Fuzzy Sets, Logic and Systems and Applications
Prof. Nishchal K. Verma
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 32
Cylindrical Extension of Fuzzy Set

So welcome to lecture number 32 of Fuzzy Sets, Logic and Systems and Applications in this lecture, we will discuss Cylindrical Extension of Fuzzy Sets. So, let us first take the normal fuzzy set which is defined in the universe of discourse X.

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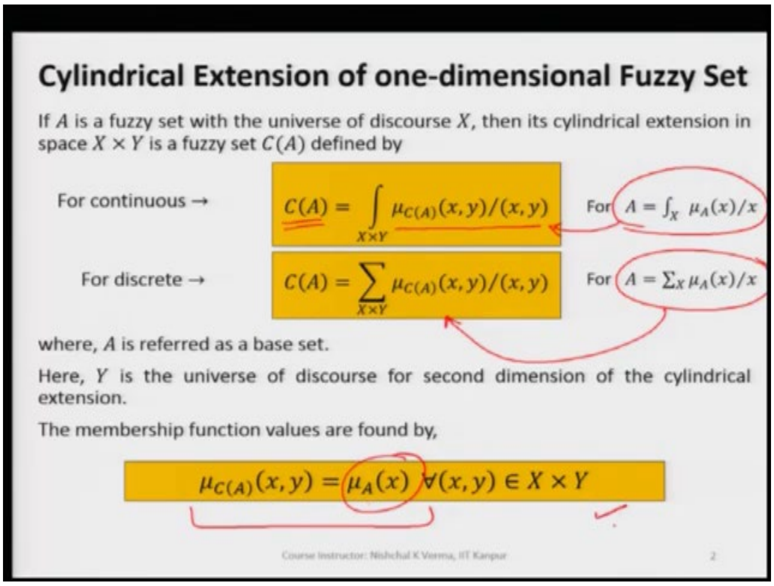

So, if we have a fuzzy set A let us see as to how we can find the cylindrical extension of this fuzzy set. If we have a fuzzy set A is defined here as we have already seen in either continuous domain or discrete domain. So, what is a cylindrical extension of fuzzy set A? So, if it is continuous fuzzy set A we see that the cylindrical extension of fuzzy set is defined by $C(A)$ this is the cylindrical extension here which is defined by $C(A)$, and $C(A)$ is nothing but it is equal to the fuzzy set which is in continuous domain and this is expressed in the increased dimension or increased universe of discourse.

So, here in this case we see that the $C(A)$ which is cylindrical extension of A is defined in two universe of discourse so the original fuzzy set that was taken here is A is defined in the universe of discourse X. When we take cylindrical extension of A you see we have added one more universe of discourse that is Y. So, or in other words we say that we have added one more generic variable y. So, the universe of discourse basically becomes capital $X \times Y$.

And since this is continuous fuzzy set so, we represent this fuzzy set with the integral sign and the membership values here you see as $\mu_{C(A)}(x, y)$ and these membership values are corresponding to (x, y) points, which is represented of course, in the universe of discourse $X \times Y$. So, this way we have increased the dimension we have added one more generic variable here and that is how the extension happens and this is why we call the cylindrical extension of fuzzy set A. So, cylindrical extension of fuzzy set A is $C(A)$ which is defined by

$$\int_{X \times Y} \mu_{C(A)}(x, y)/(x, y)$$

And when A is a discrete fuzzy set for which we are taking the cylindrical extension. So, for this we are getting

$$C(A) = \sum_{X \times Y} \mu_{C(A)}(x, y)/(x, y)$$

So, this is very simple and we can clearly understand as to how the dimension, extension happens. And by the name itself it is very clear that the extension is happening the one more dimension is happening every time whenever we use the cylindrical extension over any fuzzy set.

So, if we will look at the fuzzy set A that we have taken it has only one generic variable in both the cases either discrete or continuous, but when we have taken the cylindrical extension we have added one more generic variable that is y and it is needless to say that the initially A had the universe of discourse as capital X, but after the cylindrical extension the universe of discourse becomes $X \times Y$ as you know the space of the universe of discourse.

