Fuzzy Sets, Logic and Systems and Applications
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Lecture – 34
Properties of Fuzzy Relation

So, welcome to lecture number 34 of Fuzzy Sets, Logic and Systems and Applications. This lecture is in continuation to our previous lecture, where we discussed the properties that was relation with the Fuzzy Relation and the Crisp Relation.

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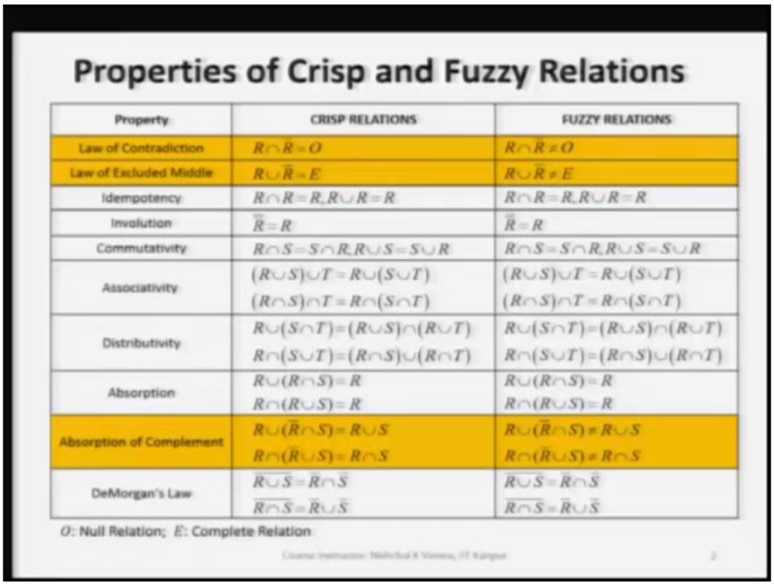
Properties of Crisp and Fuzzy Relations

Property	CRISP RELATIONS	FUZZY RELATIONS
Law of Contradiction	$R \cap \bar{R} = O$	$R \cap \bar{R} \neq O$
Law of Excluded Middle	$R \cup \bar{R} = E$	$R \cup \bar{R} \neq E$
Idempotency	$R \cap R = R, R \cup R = R$	$R \cap R = R, R \cup R = R$
Involution	$\bar{\bar{R}} = R$	$\bar{\bar{R}} = R$
Commutativity	$R \cap S = S \cap R, R \cup S = S \cup R$	$R \cap S = S \cap R, R \cup S = S \cup R$
Associativity	$(R \cup S) \cup T = R \cup (S \cup T)$ $(R \cap S) \cap T = R \cap (S \cap T)$	$(R \cup S) \cup T = R \cup (S \cup T)$ $(R \cap S) \cap T = R \cap (S \cap T)$
Distributivity	$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$ $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$	$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$ $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$
Absorption	$R \cup (R \cap S) = R$ $R \cap (R \cup S) = R$	$R \cup (R \cap S) = R$ $R \cap (R \cup S) = R$
Absorption of Complement	$R \cup (\bar{R} \cap S) = R \cup S$ $R \cap (\bar{R} \cup S) = R \cap S$	$R \cup (\bar{R} \cap S) \neq R \cup S$ $R \cap (\bar{R} \cup S) \neq R \cap S$
DeMorgan's Law	$\overline{R \cup S} = \bar{R} \cap \bar{S}$ $\overline{R \cap S} = \bar{R} \cup \bar{S}$	$\overline{R \cup S} = \bar{R} \cap \bar{S}$ $\overline{R \cap S} = \bar{R} \cup \bar{S}$

O: Null Relation; E: Complete Relation

So, in the previous lecture, we discussed some of the properties listed here. So, we discussed law of contradiction, law of excluded middle, idempotency, involution and commutativity with respect to fuzzy relations. And we saw that we had law of contradictions with respect to the fuzzy relations, law of excluded middle with reference to fuzzy relations.

So, these were not behaving same as the crisp relations these two properties and now the idempotency, involution, commutativity all these three properties were satisfied, were holding good when we took fuzzy relations.

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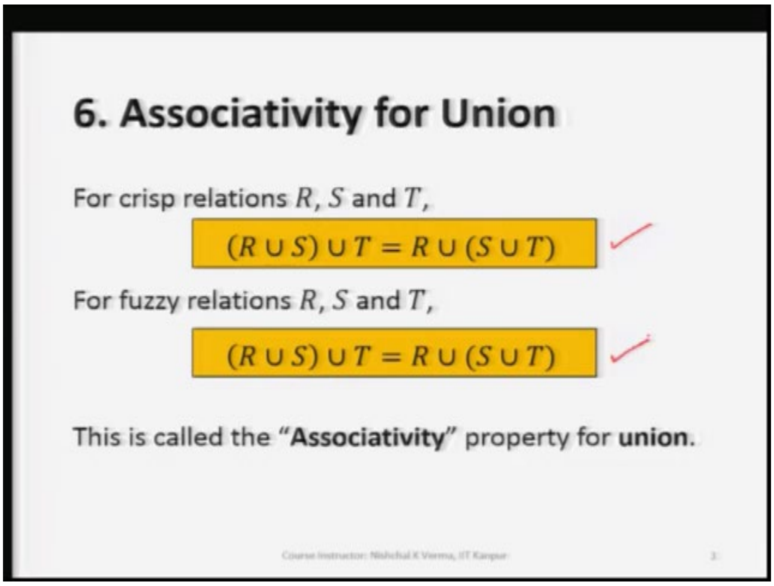

So, let us now move in continuation to these properties that has come to associativity and when we take crisp relations R, S, T. So, these crisp relation shows the associativity for Union. So, as it is mentioned here that if we take the $(R \cup S) \cup T = R \cup (S \cup T)$. So, this is true for, this holding good for the crisp relations. Now, when it comes to fuzzy relations $R, S\ and\ T$, here also the associativity for union holds good. So, let us verify this by taking one example.

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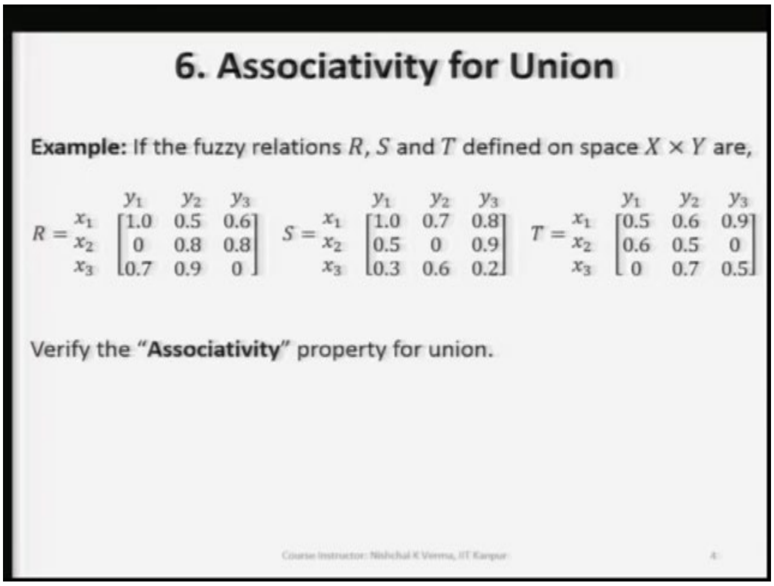

And in this example here, we have taken 3 fuzzy relation sets. So, we have taken R fuzzy relations set, S fuzzy relation set and T fuzzy relation set.

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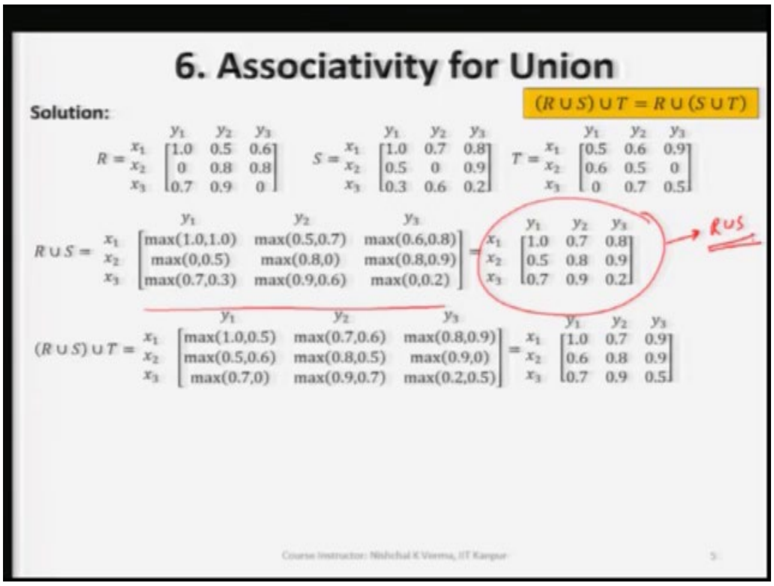

Now, when we take the $R \cup S$, we are getting here this fuzzy relation set and as we already know that we take the maximum of the respective membership values and with this we get the $R \cup S$ and the process is shown here you can follow. And then here, we take the $(R \cup S) \cup T$. So, here when we take the whatever we have got here as $R \cup S$, now with further take the union with T here, we get here this fuzzy set as result.

