Fuzzy Sets, Logic and Systems and Applications
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Lecture - 33
Properties of Fuzzy Relation

So, welcome to lecture number 33 of Fuzzy Sets, Logic and Systems and Applications in this lecture we will discuss the Properties of Fuzzy Relations.

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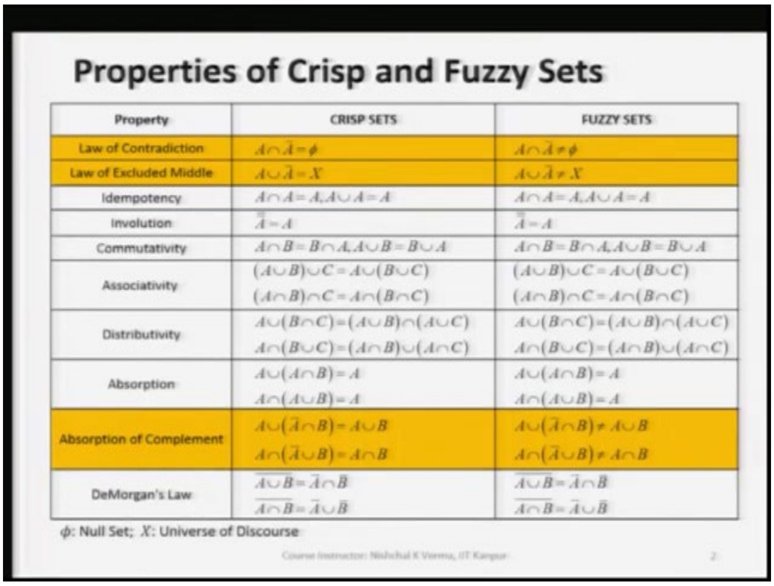

Property	CRISP SETS	FUZZY SETS
Law of Contradiction	$A\cap\bar{A}=\phi$	$A\cap\bar{A}\neq\phi$
Law of Excluded Middle	$A\cup\bar{A}=X$	$A\cup\bar{A}\neq X$
Idempotency	$A\cap A=A, A\cup A=A$	$A\cap A=A, A\cup A=A$
Involution	$\bar{\bar{A}}=A$	$\bar{\bar{A}}=A$
Commutativity	$A\cap B=B\cap A, A\cup B=B\cup A$	$A\cap B=B\cap A, A\cup B=B\cup A$
Associativity	$(A\cup B)\cup C=A\cup(B\cup C)$ $(A\cap B)\cap C=A\cap(B\cap C)$	$(A\cup B)\cup C=A\cup(B\cup C)$ $(A\cap B)\cap C=A\cap(B\cap C)$
Distributivity	$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$	$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
Absorption	$A\cup(A\cap B)=A$ $A\cap(A\cup B)=A$	$A\cup(A\cap B)=A$ $A\cap(A\cup B)=A$
Absorption of Complement	$A\cup(\bar{A}\cap B)=A\cup B$ $A\cap(\bar{A}\cup B)=A\cap B$	$A\cup(\bar{A}\cap B)\neq A\cup B$ $A\cap(\bar{A}\cup B)\neq A\cap B$
DeMorgan's Law	$\overline{A\cup B}=\bar{A}\cap\bar{B}$ $\overline{A\cap B}=\bar{A}\cup\bar{B}$	$\overline{A\cup B}=\bar{A}\cap\bar{B}$ $\overline{A\cap B}=\bar{A}\cup\bar{B}$

So, here let us have a table of all the properties that normally the crisp and fuzzy sets you know follow or un-follow. So, we have out of these properties that are listed here, we have a law of contradiction, law of excluded middle and absorption of complement these three properties are not followed by the fuzzy sets.

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Whereas, these properties are followed by the crisp sets. So, as I have already discussed this thing in my previous lectures all of these like a law of contradiction, law of excluded middle and absorption of complement do not hold good for fuzzy sets. Now, when it comes to the crisp relations and fuzzy relations. So, crisp since crisp relation is again a crisp set and fuzzy relation also a fuzzy set these properties will be followed by crisp relations and these properties will not be followed by the fuzzy relations.

So that means, when we talk of fuzzy relations. So, fuzzy relations do not hold good for law of contradiction, law of excluded middle, absorption of complement.

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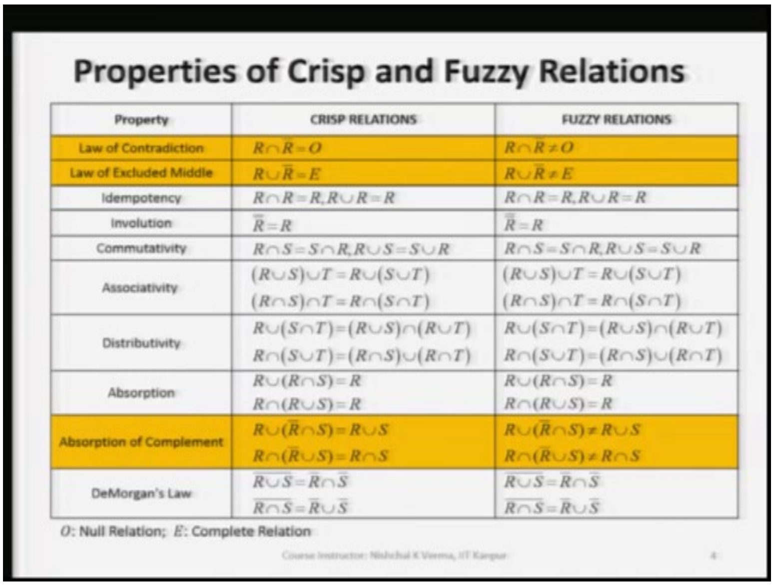