So, it is very interesting to note here that the increased membership values corresponding to the increased dimension as to how we can get the values of membership in this new generic variable or the combination of generic variables.

So, we see that its very simple here what is happening is that we take the original membership value which is $\mu_A(x)$ which was earlier in this generic variable space capital X and this is going to be equal to the new membership value which is corresponding to the point (x, y). So, $\mu_{C(A)}(x, y)$ is nothing but the original membership value corresponding to x. So, here this does not depend upon y. So, no matter what dimension is added we just write this and original membership value is retained this is called a cylindrical extension because (x, y) is $\mu_A(x)$.

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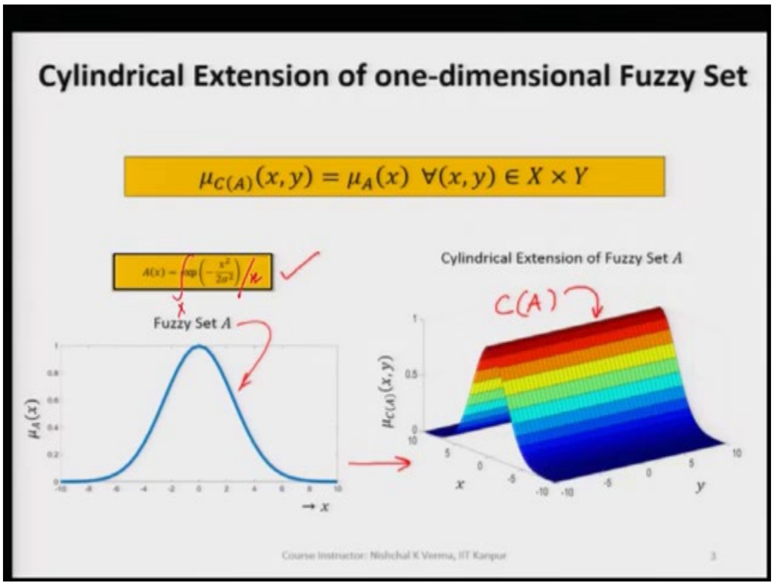

So, this is very important to note we'll understand this better by one example here. If we have a fuzzy set A which is defined by a Gaussian membership function. So, this fuzzy set A we see that it is here fuzzy set A. And the cylindrical extension of this fuzzy set A will look like this here this what is this cylindrical extension of the original fuzzy set A. So, $C(A)$ which is nothing but the $C(A)$ and this looks like this. So, we see that we have the generic variable only x and when we extend it cylindrically y gets extended. So, here we have taken the fuzzy set A which is with the generic variable x.

So, the extension is happening in the other dimension or we are adding one more dimension that is y.

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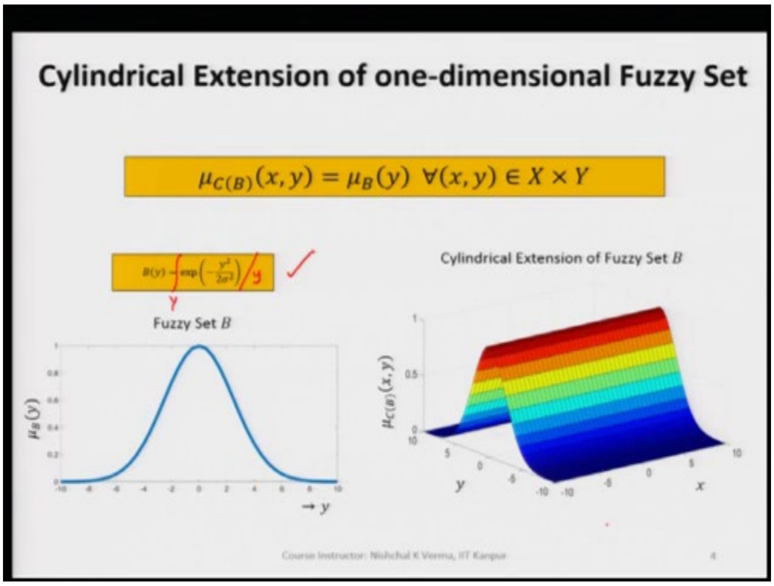