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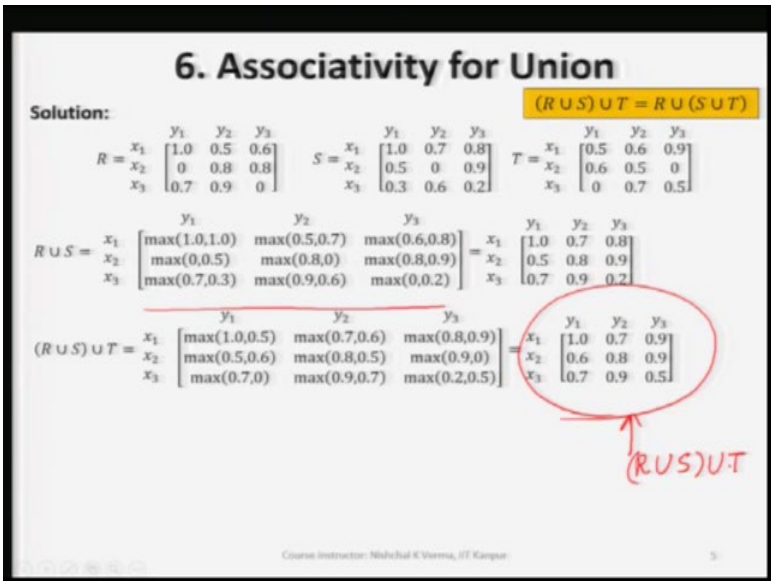

So, this is nothing but the $R \cup S$ and then, the union of it with T. So, this is what we are getting.

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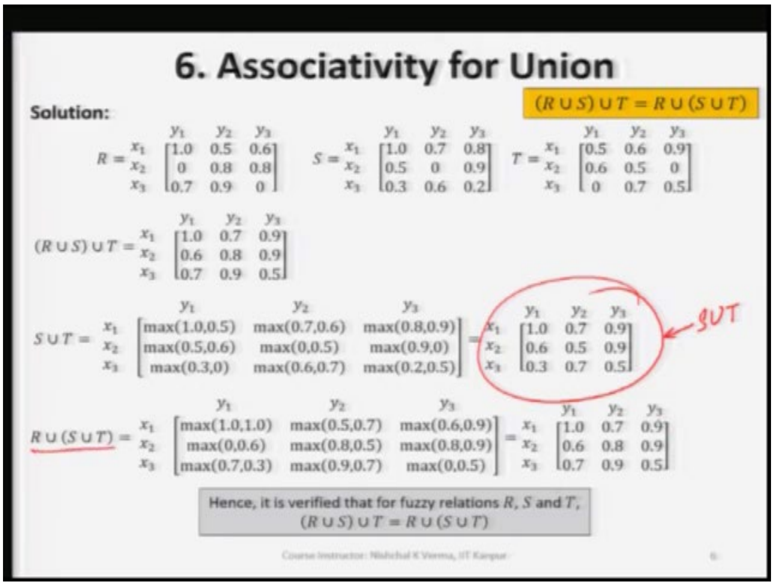

Now, let us take the $S \cup T$. So, $S \cup T$ is here;$R \cup (S \cup T)$.

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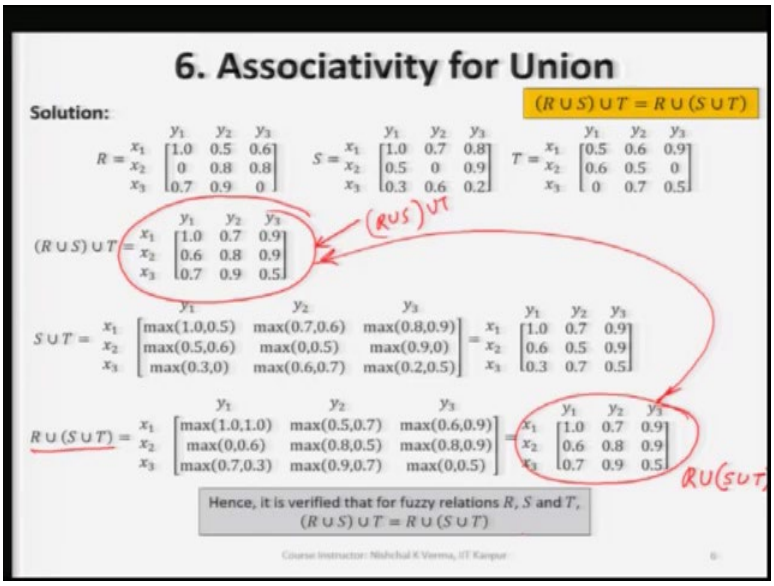

So, if we take this, we are getting here this fuzzy relation set and in the matrix representation, we see that we get R union, $R \cup (S \cup T)$. So, if we compare this with the previous outcome which was nothing but the $(R \cup S) \cup T$. So, we see that these two, these both the sets are same.

So, we can say that $(R \cup S) \cup T = R \cup (S \cup T)$ and these R, S and T all are the fuzzy relation sets. So, this way we see that the associativity property for union holds good for fuzzy relation sets R, S and T. Now, let us check the associativity for intersection.

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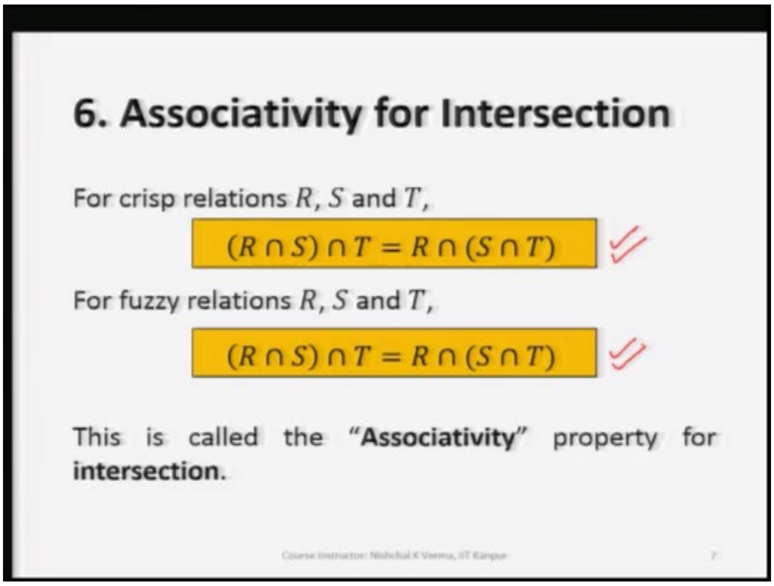

And we already know that if we take crisp relations R, S and T, this relation this property that means, the associativity for intersection is satisfied. So, we are not going into it. Now, let us understand this relation that when we take fuzzy relations R, S and T the associativity for intersection is also holding good. So, let us verify this by taking an example for this property.

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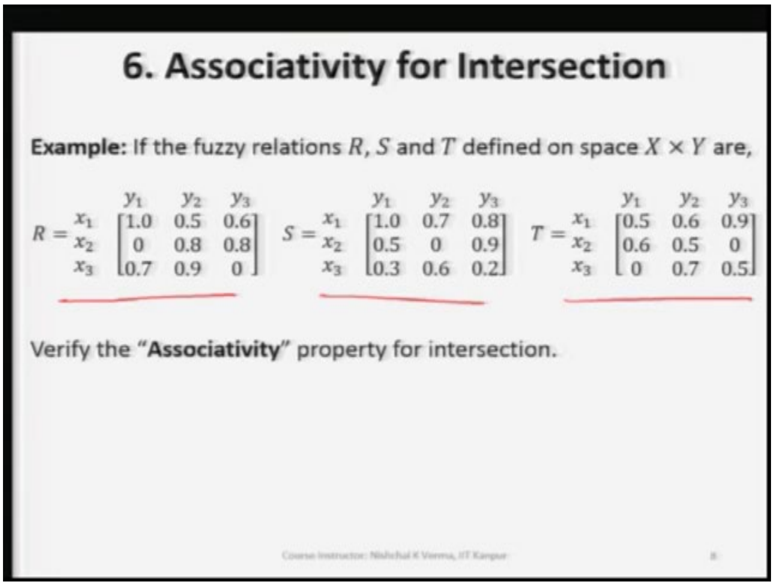

So, here also we take R, S and T, as you can see here these 3 fuzzy relation sets, we have taken.

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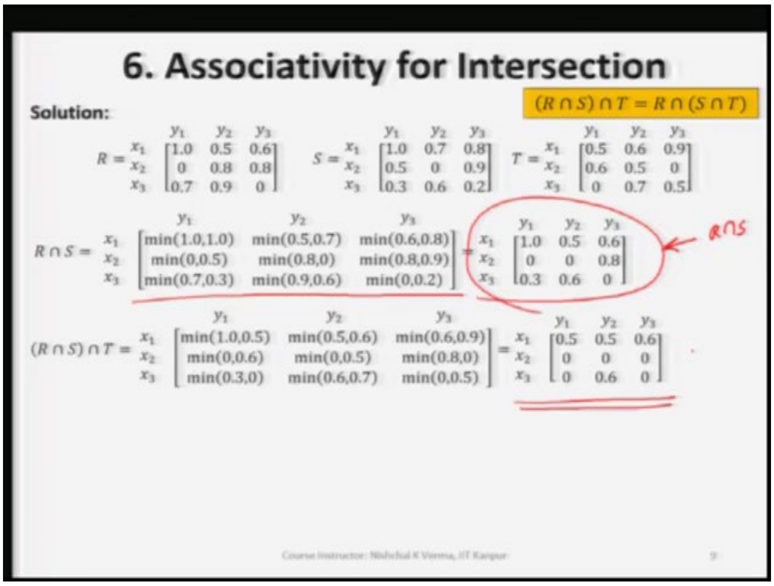

And let us first find the $R \cap S$. So, when we use the min criteria here, we find here this outcome as the $(R \cap S) \cap T$. So, when we try to find this, again the intersection of $R, S \cap T$. So, we find here this outcome. So, this is nothing but the intersection of R intersection S and T.

