in this discrete fuzzy set we have these generic variable values -2 ,-1, $0, 1, 2$. So, let us first compute the generic variable values, that mean, the y_1, y_2, y_3, y_4 *and* y_5 in the universe of discourse Y. And this is very easy to compute because we have the mapping function with us.

So, what we need is here let say the mapped fuzzy set that we are interested in is $B(y)$, then what we need here is corresponding to $A(x)$ we need here is $\mu(y_1)/y_1$, $\mu(y_2)/y_2$, $\mu(y_3)/y_3, \mu(y_4)/ y_4, \mu\ (y_5)/ y_5$. So, the unknowns are y_1, y_2, y_3, y_4, y_5 mapping function is given. So, let us first compute y_1. So, our y_1 is here. So, since the mapping function is known we just substitute the value of x, that means, for $x_1 = -2$ we are getting our $y_1 = -1$.

Similarly, for $x_2 = -1$ we are getting $y_2 = -3$. For x we are getting y for x_3 we are getting $y_3 = -3$, $x_3 = 0$ here. So, for $x_3 = 0$we are getting $y_3 = -3$, $x_4 = 4$ we are getting $y_4 = -1$, $x_5 = 2$ we are getting y_5, 3. So, this means that the mapping function helping us to compute y_1, y_2, y_3, y_4, y_5 corresponding to the generic variable values defined in the universe of discourse X.

So, when this is known then we directly substitute this in the equation $B(y)$ here this equation this fuzzy set. So, all y, y_1, y_2, y_3, y_4, y_5 are basically substituted.

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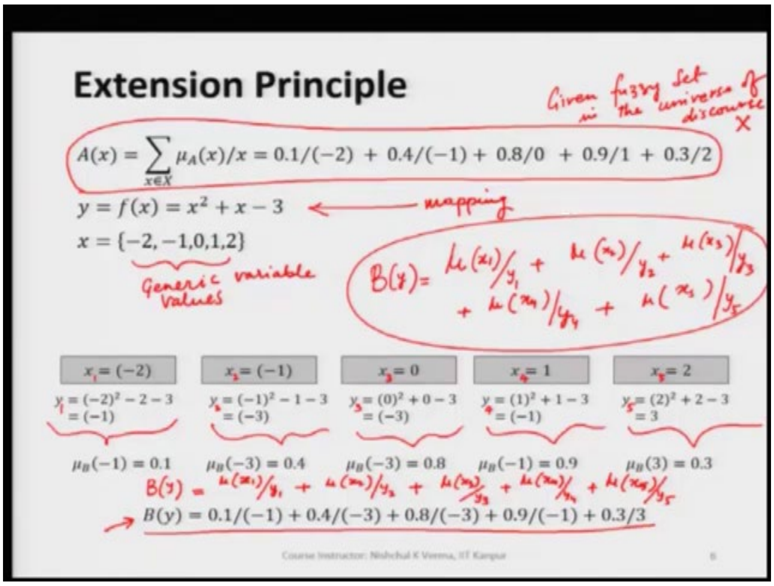

And please understand that here this is $\mu(x_1), \mu(x_2), \mu(x_3), \mu(x_4), \mu(x_5)$. So, that means, when we substitute the values of $\mu(x_1), \mu(x_2), \mu(x_3), \mu(x_4), \mu(x_5)$ and y_1, y_2, y_3, y_4, y_5. Then we get this equation. So, this means we get a transform set like this. So, transform fuzzy set is $B(y)$ fuzzy set.

So, I am again writing here how are we getting just to make you understand better. So, this we have got by just substituting these values $\mu(x_2)$, $\mu(x_3)/\,y\,3$, $\mu(x_4)/\;y_4$ then $\mu(x_5)/\;y_5$. So, when we substitute this these values $\mu(x_1), \mu(x_2), \mu(x_3), \mu(x_4)$ and $\mu(x_5)$ these values we have already got with the equation $A(x)$, that means in fuzzy set $A(x)$ we already have these values. So, we do not have to compute this.

We only have to compute y_1, y_2, y_3, y_4, y_5, that means the generic variable values in the universe of discourse Y. And this has to be computed by using the generic variable values in the universe of discourse given, that means the X and this is possible because we have the mapping function available, that means $y = f(x)$ is available by using this here we get y_1, y_2, y_3, y_4 and y_5.