But if we take fuzzy set B which is here and this fuzzy set is defined in the universe of discourse Y. So, we see that when we add one more dimension in the x direction we see that the cylindrical extension happens when we take the cylindrical extension of the fuzzy set B. So, it is very simple to understand and in nutshell we increase the generic variables in the cylindrical extension of fuzzy sets.

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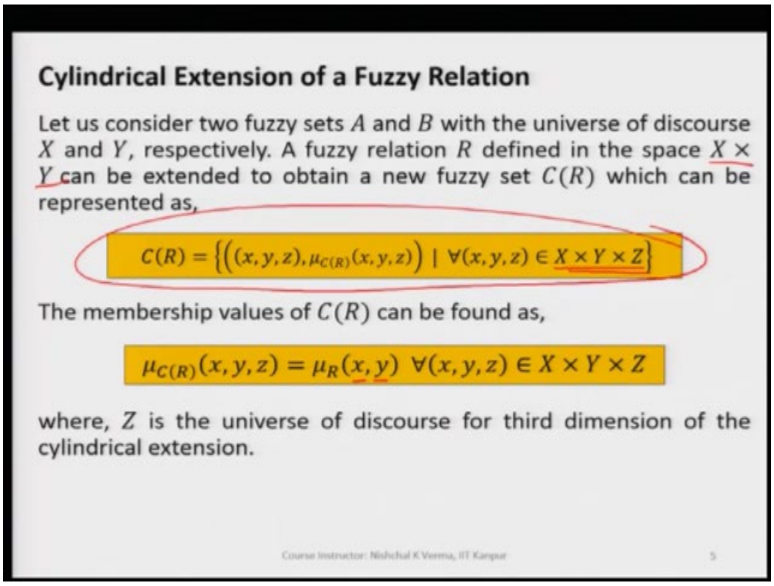

So, now if we have the fuzzy relation sets then what will happen? So, on the same lines as we have discussed the cylindrical extension of fuzzy set A, B so, on the same lines we can apply the cylindrical extension on the relation fuzzy set as well. So, we can define a relation fuzzy set by R which is let's say in the universe of discourse $X \times Y$ which is here. So, R is defined in the universe of discourse $X \times Y$. R is a relation set this has generic variable variables x, y and if we are interested in extending it the dimension extending the generic variable let us say z.

So, we can go for it and the cylindrical extension of R can be expressed by the $C\ (R)$ which is here and then the same relation fuzzy set is now I expressed in the three generic variable and the universe of discourse here will be capital $X \times Y \times Z$, here this is defined by this expression. So, this $C\ (R)$ is nothing but the cylindrical extension of R and R as we already know that this is defined in the $X \times Y$ universe of discourse.

So, here also as we have seen in the previous case where we have taken A fuzzy set and B fuzzy sets, so here since we are taking R. So, R is already defined with two generic variables like x and y all the membership values will be based on the points in the space (x, y). So now, when we are extending the dimension let us say z after taking the $C(R)$ mu cylindrical extension of R. So, here the membership values will be now dependent on the three points three generic values.

So, this $\mu_{C(R)}(x, y, z)$ means the membership value at any point in the space (x, y, z) all these are generic values. So, here this will be equal to $\mu_R(x, y)$ and of course, this is needless to say that all these x, y, z generic variable values will lie in the universe of discourse $X \times Y \times Z$. So, here by taking the cylindrical extension of a fuzzy relation we are adding one more dimension. And please note that by doing this we can increase any number of dimensions. So, this is a cylindrical extension of R which is increasing one dimension. Now, further we can go ahead we can increase one more dimension if it is needed.