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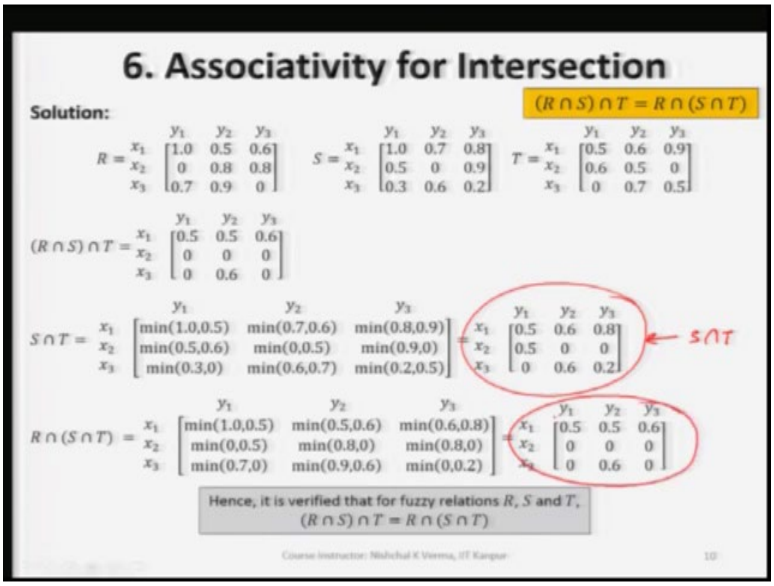

Now, let us find the $S \cap T$. So, $S \cap T$ is here and when we take the intersection of this fuzzy set which is which we have found here as $S \cap T$, $S \cap T$ and the $R \cap (S \cap T)$ here. So, if we find that we find this outcome.

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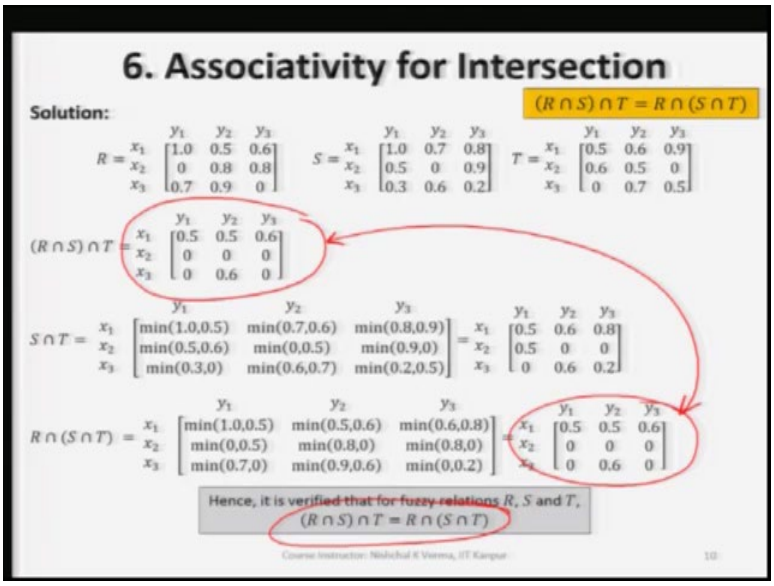

So, now when we compare this, we see that these both the outcomes are same. So, this means that the $(R \cap S) \cap T = R \cap (S \cap T)$. So, this way we can say that associativity for intersection is satisfied for fuzzy relation sets.

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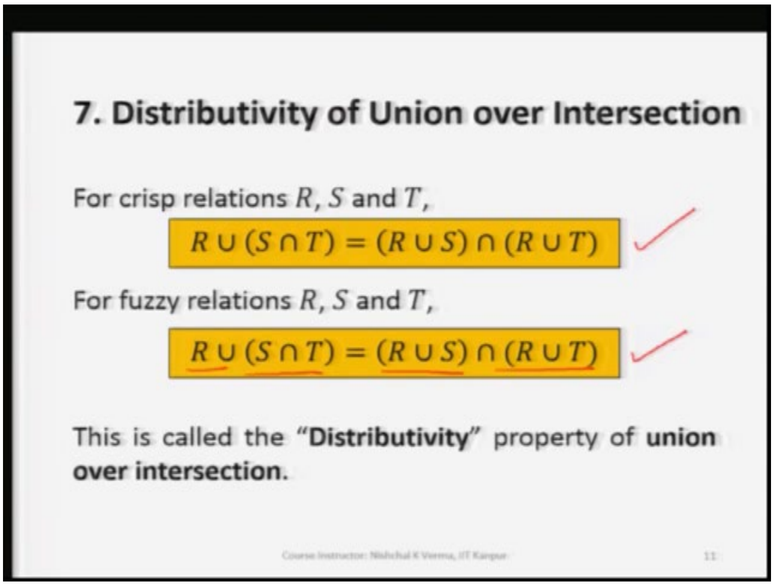

Now, let us move to the distributivity of union over intersection. So, when we take crisp relations R, S and T, we know that this is satisfied. So, we are not going into it, we are not going to discuss anything about it further. However, if we take fuzzy relations R, S and T we see that the $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$. So, let us now verify this. So, this holds good for fuzzy relations R, S and T. So, this is called distributivity of union over intersection property and this holds good for fuzzy relation sets.

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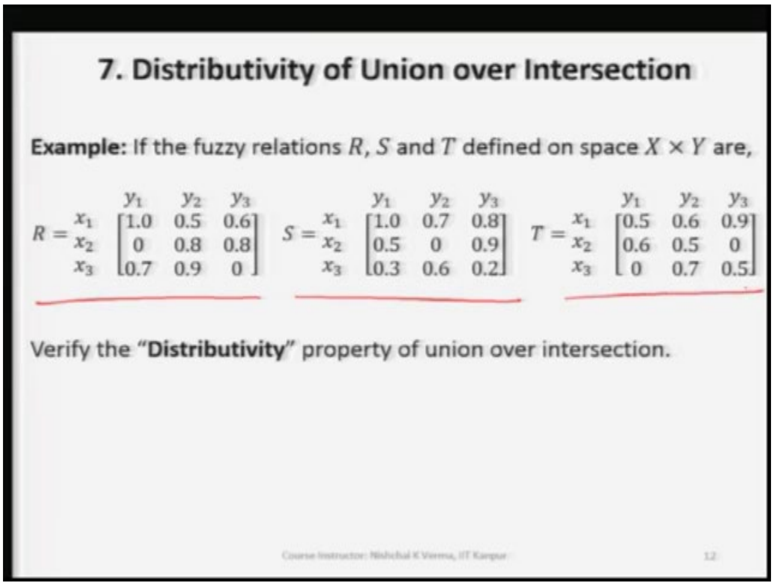

Let us take an example and verify this quickly. So, here we have taken 3 sets; 3 fuzzy relation sets.

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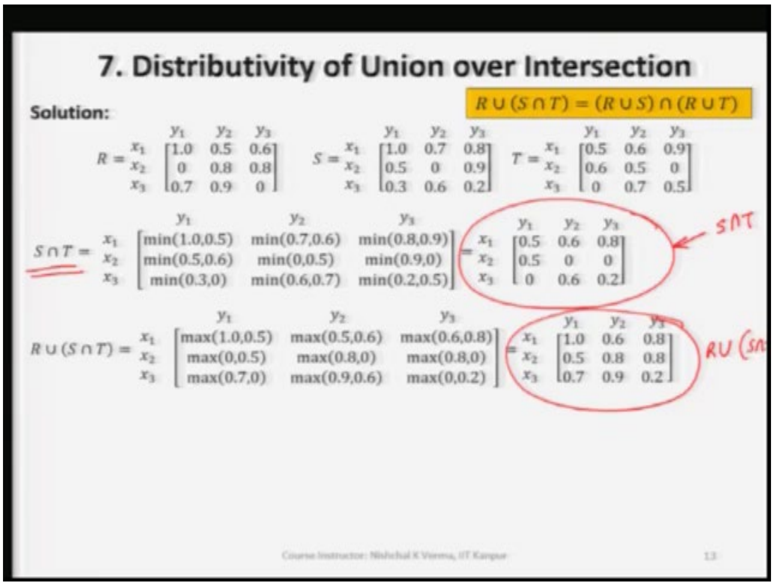

And let us take the $S \cap T$. So, $S \cap T$ which is here which is which we get like this, this the outcome of the $S \cap T$ and let us take the $R \cup (S \cap T)$. So, when we take this, we are getting this as the $R \cup (S \cap T)$.

