Fuzzy Sets, Logic and Systems and Applications
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Lecture – 31
Projection of Fuzzy Relation Set

So, welcome to lecture number 31 of Fuzzy Sets, Logic and Systems and Applications. In this lecture we will cover projection of fuzzy relation set. So, let us consider a fuzzy relation first represented by capital R.

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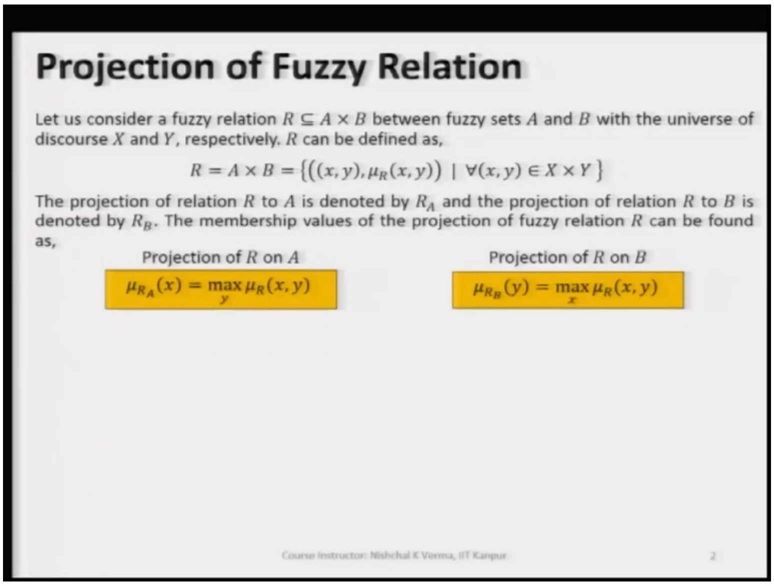

And this $R \subseteq A \times B$; that means R basically is either a subset of A cross B or R is equal to the Cartesian product of A and B; that means, $A \times B$.

So, we have already seen that how this R looks like. So, when we take the Cartesian product, A of course is defined in the universe of discourse in X and B is defined in the universe of discourse Y. So when we define R, when we define a fuzzy relation you can see here that is R.

So, $R = A \times B = \{((x, y), \mu_R(x, y)) | \forall (x, y) \in X \times Y\}$

So, R is a multidimensional fuzzy set. So, here in this case we have R, $A \times B$ and $A \times B$ as I mentioned is in terms of x and y. Means, we have the universe of discourse x and y. So, the Cartesian space that we have here is the $X \times Y$.

Now, let us discuss the projection of fuzzy relation; what is a projection of fuzzy relation? As I have already mentioned that a fuzzy relation is a multidimensional fuzzy set. When we say multidimensional, it means that we have the relation fuzzy set is defined on multiple universe of discourse like X and Y are even more even further like X, Y, Z and so on.

So, when we have fuzzy relation R, if we are interested in projecting this R to its constituents sets, like here in this case, we have A and B. So, if we are interested in projecting the relation set R on A and we know that the universe of discourse of the fuzzy set A is X.

So, before I move to the membership function of this, let me tell you that the projection of any fuzzy set reduces the dimensionality. What does this mean? This means that if we have any multidimensional set and if we project this fuzzy set to some other set, the projection is going to reduce the dimensionality of the original set and it reduces the dimensionality when we project.

So, let us now understand the projection. So, projection basically here when we talk of there are fuzzy relation; fuzzy relation is defined in the universe of discourse $X \times Y$. So, it means its a multidimensional fuzzy set.

When we are going to project this R either on A, which is a constituent fuzzy set or on B fuzzy set which is again of constituent fuzzy set; so, we can project this fuzzy relation set either on A or on B.

So, when we say project fuzzy relation set R on fuzzy set A it means, we are reducing the dimensionality. And when we are reducing the dimensionality, it means that when we are projecting R on A, it means we are retaining its resulting fuzzy set. It means we are retaining the universe of discourse X because we are projecting R on A.