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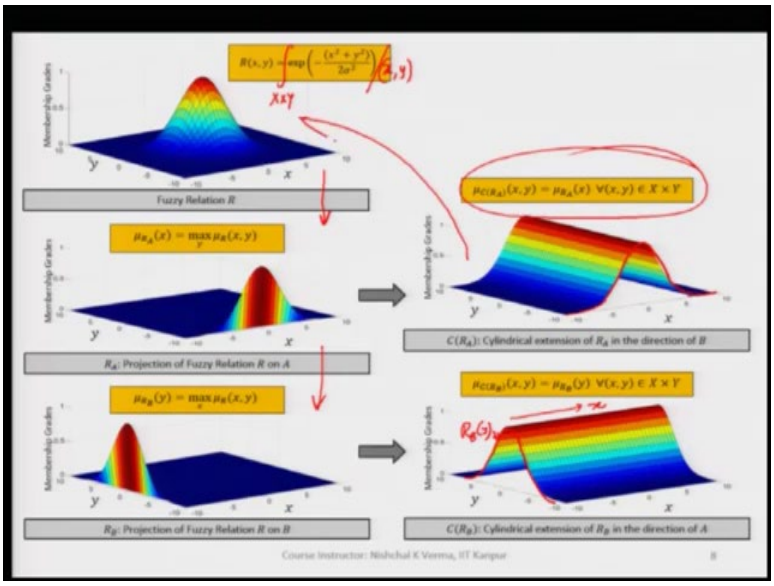

So, it is very simple to understand. Here if we take an example where we have a relation continuous fuzzy set which is defined by this expression here and we have universe of discourse $X \times Y$ and all these corresponding membership values are defined by the

$$R(x, y) = \int_{X \times Y} exp^{\left(-\frac{(x^2+y^2)}{2\sigma^2}\right)} \Big/ (x, y)$$

So, here this two dimensional membership function and here since we have one more dimension for membership values. So, we have three dimension, so that we see very clearly that we have a surface the Gaussian surface being shown.

And this is defined in $X \times Y$ universe of discourse. So, here we are taking the fuzzy relation in the beginning and now we first try to find the projection of this fuzzy relation on A. So, we have already studied the projection of fuzzy relation on A. So, if we do that we see that we apply this formula and by doing this we can very easily find where the projection of this fuzzy relation set on A and let me just remind you that this projection of any fuzzy set reduces the dimension. So, it is eliminating the dimension. So, when we say dimension it means the generic variable value with which the fuzzy set is defined.

So, in this case R is originally defined with generic variables (x, y) and when we take the projection of R on A it means we are eliminating the y generic variable and retaining the x generic variable. So, you can see here as after taking the projection of R on A the resulting fuzzy set will look like this and then similarly, we can go for the projection of the original fuzzy relation here on B. So, when we do that, obviously B fuzzy set is defined in the space of the generic variable y. So, here when we project the relation fuzzy set on B. So, x is eliminated means the y generic variable is retained.

So, this way we see that we are going to get this fuzzy set is a reduced dimensional fuzzy set as a result. So, originally we had started with the fuzzy set with two generic variables (x, y) and when we take the projection on A we are only retaining x as the generic variable we are eliminating y. And similarly, when we are taking the projection on B so, we are only retaining y generic variable and x we are eliminating, but in both the cases here we are eliminating one generic variables. So, this means that we are taking projection of fuzzy set whether it is a relation fuzzy set or any fuzzy set the dimension is getting reduced. The generic variable is getting reduced.

Now, since we have started with fuzzy relation set R which is a multidimensional multi generic variable fuzzy set and after taking projection we have reduced the dimension. Now, if we are further interested in increasing the dimension then we have already seen that we can take a cylindrical extension of the fuzzy sets to increase the dimension, but in this exercise if we do both we are not going to get the original fuzzy set, which with which we started and we took projection and then we took cylindrical extension and during this process we are not going to retain the original fuzzy set.