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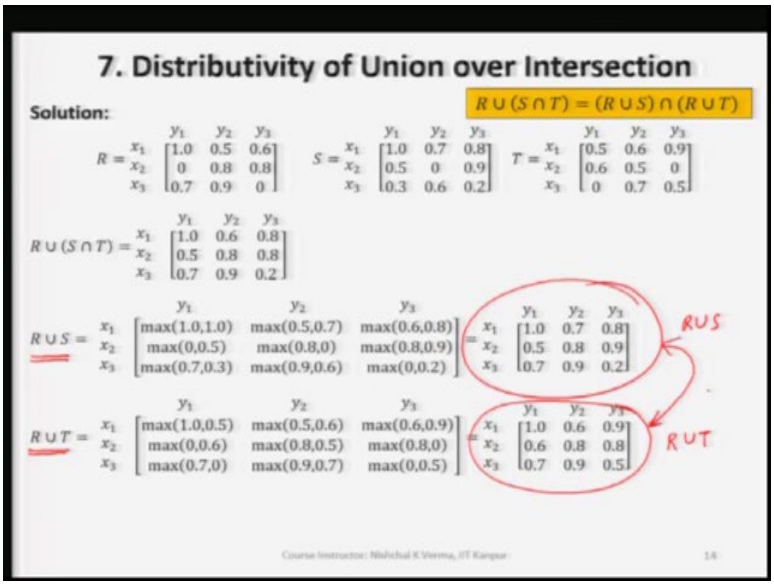

So, lets us move further and see whether the distributivity of union over intersection is verified here. So, in order to do that let us first find the $R \cup S$. So, $R \cup S$ outcome is here, then we go to find the $R \cup T$. So, $R \cup T$ is here; $R \cup T$, and now we take the intersection of these two fuzzy relations. So, let us take the intersection of these two.

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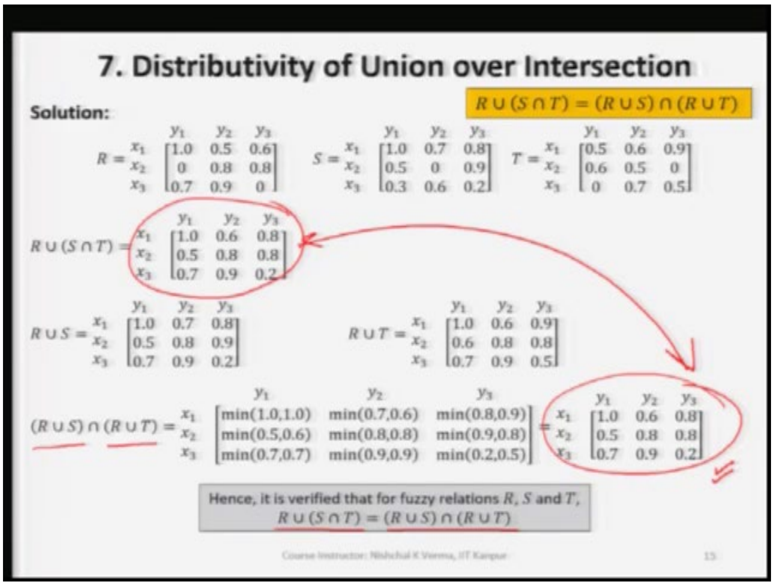

So, when we take the intersection of these two here, we get as a result here, a new fuzzy relation set which is represented by fuzzy relation matrix. So, we see that this outcome is same as this outcome, means the $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$. So, this way we can say that the distributivity of union over intersection is verified for fuzzy relations R, S and T.

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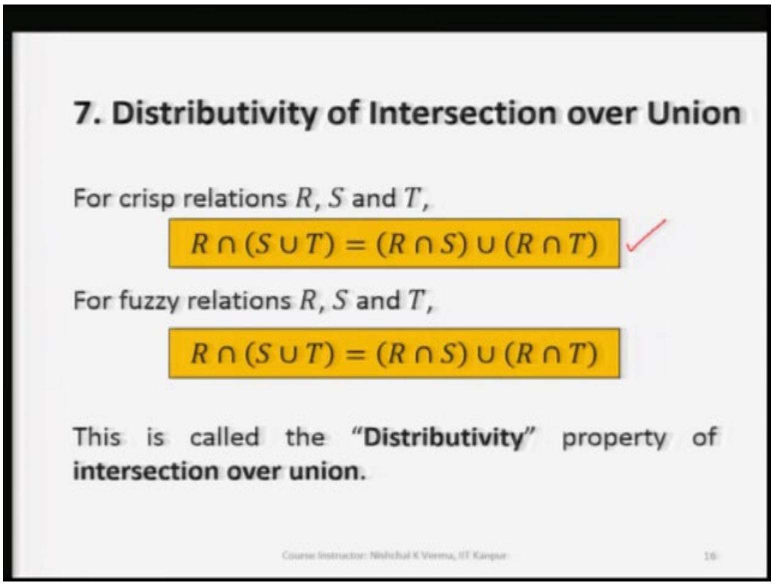

So, let us now go to the next property which is Distributivity of Intersection over Union and we all know that this holding good for crisp relations R, S and T and also we know that the, this property the distributivity of intersection over union is satisfied is holding good for the fuzzy relations R, S and T also. Now, let us verify this relation by taking an example like previously we have done for other properties.

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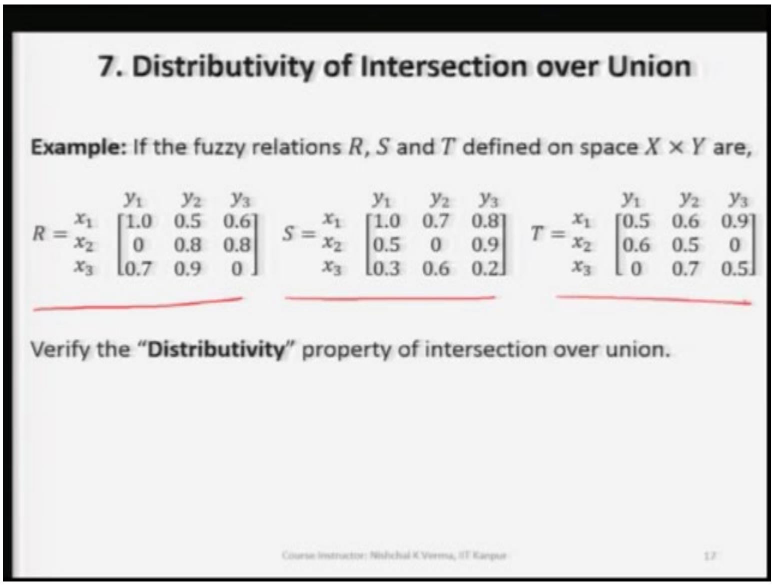

So, let us once again take the fuzzy relation sets R, S and T.

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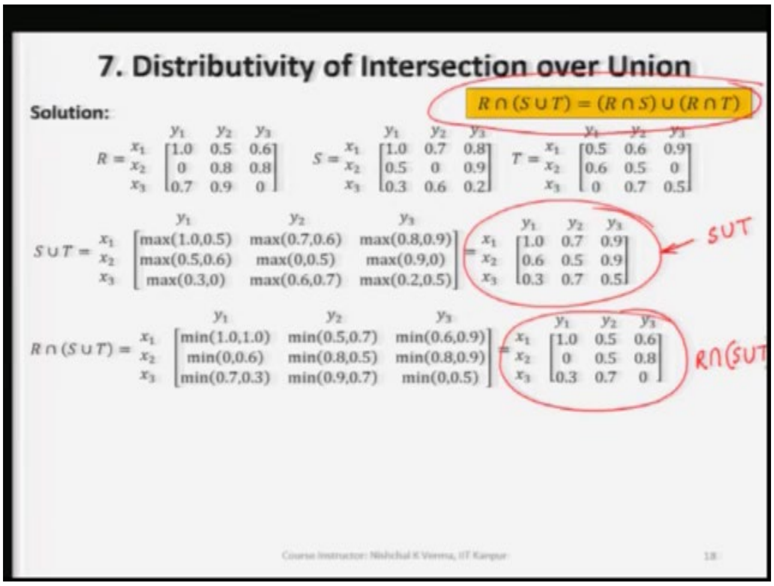

And then, since we have to verify this property that means, the distributivity of intersection over union that means, the $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$. So, let us first find the $S \cup T$. So, $S \cup T$ is here. By applying the max criteria, we get the $S \cup T$.

Similarly, when we take the fuzzy relation set R and we take the $R \cap (S \cup T)$, we get this outcome, by applying the min criteria. So, we can write this as $R \cap (S \cup T)$. So, the outcome is here.

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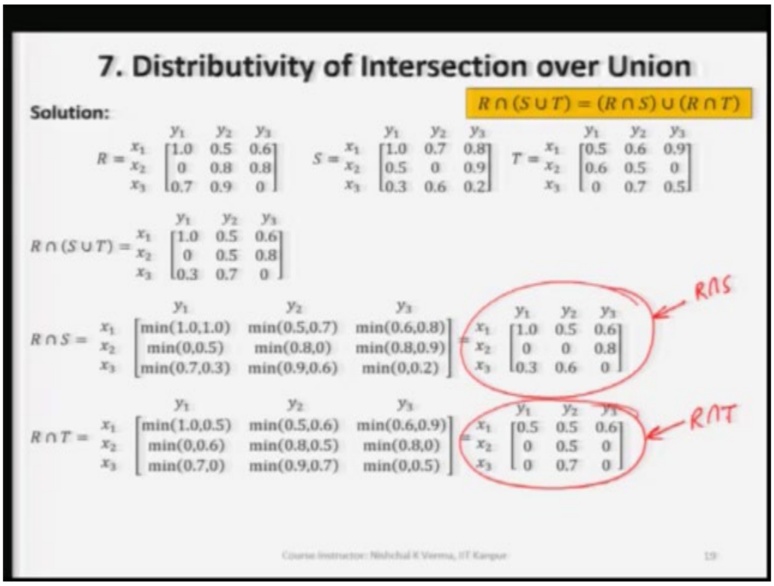

And now let us go to $R \cap S$. So, $R \cap S$ is here. When we take R and S take the intersection. So, this is $R \cap S$ and here we have the $R \cap T$. Now, if we take the union of these two.