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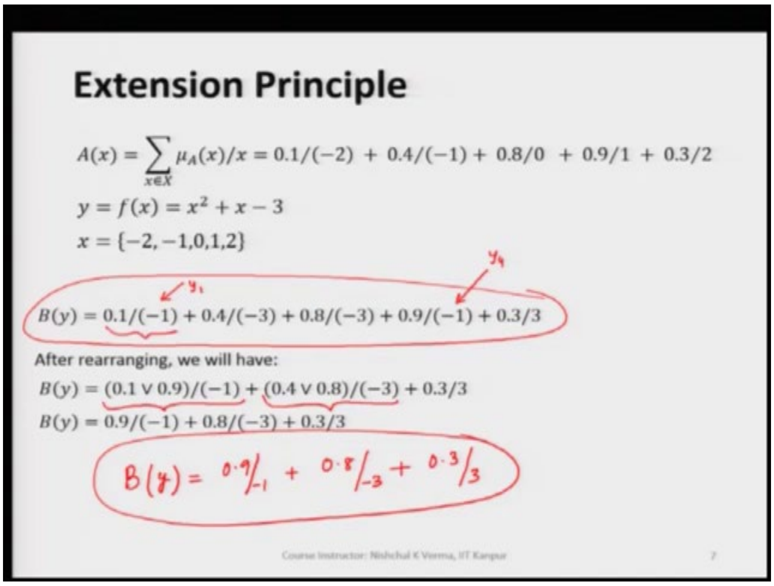

And when we substitute finally we get the transform fuzzy set $B(y)$. And it is very clear that this B of y is defined in the universe of discourse Y. So, when we substitute $\mu(x_1), \mu(x_2), \mu(x_3), \mu(x_4)$ and $\mu(x_5)$ and also y_1, y_2, y_3, y_4, y_5. Then we get here this fuzzy set, but when we rearrange this what we see is this situation. So, this situation is situation of conflict, that means that for the generic variable here for the same generic variable value; that means, $y = -1$ we have two membership values. So, $y_1 = -1$ we have 0.1 as the membership value. And again here also y this is y_4. So, $y_4 = -1$ we have 0.9.

So, this means for the same value of the generic variable we are getting two membership values. And this is the conflicting situation, so which one to keep because we can only have one membership value. So, we take the max of 0.1 and 0.9. So, max of 0.1 and 0.9 is 0.9. So, that why how we avoid this conflicting situation by keeping 0.9 for the generic variable value -1.

And then finally, we have our $B(y)$ the transformed fuzzy set in the universe of discourse that is $B(y) = 0.9/(-1) + 0.8/(-3) + 0.3/3$. So, here we have two conflicting situation one was for minus 1 as the generic variable value another one for $y = -3$. So, these two have been avoided and then finally, we have the transform fuzzy set which is $B(y)$. And this is very clear that we have got it from $A(x)$ which was given and this $A(x)$ was defined in the universe of discourse X and this was possible or this is possible with the help of extension principle.

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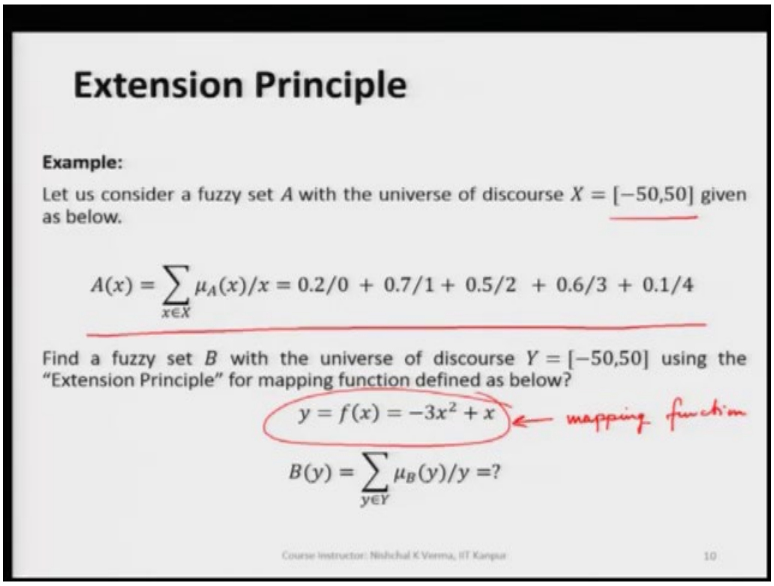

Now, let us take an other example here, we are taking another fuzzy set which is again defined in the universe of discourse X. And this X range here is given that is -50 to 50. So, all the generic variable values will be within the limit that is given that is -50 to 50. So, we have the fuzzy set A and let us now use the mapping function which is also given that is $y = -3x^2 + x$. So, this is the mapping function if the mapping function is not given then the conversion is not possible.