So here, but in many cases we need to do that because we are many cases for processing of the fuzzy sets for you know inferencing we are interested in projection and cylindrical extension, but not simultaneously. So, here the fuzzy set that we have got after taking the projection R_A and if we are let's say increasing the dimension by taking the cylindrical extension. So, you see how the resulting fuzzy set will look like. So, we had here only the original A fuzzy set here. And now you see that we are extending towards y. So, this y dimension is getting extended. So, that is how it is called cylindrical extension.

And we used this formula for extending the dimension. So, mu $C_{R_A}(x, y)$ all these mu values of the cylindrically extended fuzzy set is going to be equal to the $\mu_{R_A}(x)$ and this is for all (x, y) belonging into the universe of discourse $X \times Y$. So, similarly you see here I can just over write this here just to make sure that what was R_B earlier. So, this was the R_B which was defined in the universe of discourse Y only. And here is the extension cylindrical extension in the direction X. So, we can clearly see that whenever we increase the dimension we see that it is the extension is here the cylindrical means whatever membership value that we have for the generic variable let us say the same value is going to be obtained for membership value (x, y) as well.

So, we can clearly understand here that how the cylindrical extension will look like and also before that we see the projection of a multidimensional fuzzy set is going to look like. So, we have done a very interesting thing here that we took projection on A and projection on B and then we further tried to extend the dimension. So, we clearly see that we are not going to get the original relationship set with which we started. So, this is what is very interesting to note, but what we gain here is that we are able to get the dimension extended.

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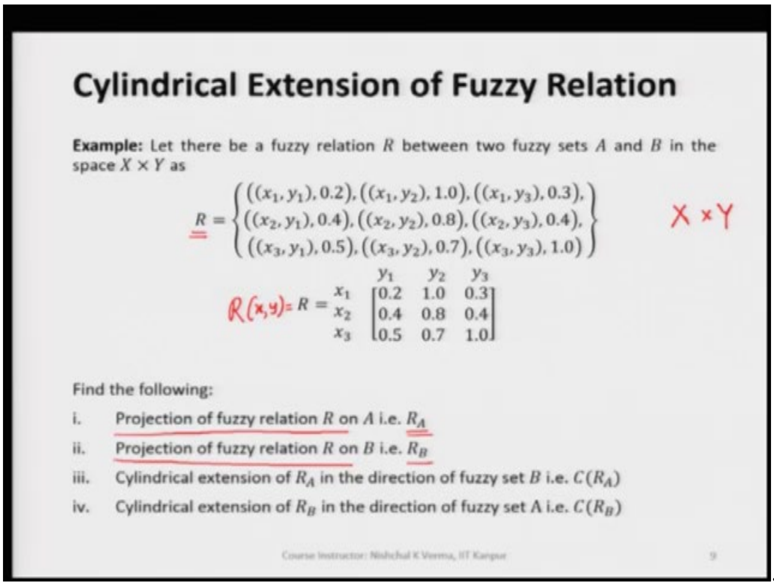

Now, this was the understanding of cylindrical extension with the continuous relation fuzzy set. Now, let us take a discrete relation fuzzy set. So, with discrete fuzzy set we have the let us say we have a fuzzy set R which is a relation fuzzy set and again this is defined in the universe of discourse $X \times Y$. So, of course, we can represent this discrete fuzzy set in the matrix form. So, we have a relation matrix here R and this R, I can just write that this is dependent on x and y. So, we see that the we have all the column values are with respect to x_1, x_2, x_3 and all the row values are with respect to y_1, y_2, y_3.

And now let us take this fuzzy set this relation fuzzy set and find the projection of this fuzzy set on A, so that means, when we take projection of R_A on A we call this as R_A. And similarly projection of fuzzy relation R on B we call this as R_B, and then finally, the cylindrical extensions of both R_A and R_B. So, let us now go and see with this discrete relation fuzzy set as to how we get all these projections and the cylindrical extensions.