Property	CRISP RELATIONS	FUZZY RELATIONS
Law of Contradiction	$R \cap \bar{R} = O$	$R \cap \bar{R} \neq O$
Law of Excluded Middle	$R \cup \bar{R} = E$	$R \cup \bar{R} \neq E$
Idempotency	$R \cap R = R, R \cup R = R$	$R \cap R = R, R \cup R = R$
Involution	$\bar{\bar{R}} = R$	$\bar{\bar{R}} = R$
Commutativity	$R \cap S = S \cap R, R \cup S = S \cup R$	$R \cap S = S \cap R, R \cup S = S \cup R$
Associativity	$(R \cup S) \cup T = R \cup (S \cup T)$ $(R \cap S) \cap T = R \cap (S \cap T)$	$(R \cup S) \cup T = R \cup (S \cup T)$ $(R \cap S) \cap T = R \cap (S \cap T)$
Distributivity	$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$ $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$	$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$ $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$
Absorption	$R \cup (R \cap S) = R$ $R \cap (R \cup S) = R$	$R \cup (R \cap S) = R$ $R \cap (R \cup S) = R$
Absorption of Complement	$R \cup (\bar{R} \cap S) = R \cup S$ $R \cap (\bar{R} \cup S) = R \cap S$	$R \cup (\bar{R} \cap S) \neq R \cup S$ $R \cap (\bar{R} \cup S) \neq R \cap S$
DeMorgan's Law	$\overline{R \cup S} = \bar{R} \cap \bar{S}$ $\overline{R \cap S} = \bar{R} \cup \bar{S}$	$\overline{R \cup S} = \bar{R} \cap \bar{S}$ $\overline{R \cap S} = \bar{R} \cup \bar{S}$

So, we have a table here as we have already seen this table in case of you know when we have discussed crisp set and the fuzzy sets.

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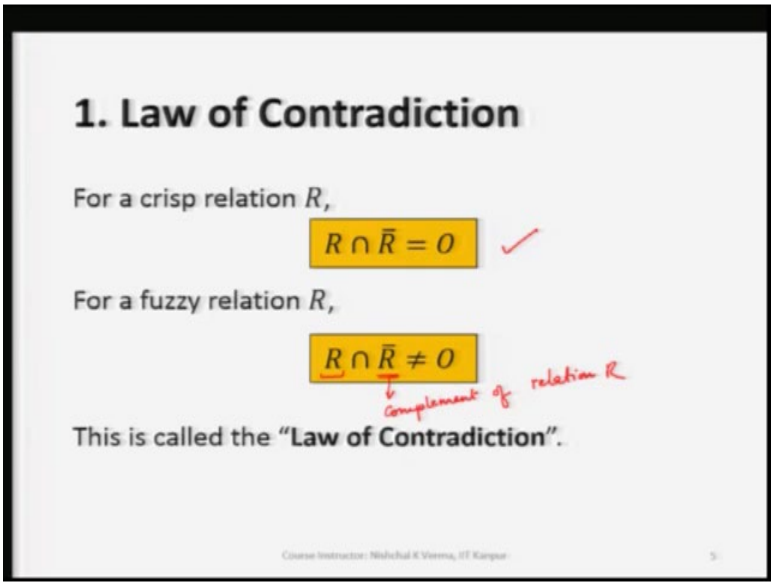

So, let us discuss all these properties one by one by taking the fuzzy relations and the crisp relations. So, for crisp relation let us say which is represented by R. So, $R \cap \bar{R}$, so, under any universe of discourse let us say X. So, here the intersection of $R \cap \bar{R} = 0$. This means when we take the intersection of any crisp relation and its complement this is going to be the null set, but when it comes to the fuzzy relation R. Let us say if we take some fuzzy relation R and then we take the intersection of $\bar{R}$. So, this is what is the complement of relation R of relation R.

So, if we take the intersection of these two it is not coming as the null set. So, this is called the law of contradiction. So, in nutshell basically this when we take a crisp relation we are getting the intersection of crisp relation, and its complement it is coming out to be a null set whereas, when we take fuzzy relation R set and take the intersection of it with it's complement it is not going to be a null set and this is called as the law of contradiction.

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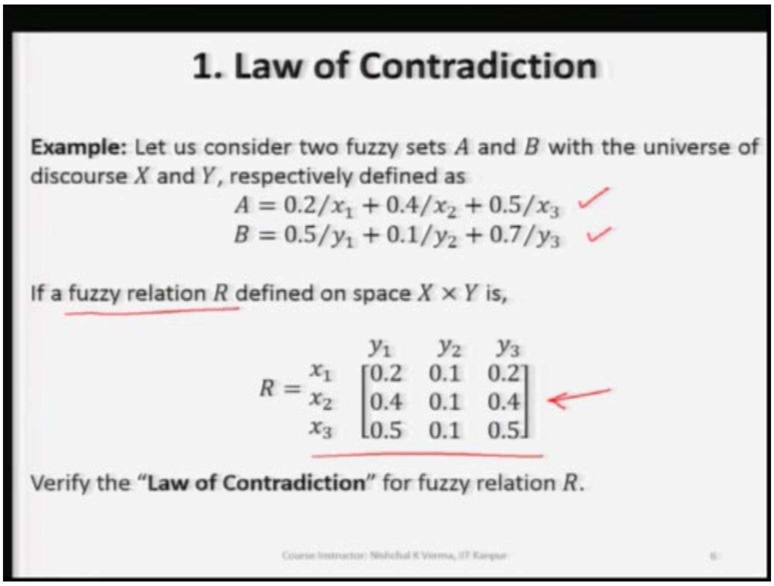

So, let us understand this again by taking one example here. So, if we have two fuzzy sets A and B both these fuzzy sets are discreet fuzzy sets. So, let us form a relation quickly. So, let R be a fuzzy relation which is represented by the relation matrix here. So, this R we have gotten just by taking the $A \times B$.

So, here we have the fuzzy relation and if we see that if we take the intersection of R and its complement, so, then it is not going to be the null matrix. So, let us first find the $\bar{R}$ that means, the complement of the relation matrix means the fuzzy relation R. So, complement of R is $\bar{R}$ which is here.