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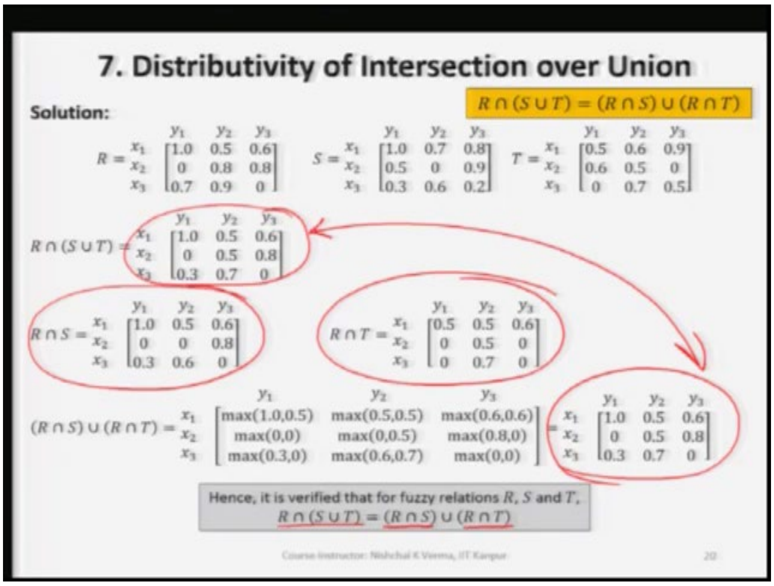

So, when we take union of these two, means the $(R \cap S) \cup (R \cap T)$. So, when we take this since we are taking union, we take the max, we take we use the max criteria and when we use this max criteria we are getting here this has the outcome.

So, we get a new fuzzy relation set and which is nothing but equal to you see here equal to this fuzzy relation set. So, this way we can say that the outcome of the $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$. So, this way we can say that the distributivity of intersection over union is verified for fuzzy relations R, S and T.

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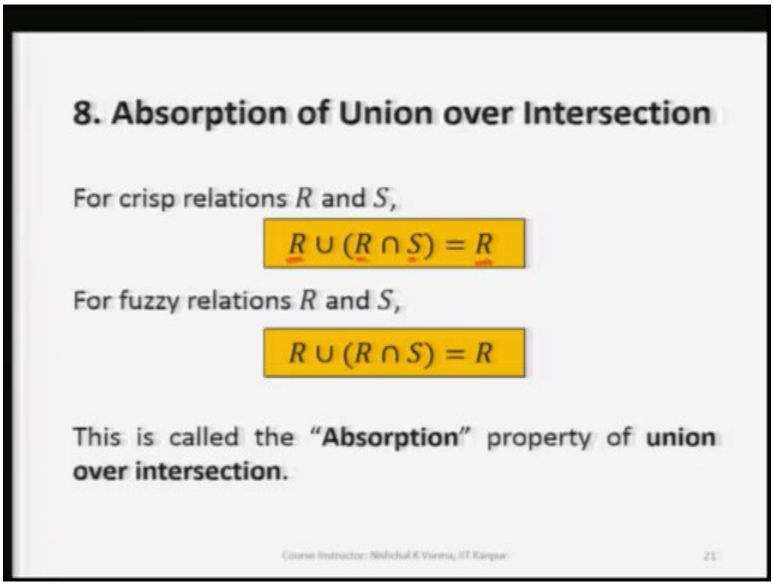

Let us move to the next property which is absorption of union over intersection. So, here for crisp relations R and S, it is like this like a we take the $R \cup (R \cap S)$, we get here R. So, this is holding good for crisp relations and when it comes to fuzzy relations R and S, here also the absorption of union over intersection holds good means the $R \cup (R \cap S) = R$. So, let us this also verify by taking an example.

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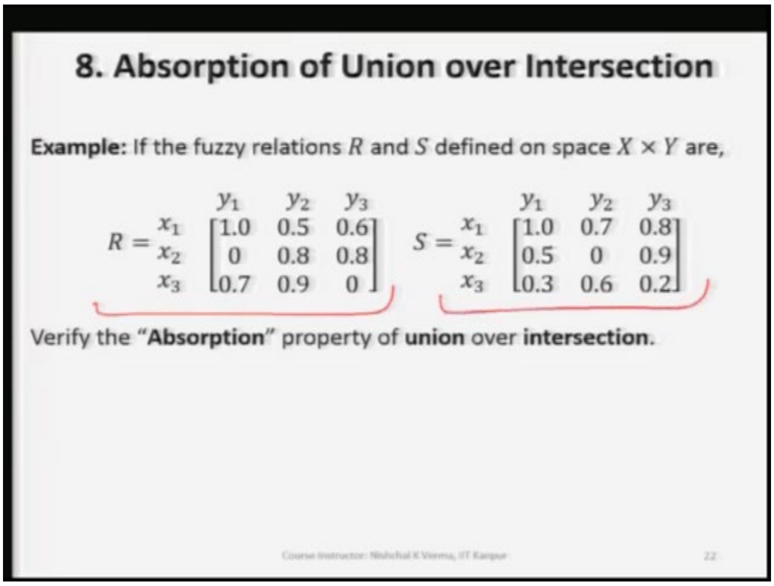

So, here we have taken one fuzzy relation set R and another fuzzy relation set S also we have taken. So, R and S both are the fuzzy relation sets and now let us verify the absorption of union over intersection.

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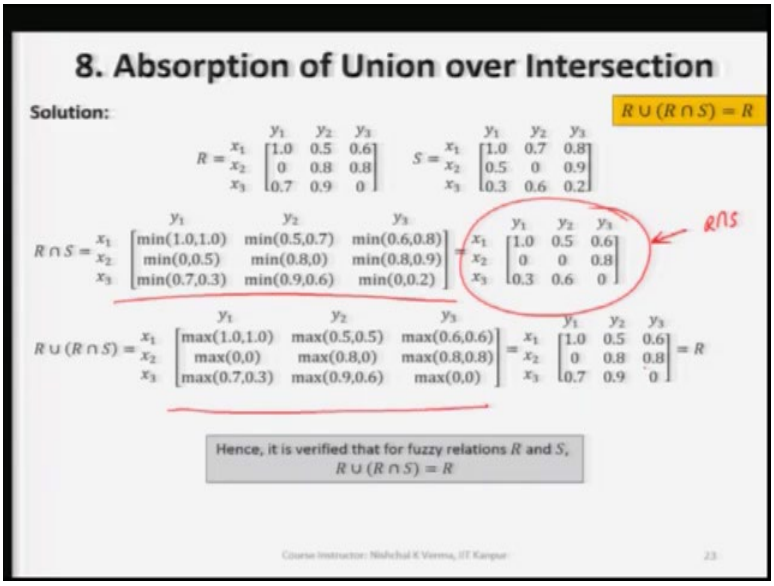

So, for this let us find the $R \cap S$. So, since here we are finding the intersection, we use min criteria you see here and when we use min criteria the outcome is this. So, this is nothing but the $R \cap S$. Now, this outcome is again used for union with R. So, when we take the $R \cup (R \cap S)$, now since we are taking union, we use the max criteria here and then, the outcome here is like this.

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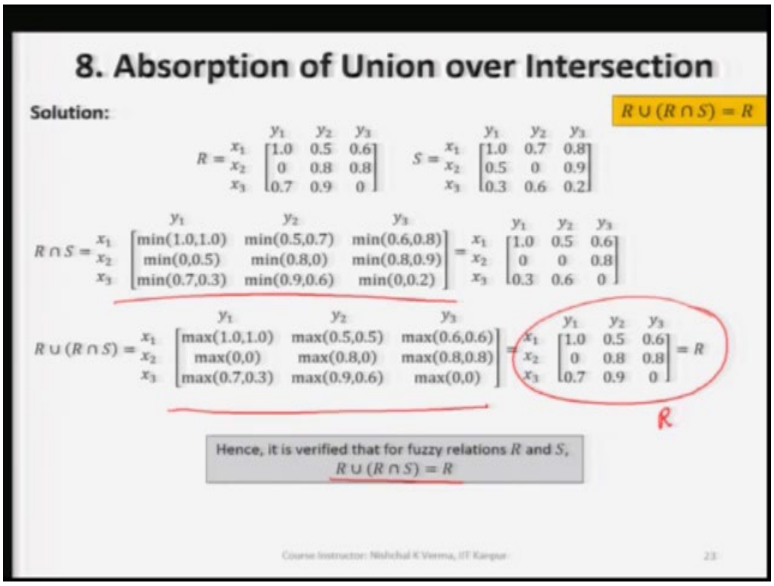

So, the outcome here is like this and which is nothing but the fuzzy set R which we have taken. So, we can say that the $R \cup (R \cap S) = R$. So, this means that the absorption of union over intersection is verified for fuzzy relations R and S.