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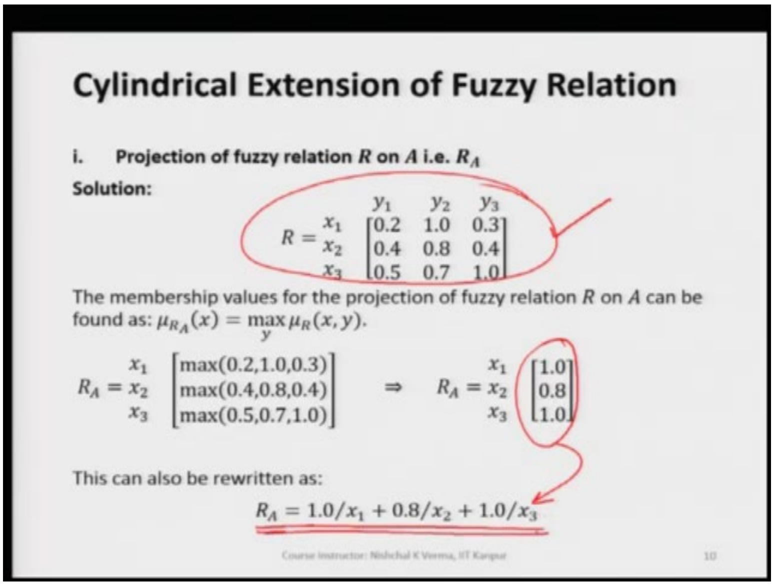

So, here we have the relation fuzzy set that has been given to us and if we apply the formula for finding the membership values for the resulting fuzzy set after taking the projection on A. So, the formula is here. So, as I already mentioned that when we take the projection we are reducing the dimension. So, you see that we take the maximum of all the membership values over the generic variable value which is going to be eliminated. So, here since we are taking the projection of R_A it means the A which is defined with the generic variable x. So, it means we are eliminating y generic variable here.

So, when we are taking y generic variable here it means we take all the for corresponding to the generic variable value we take max and then that is how we eliminate the y dimension you see here. So, we have original relation matrix. Now, we take the across all the rows we take the maximum of all the elements. So, corresponding to x_1 we get the membership value here after taking $\max(0.2, 1.0,0.3)$. So, the membership value corresponding x_1 point we are getting 1.

And then similarly x_2 we are getting 0.8 and corresponding x_3 we are getting 1. So, this way we see that we are reducing the dimension y and corresponding to x we are getting these as the membership values and which is written here in the expression of the fuzzy set. So, R_A will be nothing but $1\ /x_1$ which is the corresponding generic variable value. And similarly plus $0.8\ /x_2$ which is again the corresponding generic variable value corresponding to x_2 and similarly $1\ /x_3$ means 1 is the generic variable value corresponding to x_3. So, this way we have got here R_A.

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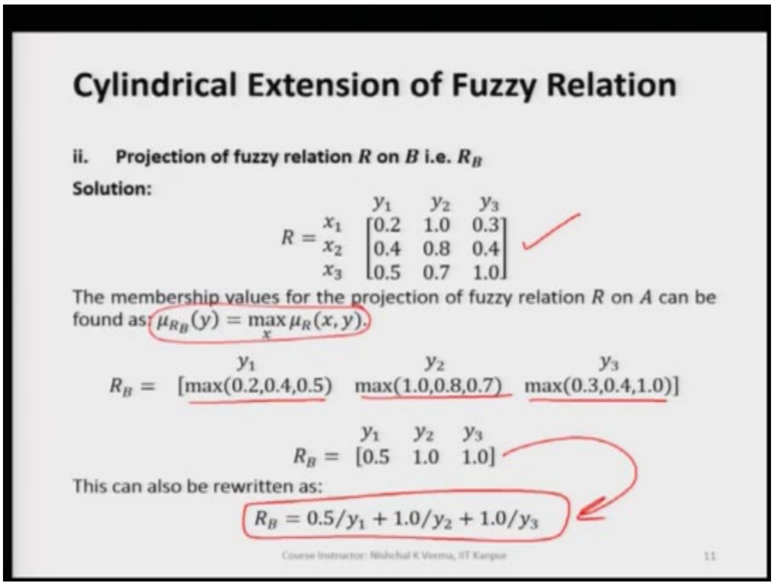