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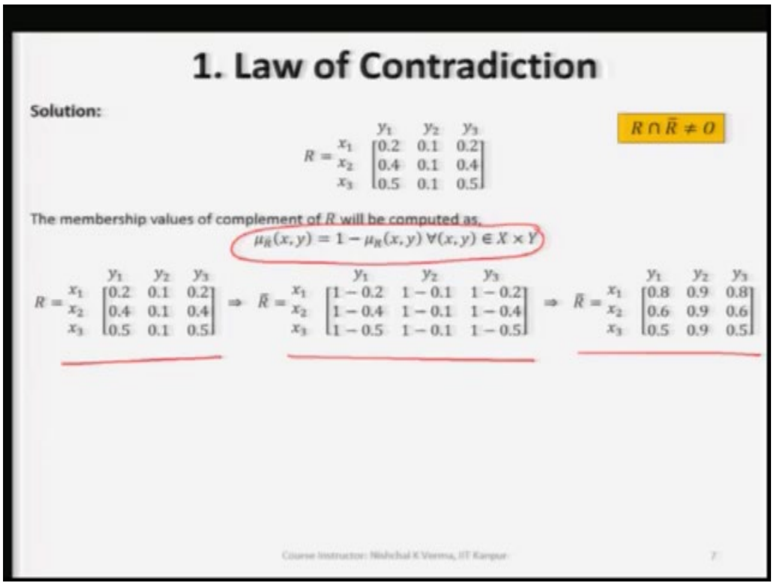

So, which we can get by just computing all its membership values by subtracting from 1. So, when we do that if we have R like this means the R the fuzzy relation matrix represented by here, R, then $\bar{R}$ is going to be this matrix which we are getting by subtracting it is all its elements from 1. So, you see $\bar{R}$ is this. So, now, here we have the complement of fuzzy relation R.

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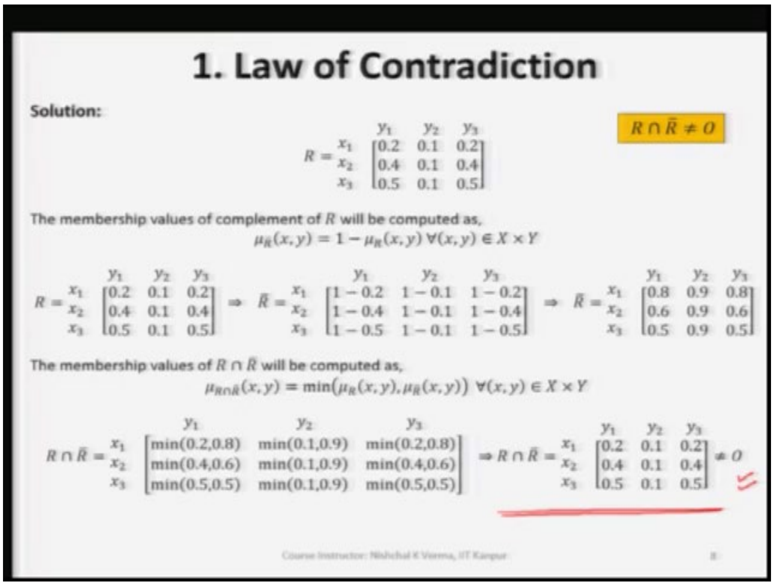

Now, let us take the $R \cap \bar{R}$. So, when we take the $R \cap \bar{R}$ which is here. So, as we already know that the basic intersection is computed by simply taking the min between all the corresponding membership values. So, if we compute the $R \cap \bar{R}$ and the basic min criteria is followed. So, $R \cap \bar{R}$ we are going to get here like this.

So, we get here fuzzy relation matrix or I would say we get as a result of $R \cap \bar{R}$ some fuzzy relation matrix which is not equal to 0. So, this way we can say that the law of contradiction is verified.

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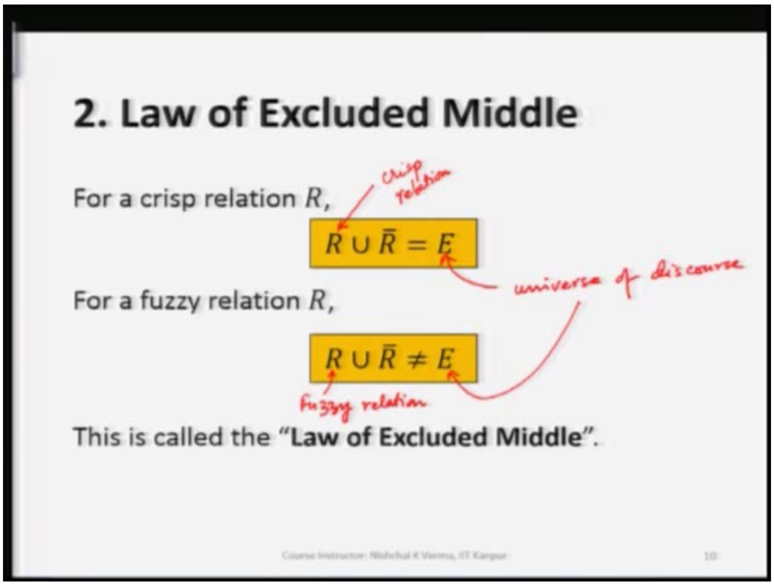

So, let us now come to the 2nd property which is law of excluded middle. So, in law of excluded middle as we all know that when we take a crisp relation we are getting if R is our crisp relation here, we are getting the universe of discourse when we take $R \cup \bar{R}$.

This means that if we take the union of crisp relation and its complement we are going to get E. E is nothing but the universe of discourse. So, this is true for the crisp relation. Now, when we take R as a fuzzy relation, so here if R is a fuzzy relation and if we take $R \cup \bar{R} \neq E$. So, this is called the law of excluded middle.

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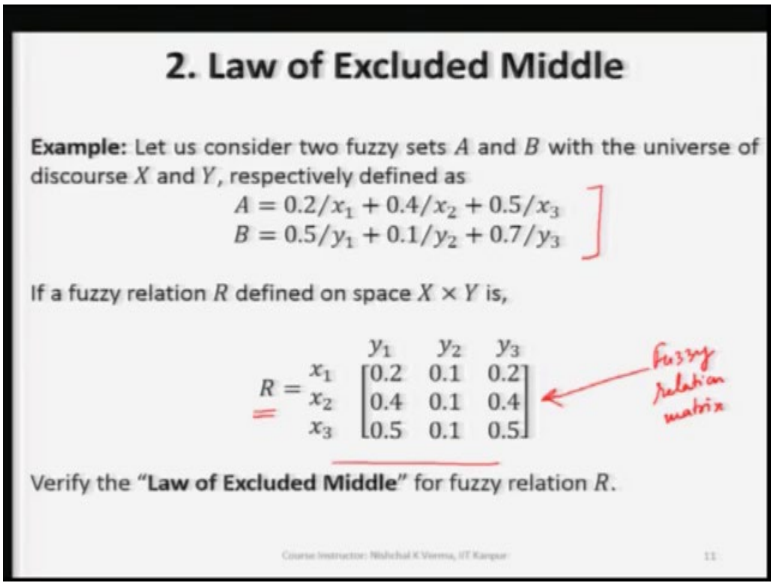

So, let us quickly understand this by taking an example here. So, if we take an example here where, we have two discrete fuzzy sets and we form a relation R out of this $A\ and\ B$ discrete fuzzy sets. So, we have a fuzzy relation matrix which is represented by R. So, this I can write here this is a fuzzy relation matrix alright. So, next is we have to here take the $R \cup \bar{R}$.