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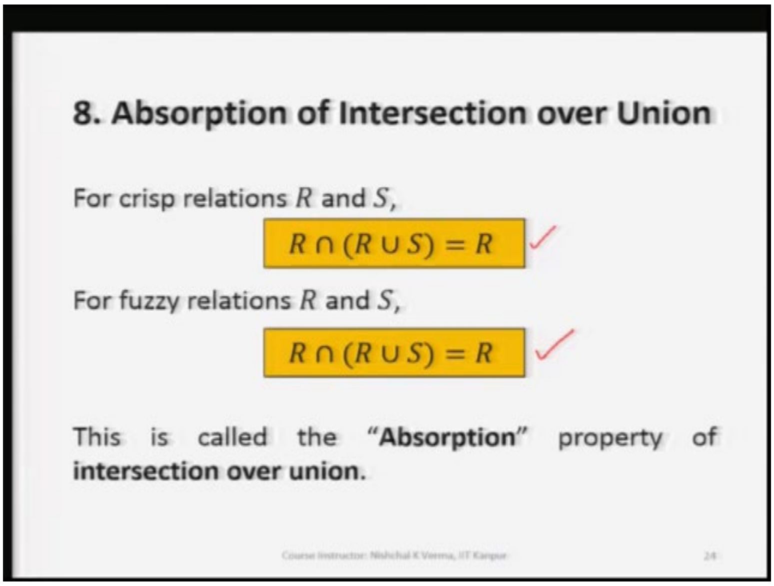

Now, let us move to another property which is absorption of intersection over union. So, here, we have the $R \cap (R \cup S) = R$. So, this is this holds good for the crisp relation sets. And the same also holds for the fuzzy relations R and S. That means, if we have two fuzzy relation sets R and S, so the intersection between the R fuzzy relation set and the union of fuzzy relation set R and fuzzy relation set S, we are going to get R. So, this means that "Absorption" of intersection over union holds good for a crisp relations as well as fuzzy relations. So, let us verify this by taking an example.

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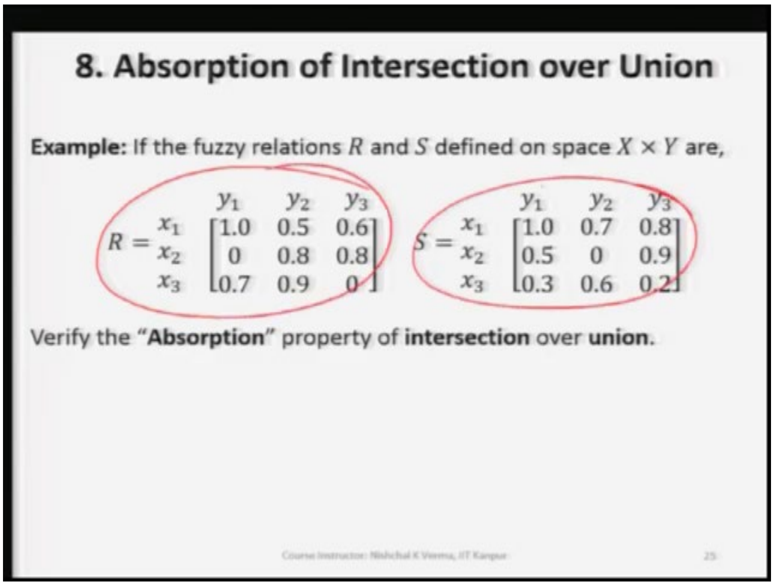

So, here we have taken two fuzzy relation sets R and S, which you can see here.

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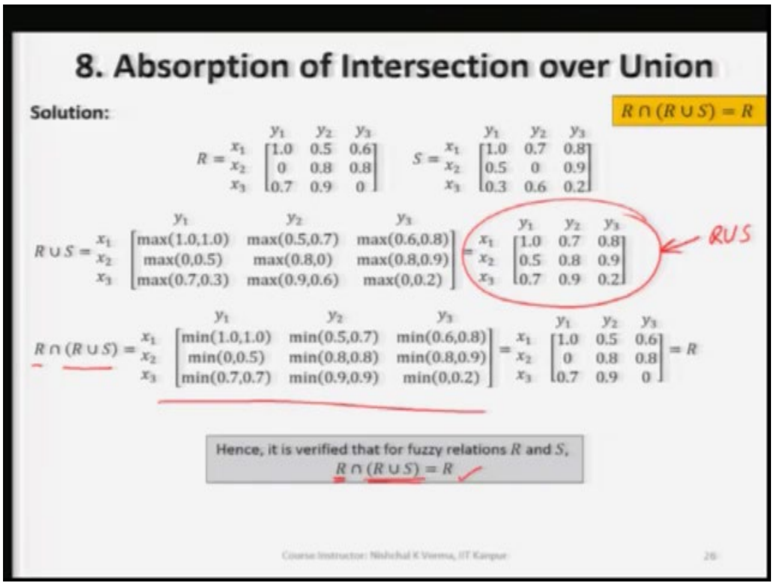

And then, now let us try to verify for absorption of intersection over union. So, when we do that in order to do that, we first find the $R \cup S$. So, $R \cup S$ is here. So, we use max criteria for union. So, we get the $R \cup S$ here and then we take intersection of this with R. So, we take R first and then we take the $R \cap (R \cup S)$.

So, since we are taking the intersection here, we use min criteria and then when we use min criteria, we are getting this as the outcome and this outcome if we compare this is nothing but the R. So, we can clearly say here that the $R \cap (R \cup S) = R$. So, here this absorption of intersection over union is satisfied or we can say it is verified.

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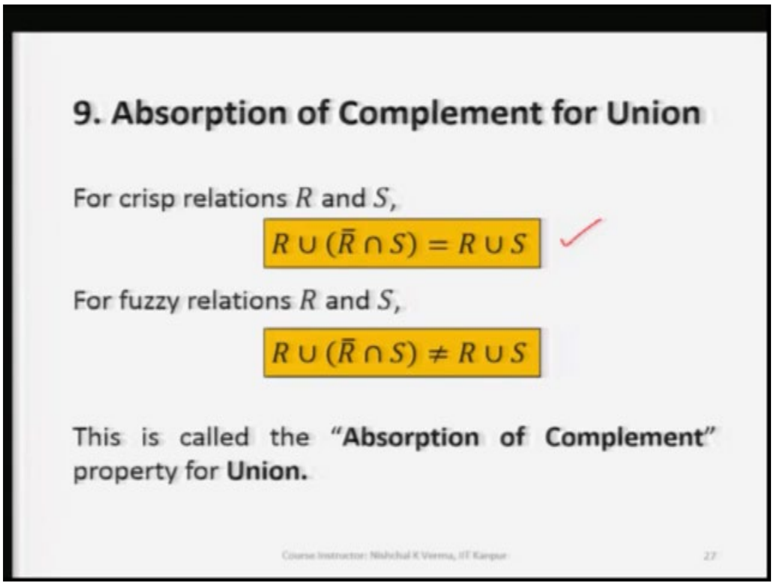

Now, let us go to the absorption of complement for union. So, here we have the relation sets, crisp relation sets R and S and we see that when we take the $R \cup (\bar{R} \cap S)$. So, this comes out to be the union of R and S.

So, this holds good for the crisp relations, but when it comes to fuzzy relations, this does not hold good. That means, when we have fuzzy relations R and S, if we take the $R \cup (\bar{R} \cap S) \neq R \cup S$. So, this way we can say that the absorption of complement property for union does not hold good for fuzzy relations R and S.

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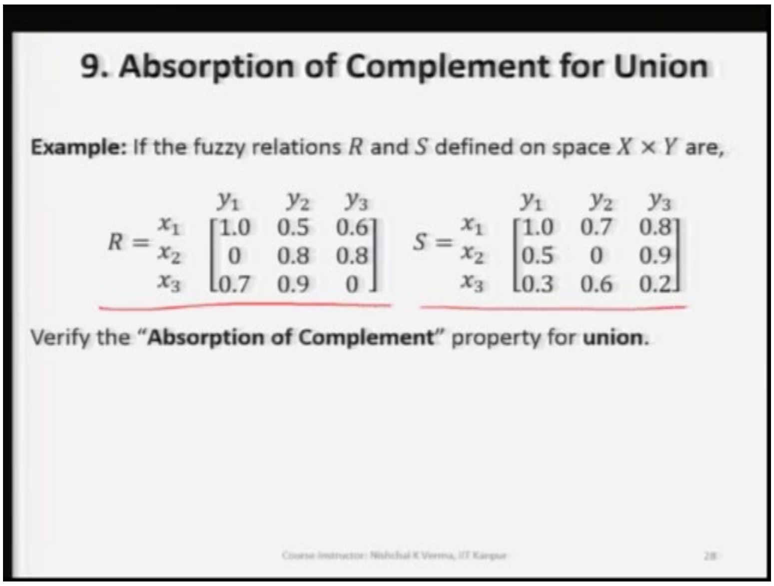

So, let us quickly verify this by taking an example. So, we have taken here in this example fuzzy relation set R and fuzzy relation set S.

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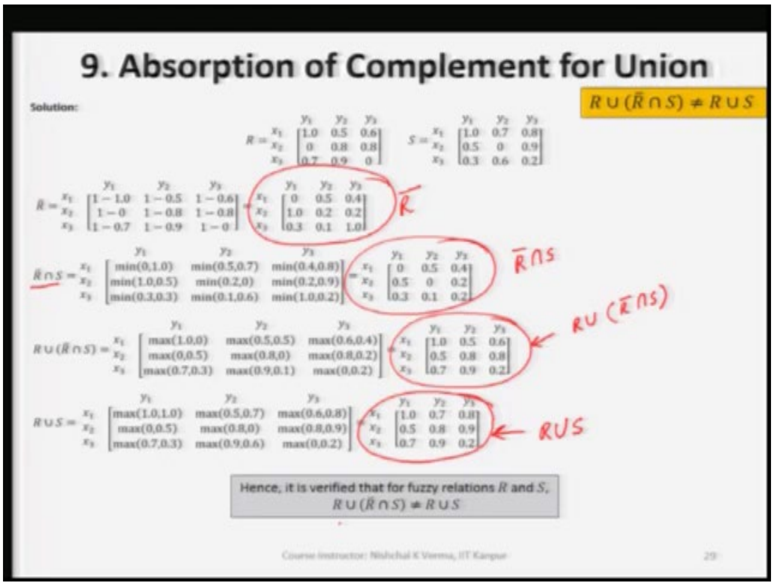

So, let us first find the complement of R. So, the complement of R is here. I am writing just complement of R and then, let us take the $\bar{R} \cap S$. So, this is the outcome. Since, we are taking the intersection, we will use the min criteria which is clearly visible here and then when we have taken this, then let us take the union of R and this outcome. This outcome is the $\bar{R} \cap S$. So, when we do that, we are getting here S, the $R \cup (\bar{R} \cap S)$. Now, let us find the $R \cup S$.