Similarly, when we are projecting the relation fuzzy set R on B, so this means that we are going to retain the universe of discourse Y, of the resulting fuzzy set. Let this be represented by the projection of R on A is represented by R_A. And here similarly, the projection of R on B is represented by R_B. So as I mentioned, that the resulting fuzzy set is going to be a fuzzy set with reduced dimensionality, it means that the universe of discourse is going to be reduced.

So, in this case since we are taking capital R, which is defined in the universe of discourse X and Y, so if we are projecting the relation fuzzy set R on A, it means we are going to have only one universe of discourse of the resulting set. And that is again since this is the projection is on A, the universe of discourse will be only A.

So, this is here the expression by which we can reduce the dimensionality, we can get the projection of R and the resulting fuzzy sets membership values can be computed by this expression.

Similarly, when we talk of the projection of relation fuzzy set R on B, we use this expression for reducing the dimensionality; that means, the projection of R on B that is R_B. And here, the membership values can be computed by this expression and that is why we are writing here as μ_{R_B}.

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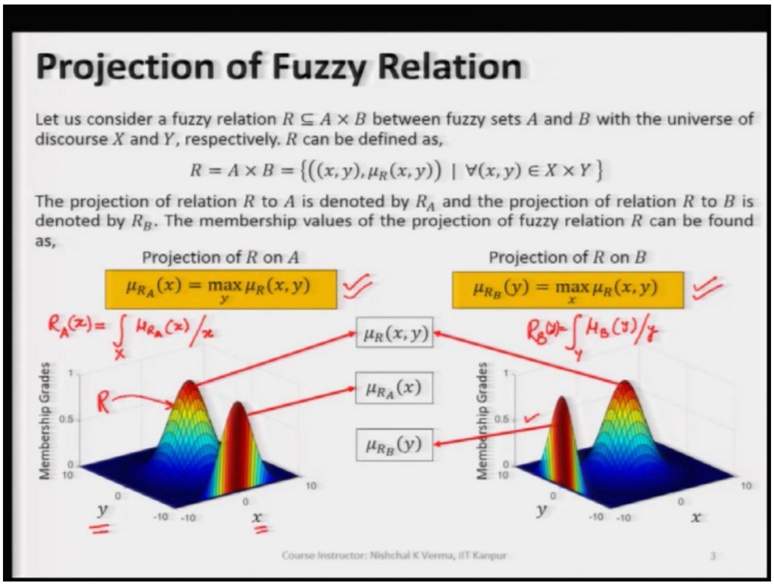

So, when we have the membership values of the resulting set when we get by this expression and here in the other case when we are projecting R on B, we get by this expression. So, when the membership values are known, obviously it is easier to write the resulting fuzzy set.

So, if let's say I am interested in writing the resulting fuzzy set in continuous domain. So, this is going to be like this. So of course, this R_A will be only the function of R, means we will have only the universe of discourse X here and this is in continuous form this will be like this, $\mu_{R_A}(x)$ and then we have small x.

So, this is the continuous form. So, the resulting fuzzy set $\mu_{R_A}(x)$ after projection we are going to get like this. Similarly, here we are going to get R_B like this, here also I have the universe of discourse only Y.

So, we can write y here and Y as the universe of discourse and $\mu_B(y)$ and then y. So, this is how we can represent the resulting fuzzy set. This exactly the same here if you are interested in writing this $\mu_{R_A}(x)$.

So, let us now better understand this by taking a very interesting example. So, let us assume that we have multidimensional relation fuzzy set. Here, in our case we have fuzzy set represented by the surface. So, or in other words, I would say we have let us say a continuous relation fuzzy set R and which is defined in two universe of discourses like X and Y.

So, we have here the generic variable value x and y, and of course, then we have the universe of discourse as $X \times Y$. So, this is our R the relation fuzzy set. Now you see here this fuzzy set is a projection of R on fuzzy set A. So you see, as I already mentioned that this is represented by $\mu_{R_A}(x)$. So this is nothing but, the projection and you see this is coming in only one dimension.