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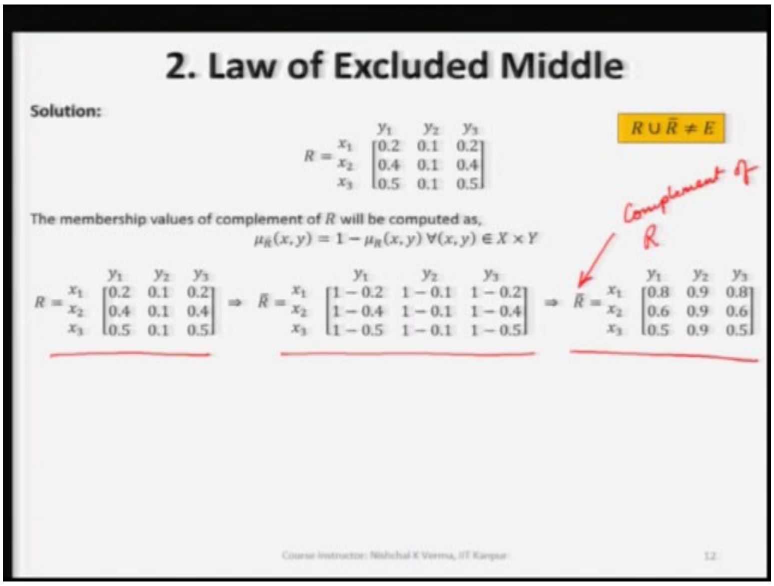

So, let us first find the $\bar{R}$. So, we know as to how we can find $\bar{R}$ that means, the complement of the fuzzy relation R. So, we have R here and then we find $\bar{R}$ by simply you know subtracting each element from 1 and of the matrix and then we get here $\bar{R}$. So, this is nothing but the complement of R is complement of R and R is nothing but the fuzzy relation.

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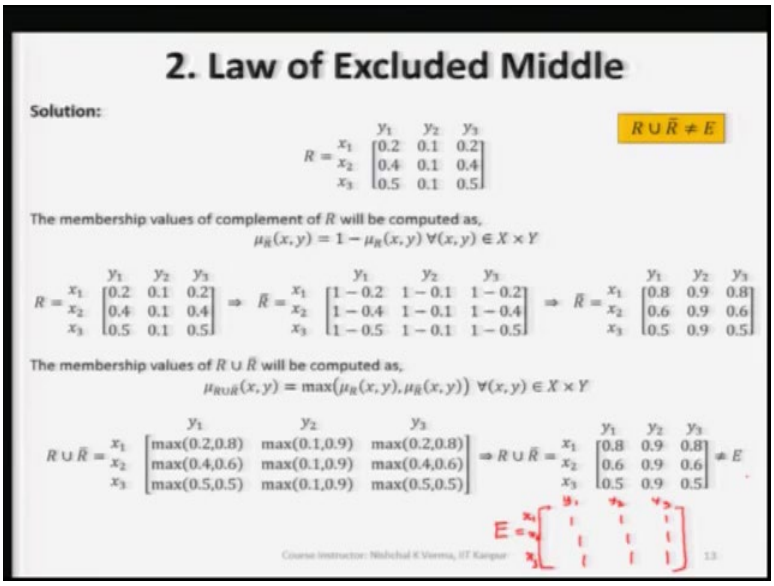

Now, when we take the max of the corresponding membership values so, we use here the basic union criteria we are getting here a fuzzy relation matrix. And we see that this is not equal to E. So, what does this mean when we say this is equal to E. So, this would have been equal to E, when all these element should have its values equal to 1. So, but here we see that not all the values of this fuzzy relation matrix is equal to 1. So, in case it would have been the E would have been like this.

So, this is our x_1, this our x_2, this is our x_3 and then y_1, y_2, y_3 if you write it like this. So, this would have been like this all the elements of the resulting fuzzy relation would have been 1. So, we can say $R \cup \bar{R} \neq E$.

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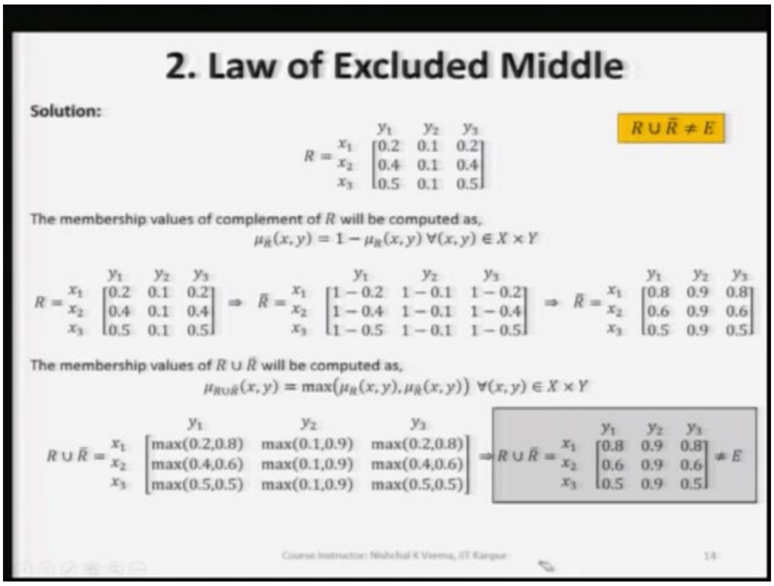