Now, on the same lines we get here R_B and we can start from here the R is in terms of x and y generic variables and when we use this criteria, when we use this expression for finding the membership value in the projection we take max over x. So, when we take the max over x we see that the R_B is going to be $\max(0.2, 0.4, 0.5)$ here with respect to the y_1 value and similarly max we are getting as you see max corresponding to y_2 we are getting $\max(1, 0.8, 0.7)$.

And then y_3 with reference to y_3 we are getting $\max(0.3, 0.4,1)$. So, this way we are getting 0.5 1.0 1.0 as the membership values corresponding to the generic variable values y_1, y_2, y_3 and this way we can quickly get the expression of the fuzzy set here for R_B which is the projected fuzzy set. Now, when we have reduced the dimensions now as I already mentioned that we can extend the dimension again.

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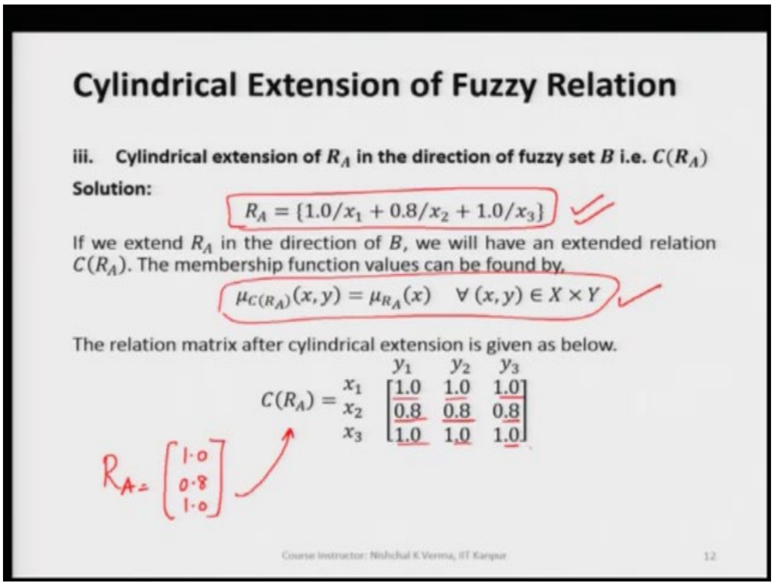

So, but that is not always necessary here since we are understanding the concept we are now adding the dimension on the projected fuzzy set. So, cylindrical extension of R_A what it is going to result we see here. So, we have here the R_A which is which we have got after the projection of R_A. So, R_A is this and then when we apply the formula for the cylindrical extension here that means, mu $C_{R_A}(x,y)$ is going to be $\mu_{R_A}(x)\forall(x,y) \in X \times Y$.

So, this way we see that when we apply this we keep adding the y_1 values to y_1 values to y_2, y_3 and so on. If there are many other points so, we see that this way we are adding the dimension and whatever was the original dimension, that means, we had in R_A we had here if I represent it like this the R_A fuzzy set in matrix form it was simply a column matrix. It was like this.

And now, when we are taking the cylindrical extension, what is happening here is that this one is getting added like for x_1 we had x_1, y_1 we have 1 and similarly x_1, y_2 will also be 1, x_1, y_3 will also be 1 which we can see here and similarly for all other elements we extend so, $0.8, 0.8, 0.8, 1, 1, 1$. So, the cylindrically extended matrix; cylindrically extended relation matrix will have either the rows or columns you know duplicated. So, that is the one thing that we just by looking at that we can say that this fuzzy set is out of the cylindrical extension.