So, we have R we have S, now since we are taking union we use max criteria and when we find this we see that we are getting here as the $R \cup S$. When we compare this outcome with the $R \cup (\bar{R} \cap S)$, we see that both of these are not same. So, when both of these are not same, we can say that the absorption of complement for union is not holding good.

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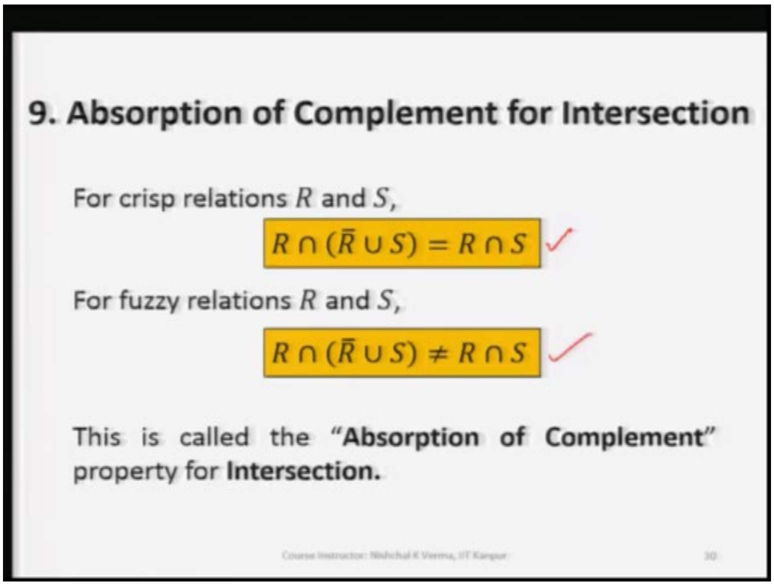

Now, let us check the absorption of complement for intersection. So, on the same lines we see that the crisp relations R and S, these relations are holding good, the absorption of complement for intersection means when we take the $R \cup (\bar{R} \cap S) = R \cap S$ and if we take the fuzzy relations R and S, so, this relation does not hold good means this property does not hold good when we take fuzzy relation sets R and S. So that means, if we have fuzzy relations sets R and S and if we take the $R \cup (\bar{R} \cap S) \neq R \cap S$.

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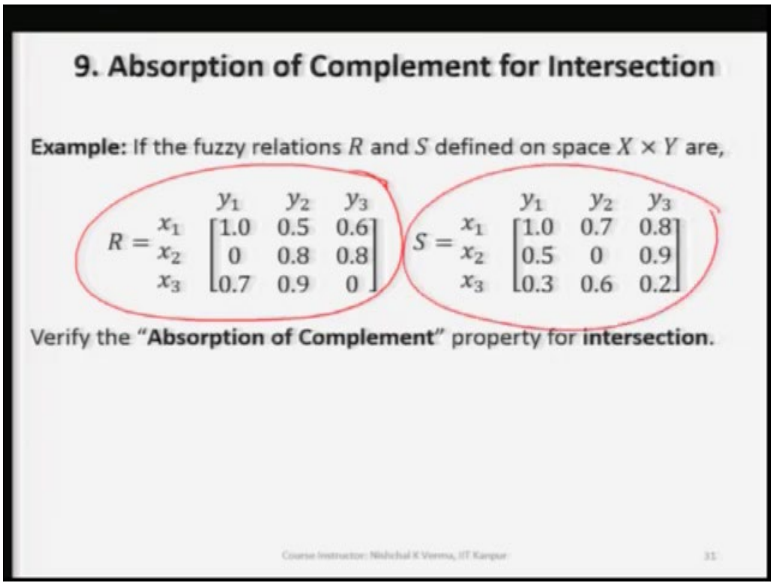

So, let us verify this also and for this, if we take our fuzzy relation set here as R, one of the fuzzy relation sets as R and then, another fuzzy relation set S here. So, when we take these two fuzzy relation sets; first fuzzy relation set is R and the second fuzzy relation set is S.

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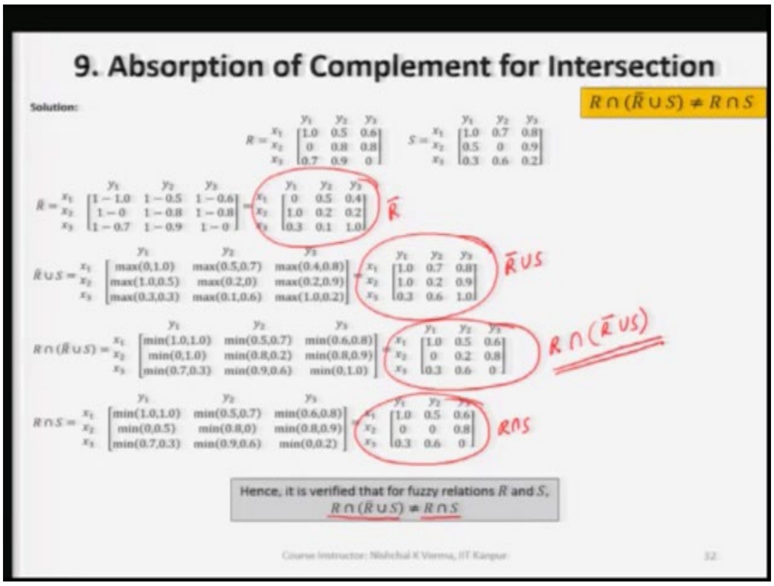

Now, let us first find the complement of R here which is coming out to be like this. So, this is $\bar{R} \cup S$. So, since we are taking union ,so we use max criteria as we can clearly see and then, the outcome of $\bar{R} \cup S$. So, this we right as the $\bar{R} \cup S$ and then, we take the intersection of it with R. So that means, the $R \cap (\bar{R} \cup S)$. So, this is the outcome of it.

And now let us further find the $R \cap S$. So, R intersection is coming out to be this. Now, we see that the $R \cap (\bar{R} \cup S) \neq R \cap S$. This clearly shows that the absorption of complement for intersection does not hold good when it comes to the fuzzy relations R and S.

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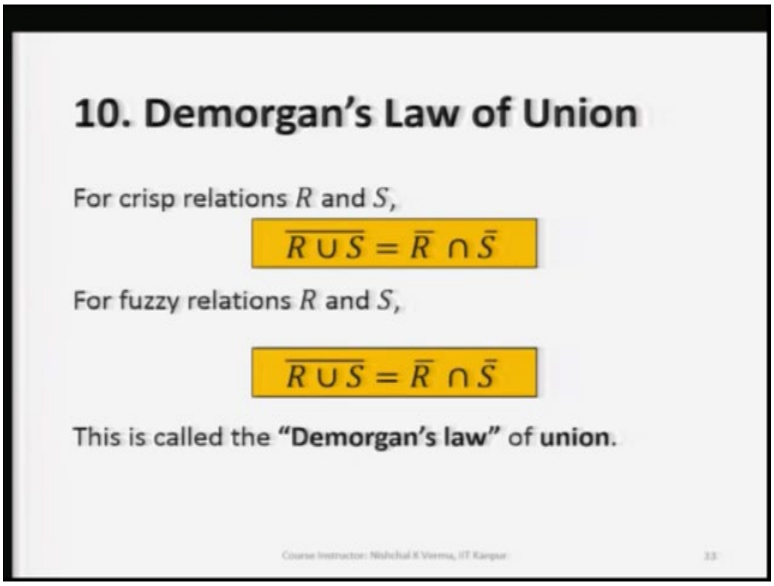

So, now let us go to another property which is Demorgan's law of union. So, we already know that when we take the $\overline{R \cup S}$, we get the $\bar{R} \cap \bar{S}$. So, when we take fuzzy relation sets R and S, this property also holds good for the fuzzy relation sets. That means, either we take fuzzy relations or we take crisp relations, for both the relations Demorgan's law of union is holding good.

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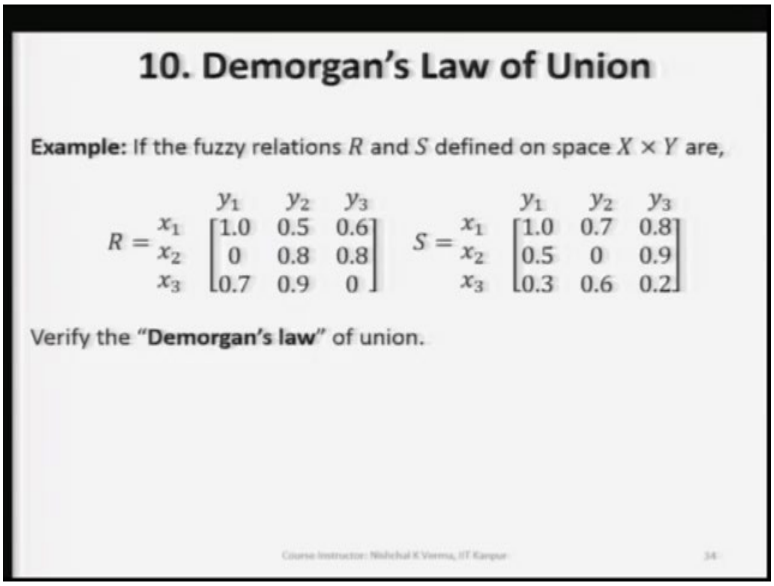

So, let us here also we take one example and we see as to how Demorgan's law of union is verified for fuzzy relations R and S.