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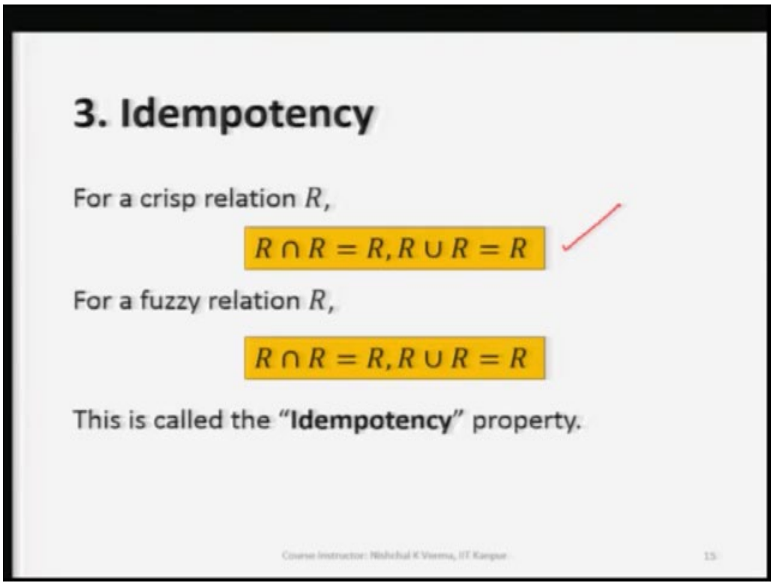

So, then we have here the 3^{rd} property which is idempotency. So, let us check this and see whether this is satisfied or not for the fuzzy relation. So, for crisp relation it is satisfied and when we have a fuzzy relation R. So, if we take the intersection or union with R. So, we see that this satisfied means this is coming out to be the same fuzzy relation.

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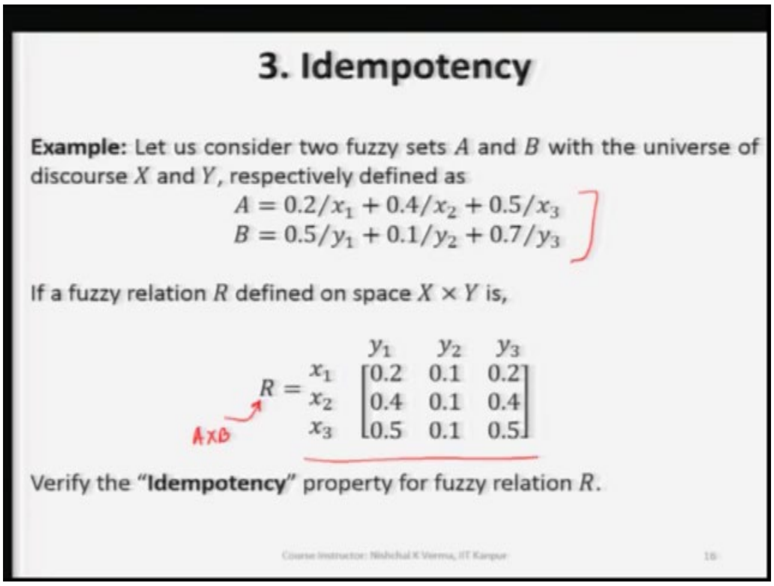

So, let us quickly check that by taking this example here. So, here also we take two discrete fuzzy sets and we find a relation a set from A and B, that means, $A \times B$ when we take this we have a fuzzy relation set here, which is represented in the form of a matrix, where the matrix elements are nothing but the belongingness the membership values.

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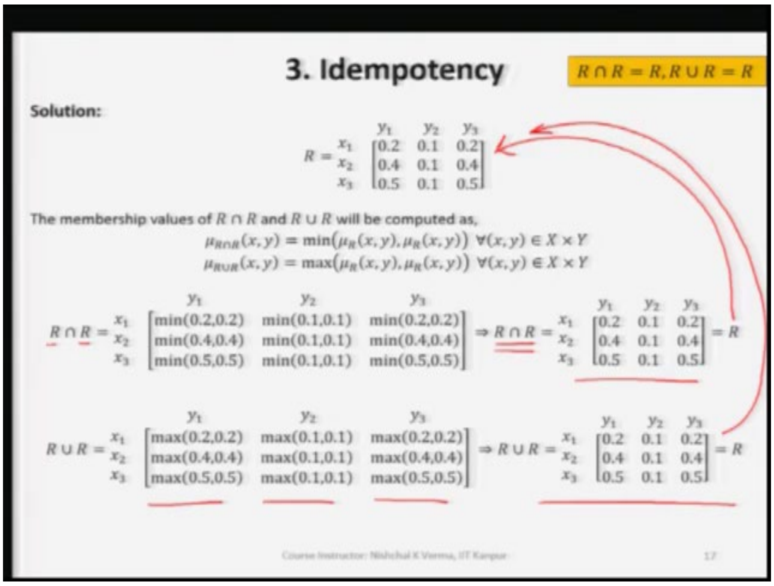

So, let us first have the intersection. So, we have the $R \cap R$. So, when we take intersection we all know that we take the min of the respective for all the resulting membership values which are nothing but the elements of the resulting relation matrix. So, $R \cap R$ is coming out to be this. So, we will be see that the resultant of $R \cap R$ is the same matrix with which we started. So, you see here that this matrix and this matrix both the matrix are same. Since this fuzzy relation matrix are same. So, we can say that $R \cap R = R$. So, this way this is satisfied.