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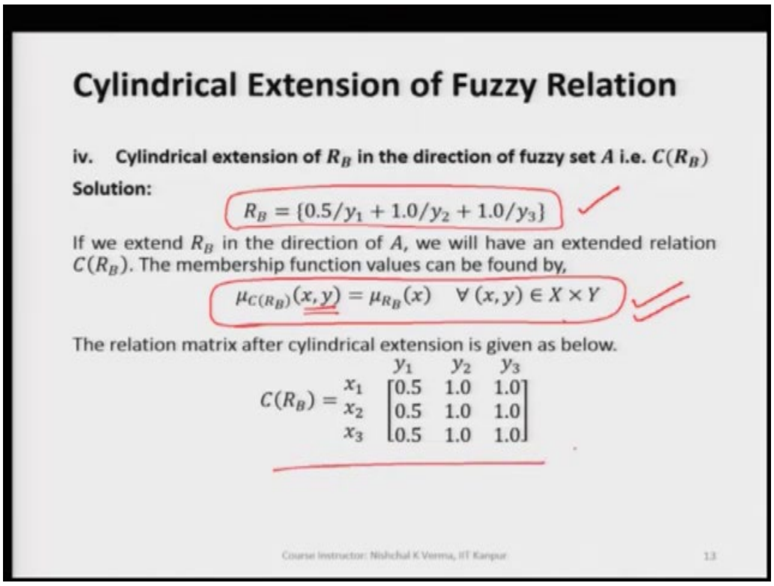

Similarly, now as we have got the expression of the fuzzy set or the relation matrix or the matrix representation for fuzzy set C_{R_A} we have C_{R_B} also this we are getting when R_B fuzzy set which we have got out of the projecting relation fuzzy set on B. So, R_B we have got and R_B is this $0.5/y_1 + 1/y_2 + 1/y_3$ and then now when we are extending the dimension or I can say the when we are extending the generic variable here instead of y we are adding one more x generic variable.

So, we are adding here x here and then you see here we use this expression we have already discussed this. So, once again I am telling that this cylindrically extended fuzzy set will have its membership values corresponding to the original fuzzy set. So, whatever membership value that we will be having here will be dependent on (x, y) both, but this is going to be equal to the membership value which was in the set that we have taken originally and with x only. So, we can see here that how we are getting the membership values of the cylindrically extended fuzzy set?

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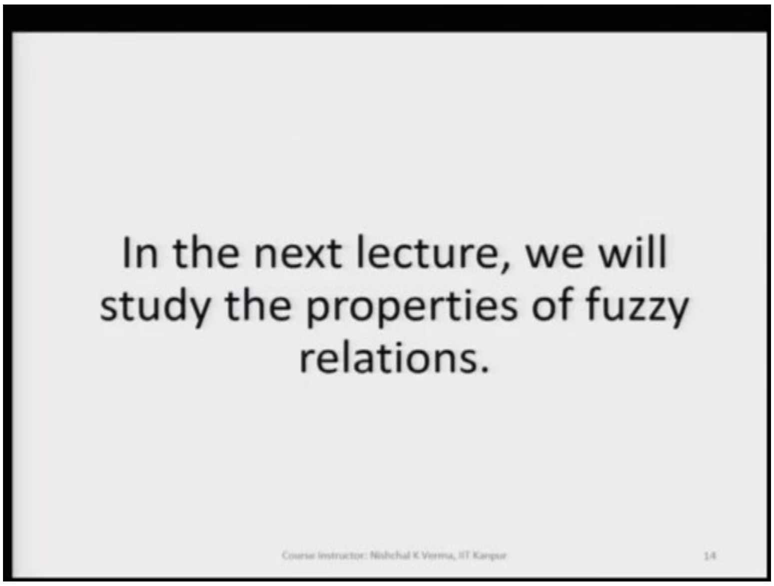

And the same we can very easily represent in the form of a matrix. So, this way we have seen that we can increase the dimension of a fuzzy set by taking the cylindrical extension of the fuzzy set and similarly we can decrease the dimension of the fuzzy set by simply projecting the fuzzy set.

So, with this I will stop here and in the next lecture we will discuss the properties of fuzzy relations.

Thank you.