So, let us use this mapping function $y = f(x) = -3x^2 + x$ to compute y_1, y_2, y_3, y_4 and y_5 in this case.

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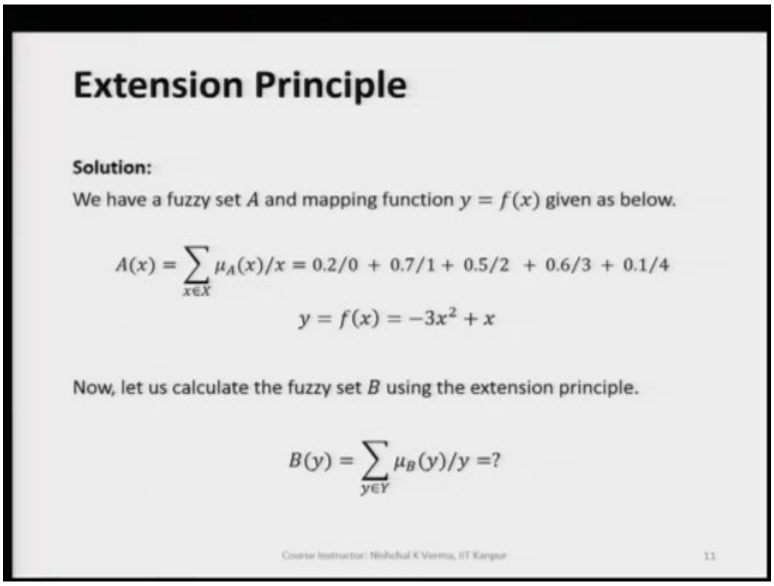

So, let us now quickly compute these values.

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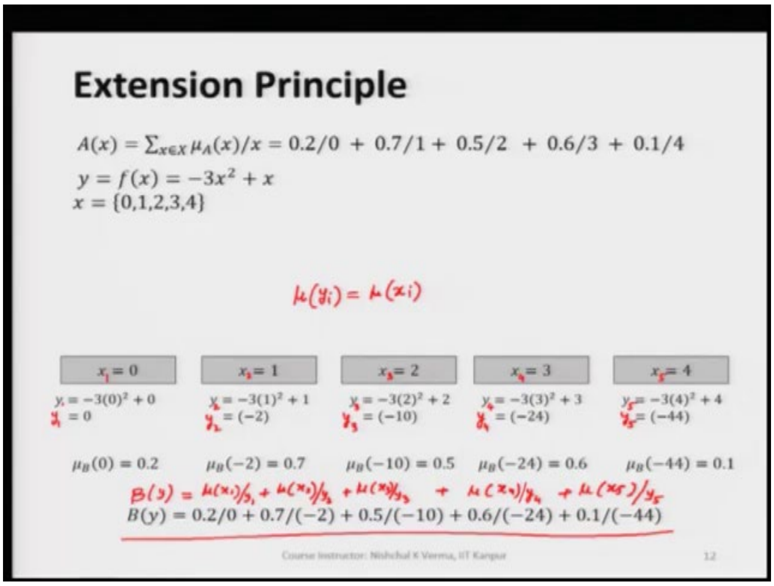

So, for $x_1 = 0$ we have $y_1 = 0$, $x_2 = 1$ we have $y_2 = -2$, $x_3 = 2$ we have $y_3 = -10$, $x_4 = 3$ we have y_4, $x_5 = 4$ we have $y_5 = -44$.

So, what we have here is $y_1 = 0$, $y_2 = -2$, $y_3 = -10$, $y_4 = -24$, $y_5 = -44$. What else do you need to write the transform fuzzy set that is $B(y)$. So, we already have y_1, y_2, y_3, y_4 and y_5 computed by using the mapping function.

So, yes so, we have to now find $\mu_B(x_1), \mu_B(x_2), \mu_B(x_3), \mu_B(x_4)$and $\mu_B(x_5)$. So, let we tell you that we do not have to do anything we do not have to do any calculation we simply write here the $\mu_B(x_1) = \mu_A(x_1)$. And as I said that this $\mu(y_1)$ is going to be equal to this, this means that we do not have to compute this we have to just keep these values just take these values. And, this gets automatically transfered into the other universe of discourse means $\mu(y_i) = \mu(x_i)$.