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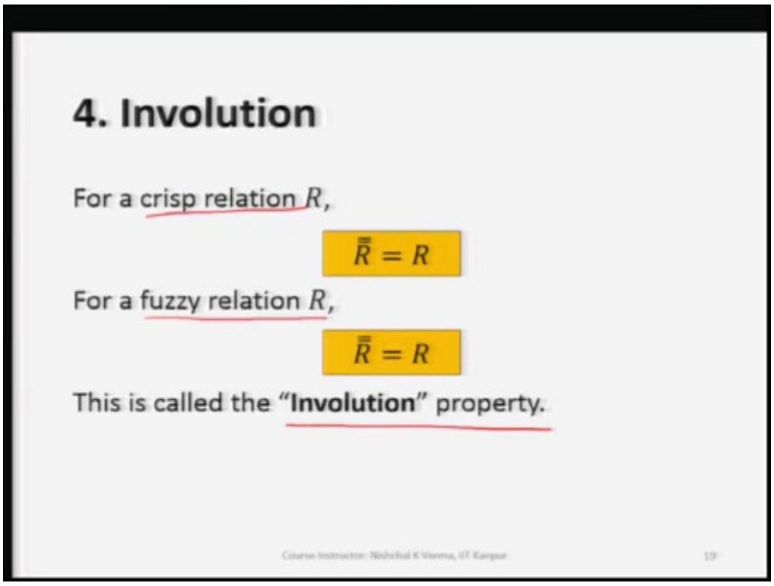

Now, let us take the union. So, when we take the union here $R \cup R$ for union we use max criteria when we apply the max criteria here you see, for computing the membership values with respect to the corresponding generic variable values. So, we find again the relation matrix which is nothing but R. So, what does this mean this means that whether we take the union or we take the intersection of the set R and the union of the same set means if we take the union of R and R.

We are going to get R or when we take the $R \cap R$ we are going to get the same set. So, this way we can say that the idempotency property is satisfied for fuzzy relation set.

Now, next comes the involution property. So, here if we have any crisp relation R and for crisp relation R if we take this complement twice, that means $\bar{\bar{R}}$.

So, when we take crisp relation R and if we take twice the, its complement we are going to get the same set with which we started means we are going to get $\bar{\bar{R}} = R$. And when it comes to the fuzzy relation set R here. So, if we take the complement twice here also we get the same set again means we get the double complement of fuzzy relations set R is equal to the fuzzy relation set again. So, this holds here for fuzzy relation also and this is called the involution property. So, let us verify this property for fuzzy relation R.

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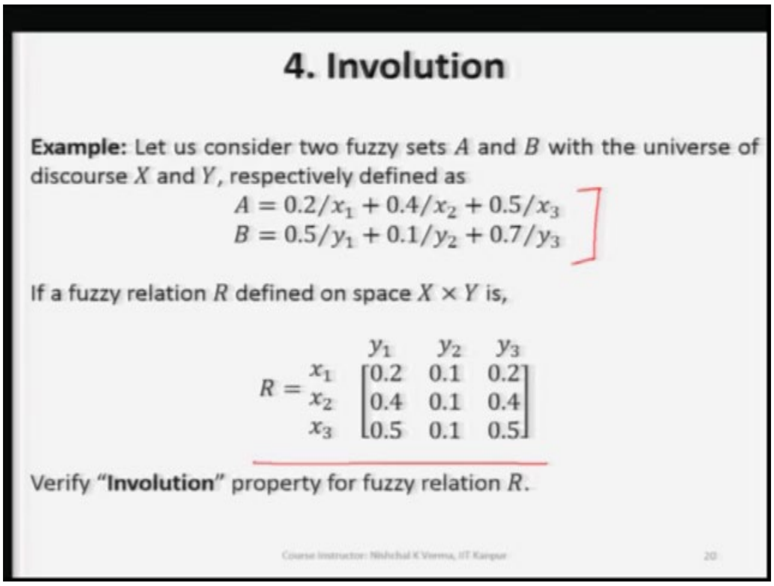

So, when we do that here with the same example as we have taken in the previous property. So, we again here we have with these two discreet fuzzy sets, we have the fuzzy relation set and with this when we go for R bar, so, if we have this as the fuzzy relation set and this is represented in the matrix form.

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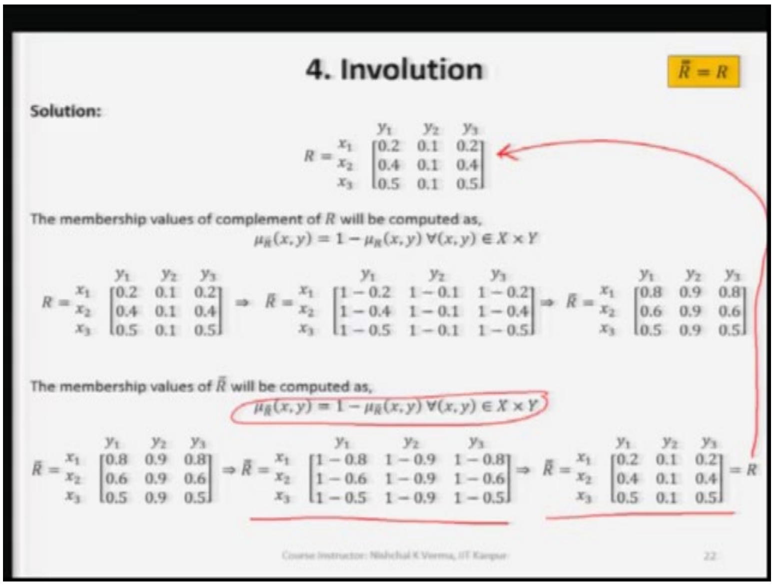

So, let us first get the $\bar{R}$; that means the $\bar{R}$. So, $\bar{R}$ is here and this as we already have done that each of the elements of this fuzzy relation matrix is found by subtracting its membership values from 1. So, this way we get $\bar{R}$ here. When we have R bar, now, let us find the complement of $\bar{R}$; that means $\bar{\bar{R}}$. So, $\bar{\bar{R}}$ with the same kind of subtraction, that means, following this criteria we get double complement of R and when we do that we are getting the fuzzy relations set here. And which is nothing but if we equate it if we see that if we compare this we see that we are getting the same set with which we started. Means we are getting the set R again means $\bar{\bar{R}}$ is going to give us R where R is the fuzzy relation matrix.