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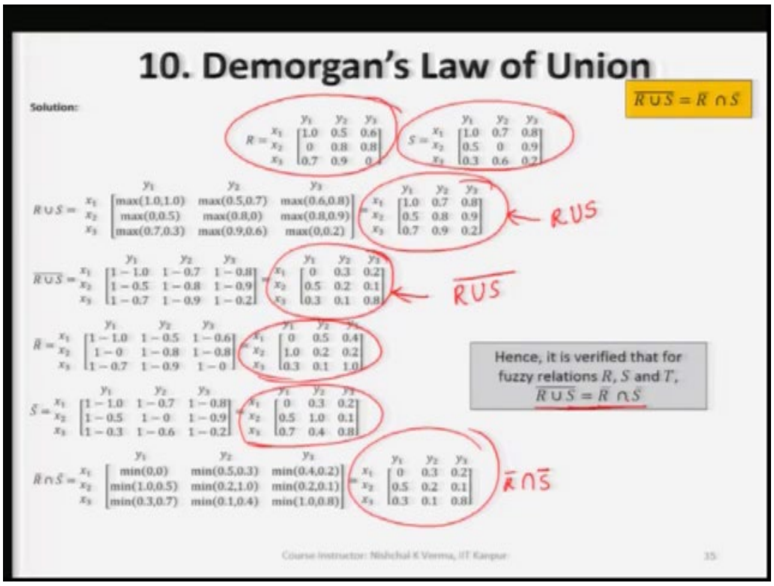

So, when we take the $R \cup S$ for this fuzzy relation set R and S. So, R and S we are getting like this. We use the union so that is why we take the we use max criteria and when we use max criteria, we get $R \cup S$ and then, when we take the complement of it, we are getting this outcome. So, this is nothing but the $\overline{R \cup S}$. Now, we take the complement of R which is this and then, we take the complement of S which is this. Now, when we take the $\bar{R} \cap \bar{S}$. So, we see that this is equal to the $\overline{R \cup S}$.

So, this way we can say that when we take two fuzzy relation sets R and S, we take the $\overline{R \cup S}$. And this is nothing, but this is equal to the $\bar{R} \cap \bar{S}$ which is here. So, Demorgan's law of union for fuzzy relation sets is holding good.

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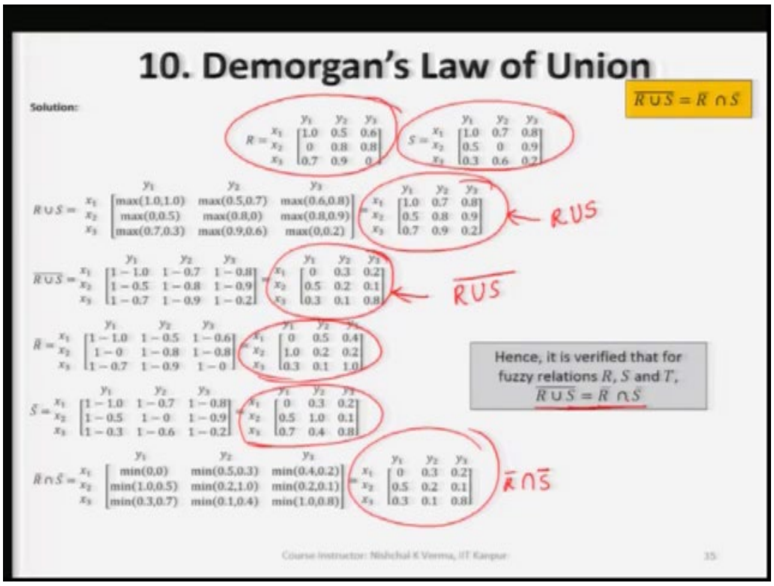

Now, let us verify the Demorgan's law of intersection. So, here also we see that if we take the $\overline{R \cap S} = \bar{R} \cup \bar{S}$ and this is holding good for the crisp relation sets as well as the fuzzy relation sets R and S. So, let us verify this also.

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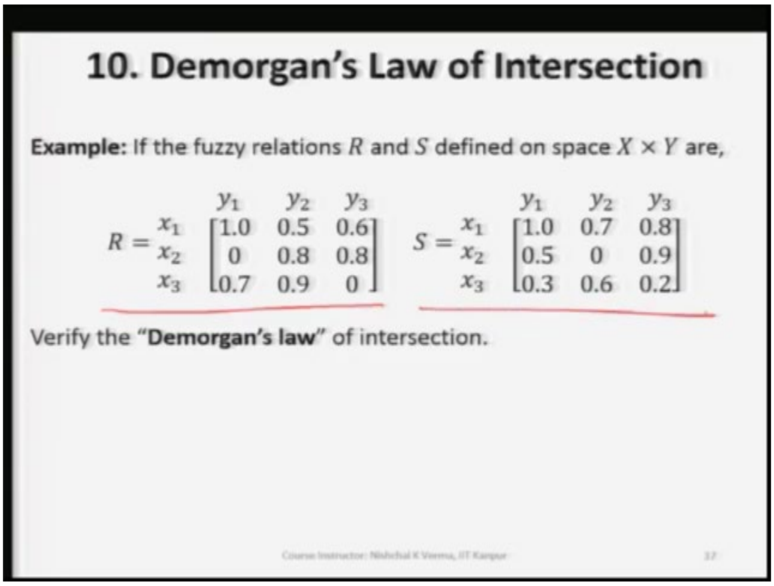

So, when we take two fuzzy relation sets here as R and S.

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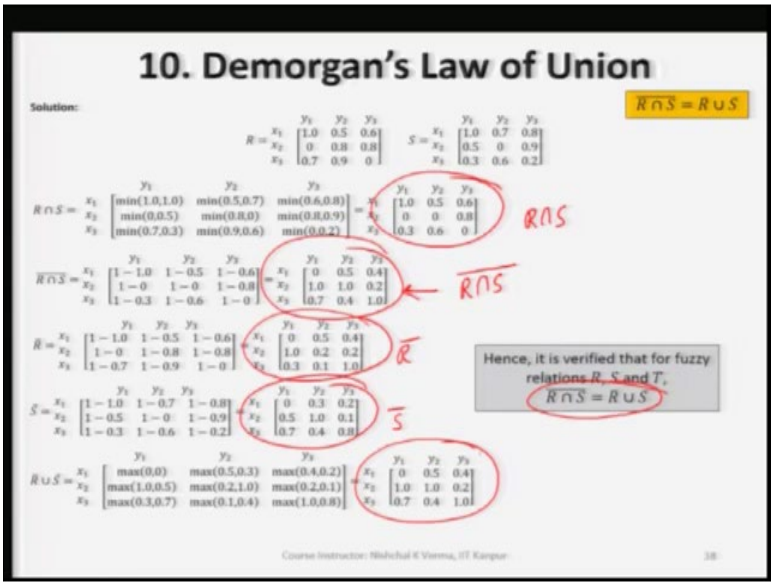

And let us find the $R \cap S$ first. So, this is the $R \cap S$ and when we take the complement of it here, so this is the outcome when we take the $\overline{R \cap S}$ and the $\bar{R}$ is here, $\bar{S}$ is here. So, we see that when we take the $\bar{R} \cup \bar{S}$, we get this as the outcome. So, we get the new fuzzy relation set as the outcome in form of the matrix here. So, this is nothing but this is equal to the $\overline{R \cap S}$. So, this way we can say that the Demorgan's law of union for fuzzy relation sets is holding good.

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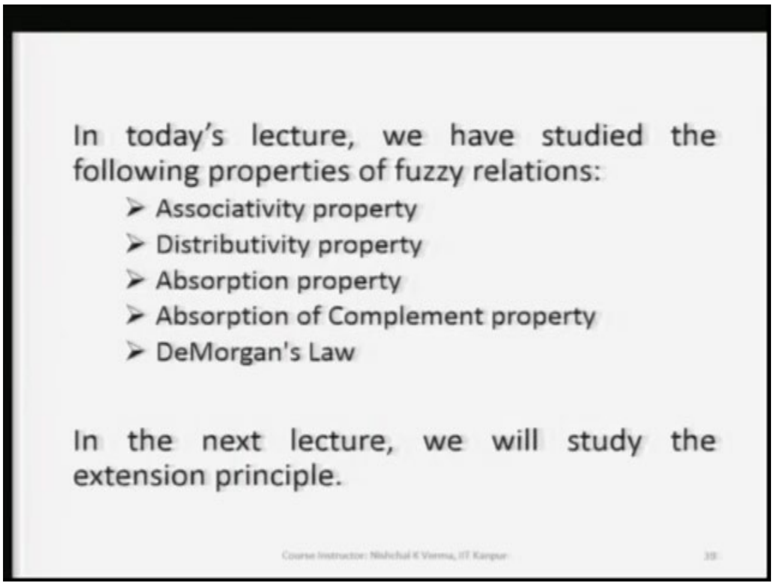

So, in today's lecture, we have studied the following properties of fuzzy relations; associativity property, distributivity property, absorption property, absorption of complement property and Demorgan's Law and this way we have seen as to how the fuzzy relation set holds good with respect to the properties that I have already discussed in this lecture. And in the next lecture, we will study the fuzzy extension principle.

Thank you.