So, when we apply this we see that $\mu_B(0) = 0.2$. And then $\mu_B(-2) = 0.7$ and $\mu_B(-10) = 0.5$, $\mu_B(-24) = 0.6$, $\mu_B(-44) = 0.1$. So, I can write here one more equation and this is $\mu(x_1)/y\ 1\ then\ \mu(x_2)/y_2\ then\ \mu(x_3)/y_3\ then\ \mu(x_4)/y_4\ then\ \mu(x_5)/y_5$. So, this way we are able to get the transformed fuzzy set $B(y)$ in the other universe of discourse that is Y, but here we have seen that for this transformation we should have the mapping function which is connecting the universe of discourse X and Y is known.

Now, let us take another example here which is the case of a continuous fuzzy set. So, here we are taking a continuous fuzzy set instead of a discrete fuzzy set.

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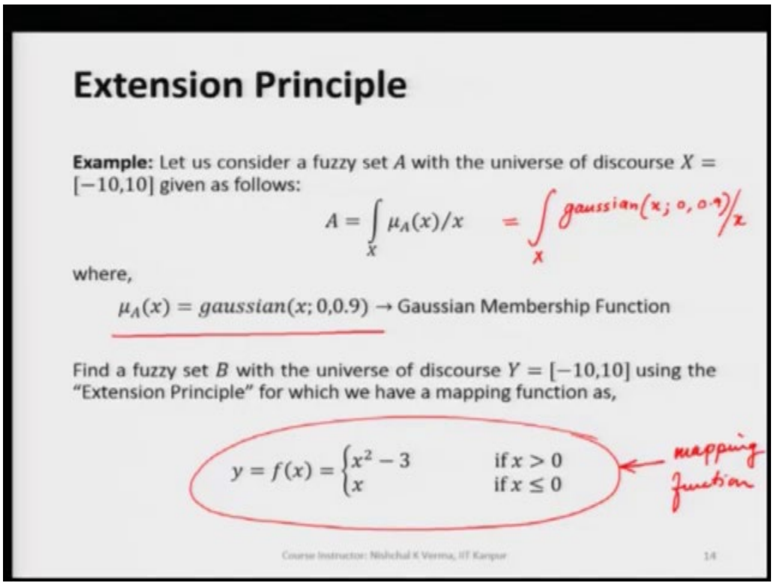

As we have seen in the previous example. So, here we are taking a fuzzy set mu fuzzy set A, where $\mu_A(x)$ is gaussian like this. So, we can write it like this and let me tell you that this fuzzy set is defined in the universe of discourse X. So, we can write it like this.

Now, the mapping function is also available. So, mapping function is this, this is our mapping function. And this mapping function says $y = f(x) = x^2 = 3$, when the generic variable values x is more than 0. And it is the $f(x)$ is x when generic variable values are either less than or equal to 0.

So, when we use this mapping function and the fuzzy set, that is given which is defined in the universe of discourse X. So, let us see how we write or how we find the fuzzy set B which is defined in the universe of discourse Y.

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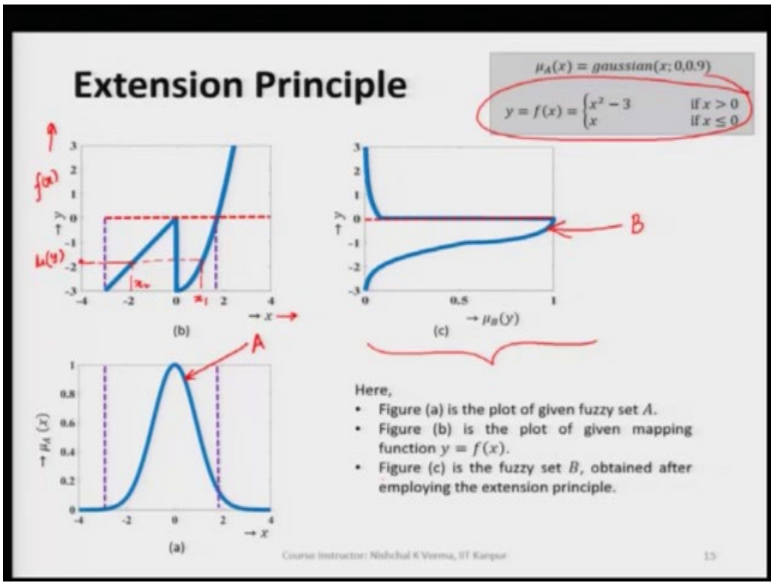

So here you see figure a is the plot of given fuzzy set A, you see here this is given that is the Gaussian fuzzy set. So, we can write here the fuzzy set A and figure b is the plot of given mapping function.