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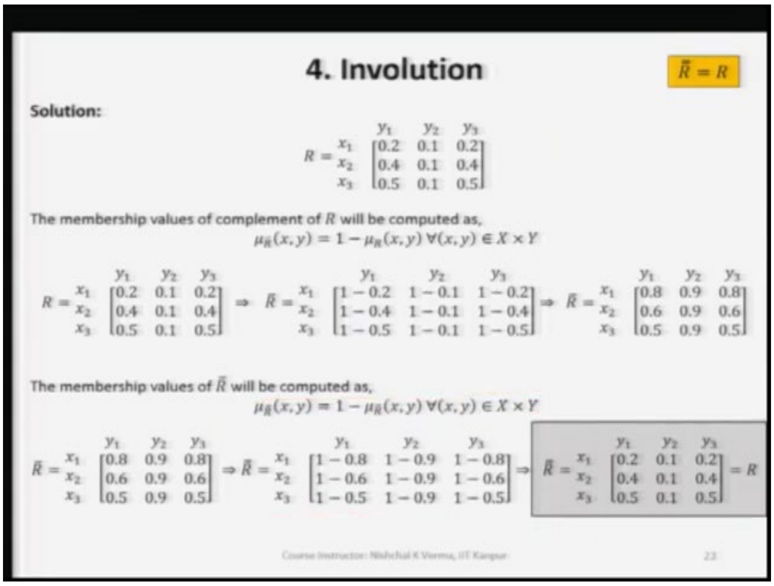

So, this way we can say that the involution property holds good for the fuzzy relation set.

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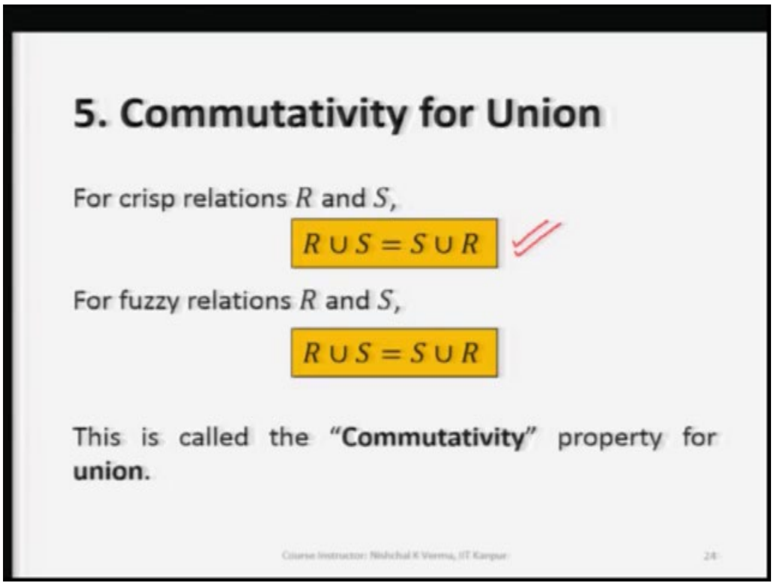

Now, let us check with the commutativity for union property. So, we all know that when we take a two crisp relations R and S and we interchange their positions means $R \cup S = S \cup R$.

So, this is where we see that this is commutative. So, crisp relations are commutative. Now, when it comes to the fuzzy relations R and S this is also commutative means if we have fuzzy relation set R and if we have another fuzzy set relation set S. So, we can write the $R \cup S = S \cup R$ it means that the commutativity for union holds good for fuzzy relation sets R and S.

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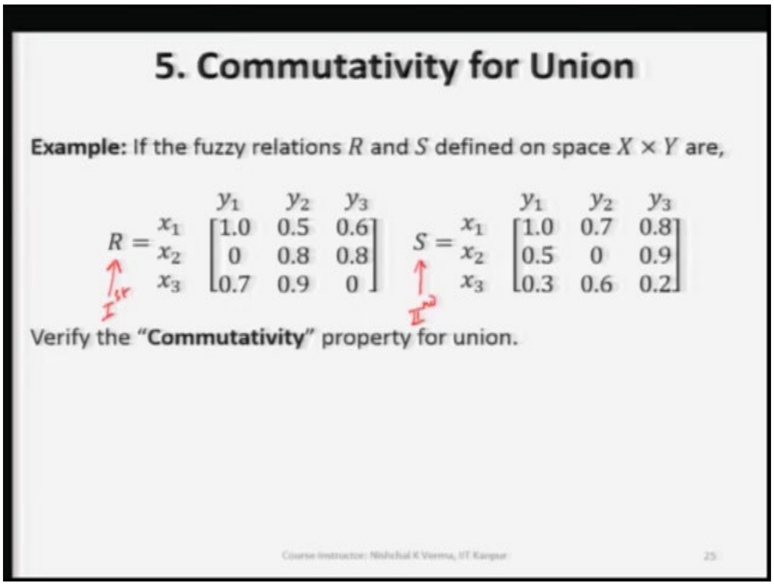

So, let us take an example to just verify this. So, here we have two relations sets this 1st fuzzy relation set R and the 2^{nd} fuzzy relation set is here 2^{nd} fuzzy relation set. So, we have two fuzzy relations sets.

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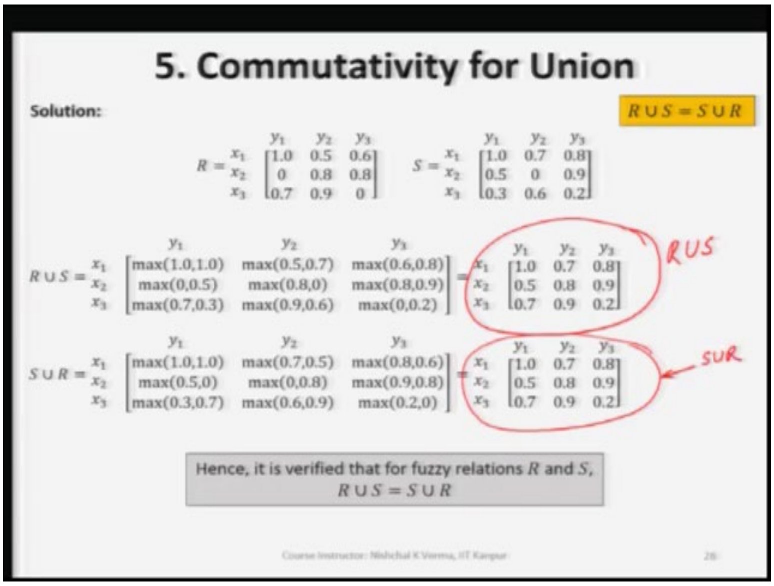

And let us now quickly take the union of these two sets means $R \cup S$. So, $R \cup S$ we get a matrix here like this and this matrix is giving us the elements like this 1,0.7, 0.8 0.5, 0.8, 0.9, 0.7, 0.9, 0.2. So, this is $R \cup S$. Now, let us find the $S \cup R$. So, here we get $S \cup R$ which is giving us a fuzzy relation set and if we compare these two we see that both of these sets remain the same. This means that we are going to get the same fuzzy set whether we take the $R \cup S$ or we take a $S \cup R$ both are same.