So, here is the mapping function which is given to us see here. So, we see that we have the $f(x)$ here we have the $f(x)$. So, we see x axis and y axis. And then we have the mapping function defined here for all values of x when it is greater than 0 the $f(x) = x^2 - 3$, that means, this part or I can say that when the x values are more than 0 the $f(x)$ become $x^2 - 3$. And when the x is either less than or equal to 0 the $f(x)$ becomes only x, means this part.

So, f x is known to me here, now let us find B. So, we clearly see that let me first tell you what is figure c here. So, figure c is the fuzzy set which we have obtained after employing the extension principle. So, I am going to explain you has to how we are going to get this fuzzy set B. And this B fuzzy set is in the universe of discourse Y. So, we see that if we take up $f(x)$ we take up the mapping function here. So, for some value of x here let us say we have x_1 and then let's say we have x_2 let us say this is $\mu(x_2)$.

So, this means that when x_1 and x_2 these two are the different values, for these two values x_1 and x_2 we see that we have here the same membership value. So, when we apply the extension principle here.

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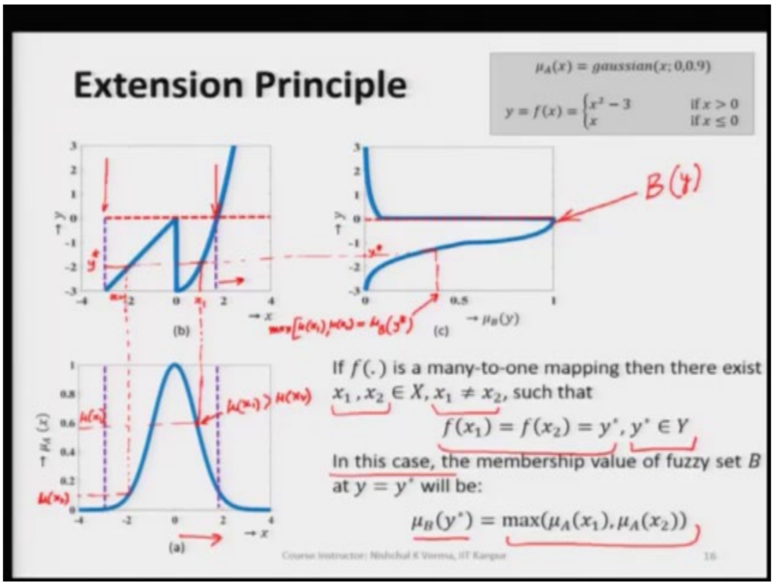

So, this says that if we have we can clearly see here if we have x_1 and x_2 like this which is belonging into the universe of discourse X. And if they are not equal as we have seen in this case, I am making this again let us say this is my x_1 and let us say this is my x_2.

And we see that we are for these two different values of x, that means, x_1 and x_2 we have getting same y, same y and we can call this as y star like this we have our x_1 and x_2 which are not equal, but we are getting $f(x_1)$ is equal to $f(x_2)$. And here y remains the same for these x_1 and x_2. So, we call this as the y star and of course, this y^* should also belong into the universe of discourse Y.

So, in this case what happens in this case what we do here is that we take the max of the $\mu_A(x_1)$ and $\mu_A(x_2)$, that means, we see here if we take the mu x what is the mu x this is x axis. So, generic variable mu here and then my $\mu(x_1)$ will be this. And my here this is

$\mu(x_2)$. So, what we are doing here is we are getting two mu values two membership values corresponding to two generic variable values, that means, x_1 and x_2 we are getting $\mu(x_1), \mu(x_2)$ and $\mu(x_1)$ and $\mu(x_2)$ are different they are not the same. So, what we do here is we take max of these two. So, when we take max of these two it is very clear that $\mu(x_1)$ is the winner.

So, we retain $\mu(x_1)$ because $\mu(x_1)$ here in this case I am talking about this case only. So, in this case $\mu(x_1)$ is greater than here $\mu(x_1) > \mu(x_2)$. So, $\mu_B(y^*)$. So, what will be the membership value corresponding to this generic variable value in the universe of discourse Y, that means this star y^*. So, $\mu_B(y^*)$ is nothing, but it is going to be the $\max(\mu(x_1), \mu(x_2))$ and by using this what we get is here.

So, we see that the corresponding y is here corresponding y is y^*. So, y^* is here and corresponding to this y^*, the mu is here this is $\mu_B(y)$ this is $\mu_B(y^*)$. And this is nothing but $\max(\mu(x_1), \mu(x_2)$. So, that is how we keep getting the corresponding μ_B values means the membership values.

So, here in this problem or in this example what is happening is that for the range here which is shown by this dotted line this and this, we see that here we see that in between this we have x values where we are getting same y values for two x values, means in other words I can say that for multiple values of x we are getting same y, I should not say multiple but I will say the pair of x.

So, this is the range in which more than one value of x we are getting the same y. So, that is why here this range in this range we apply the max condition. And then after this range we do not have to worry because we are going to get the only one to one mapping. So, that is how when we apply these we are going get this shape. So, this is what is our $\mu_B(y)$ which is nothing, but the membership function in the universe of discourse Y and this is for the fuzzy set B. So, I can write here that the resulting function B is like this. So, this way we apply extension principle in order to map from one universe of discourse to another for the fuzzy set A transforming into B.

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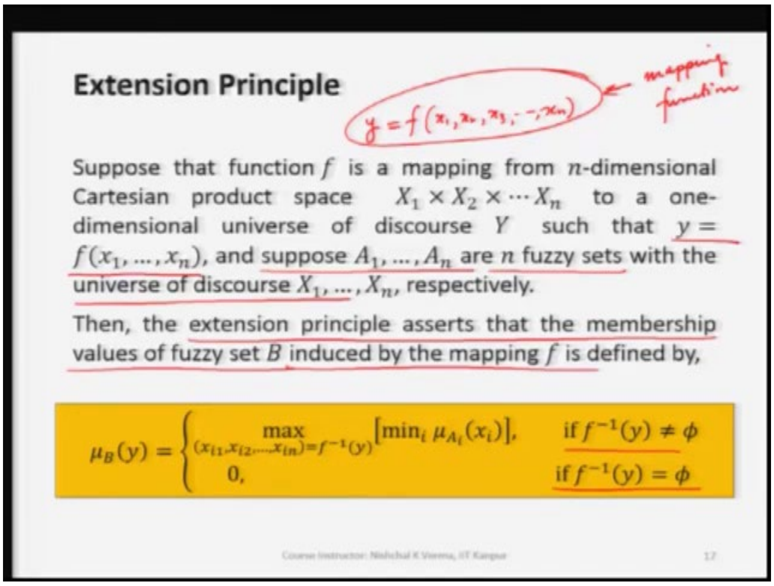

So, now if we have let say the mapping function in n dimension means if we have a mapping function like $f(x_1, x_2, x_3, \ldots\ldots, x_n)$. So, then how can be map this to y. So, suppose a mapping function f is in the n dimensional universe of discourse; that means, $(X_1,\ X_2, X_3, \ldots\ldots, X_n)$. And, we are mapping this to the universe of discourse Y. So, this means that we have Y is equal to $f(x_1, x_2, x_3, \ldots\ldots, x_n)$.

And here let's say we have a fuzzy set which is defined in the universe of discourse $(X_1, X_2, X_3, \ldots\ldots, X_n)$. So, how can we transform the fuzzy set B which is in the universe of discourse Y. And this is what is the mapping function that is given to us, this is mapping function.

So, in this case the extension principle helps us in finding the membership values, that means, the $\mu_B(y)$ where this

$$\mu_B(y) = \begin{cases} \max\limits_{(x_{i1}, x_{i2}, \ldots\ldots, x_{in}) = f^{-1}(y)} [min_i\ \mu_{A_i}(x_i)]\ , & if\ f^{-1}(y) \neq \phi \\ 0, & if\ f^{-1}(y) = \phi \end{cases}$$

So, this is how we can manage to map the n dimensional space into the single dimensional space, that means, $x_1, x_2, x_3 \ldots\ldots, x_n$ into y and with this I would like to stop here.

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And, in the next lecture we will discuss the composition of fuzzy relations.

Thank you.