So, this way we can say that the commutativity for union for the fuzzy relation set is verified R holds good. So, commutativity property for union is verified is holding good for fuzzy relations R and S.

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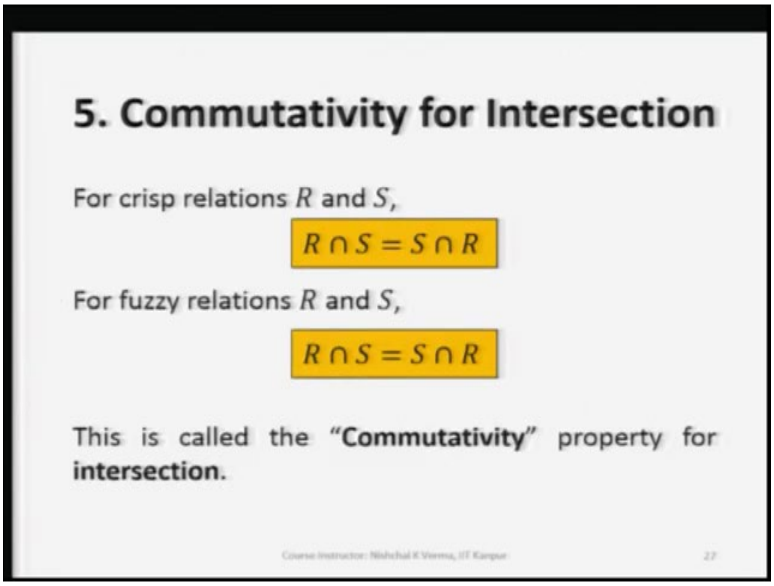

Now, let us check the commutativity property for intersection. So, when we take R and S again. So, instead of union let us check this for the intersection. So, let us quickly go ahead with R fuzzy relation set here this is R fuzzy relation set, this is S fuzzy relation set. Now, let us take the intersection of the 2.

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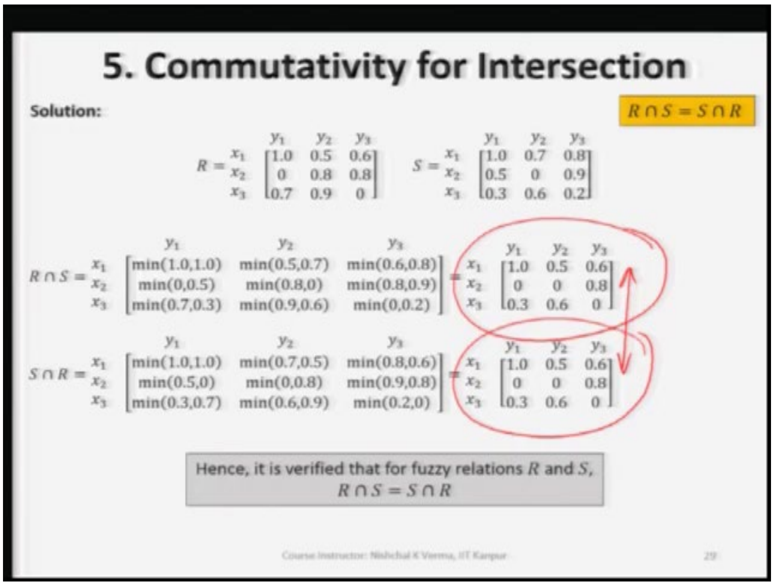

So, when we take intersection of the 2 we get here, this as the fuzzy relation set which is $R \cap S$. Now, let us take $S \cap R$. So, when we take $S \cap R$ what does this mean, this means that we take S first and then R thereafter. So, when we do that, we get here this fuzzy relation set as a result. So, when we compare these two outcomes we see that both the outcomes remain same. So, this way we can say we are getting the same fuzzy relation set. So, we can say that the commutativity property for intersection hold good for fuzzy relations sets R and S.

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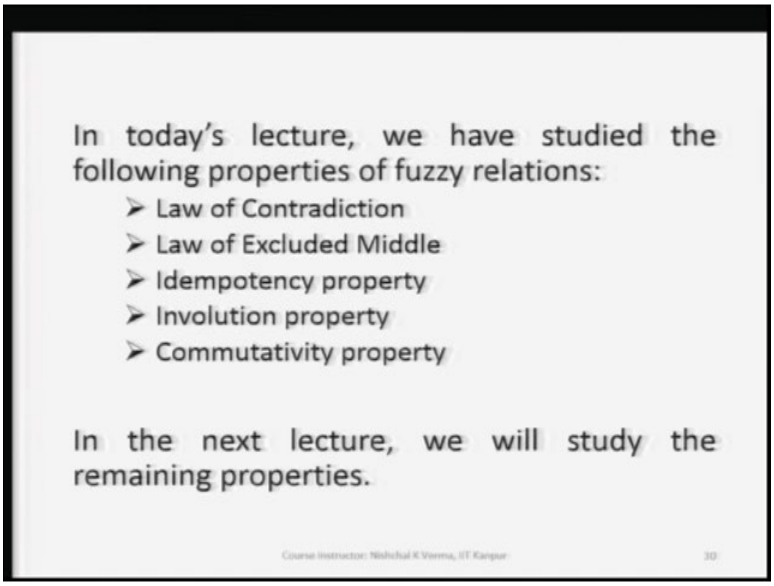

So, this way we have seen that in today's lecture, we have discussed, we have studied the following properties of fuzzy relations the law of contradiction, the law of excluded middle, idempotency property, involution property and the commutative property.

So, we will stop here and in the next lecture we will study the remaining properties which I have shown in this lecture in the previous slides.

Thank you.