Similarly, this will be our $\mu_{R_B}(y)$. So, this means that the membership values can be represented by $\mu_{R_B}(y)$ only. And here in this case, when we have the projection of R on A, the membership values will be represented by $\mu_{R_A}(x)$.

So, it is very interesting to note that projection here in this case, is also reducing the dimension. So, earlier R was defined in the universe of discourse $X \times Y$. Now, when it is projected on A, the universe of discourse Y is eliminated. So, you see here when we project

R and this R if it is projected on A, y is eliminated only x is retained. Similarly, if R is projected on B only y is retained, x is eliminated. So, this has to be noted.

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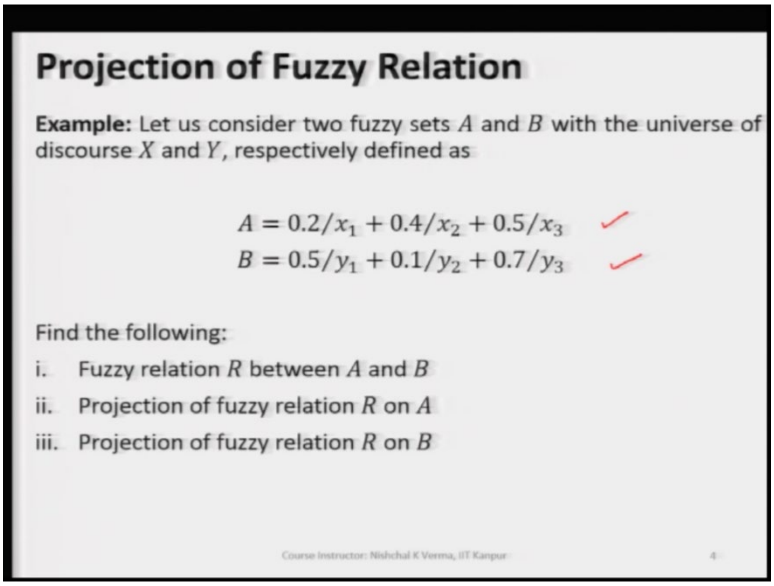

And this also can be understood by discrete fuzzy relation set example; so, we are taking here one example where we have two discrete fuzzy sets A and B. And here we will first form a relation fuzzy set and then, from the relation fuzzy set, we will see how the projection of fuzzy relation R on A and projection of fuzzy relation R on B.

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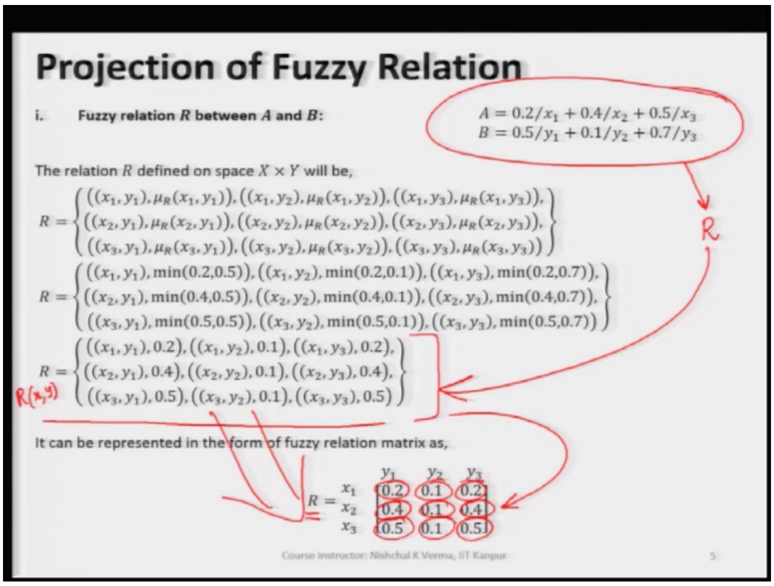

So, let's now go ahead and try this example. So, as I mentioned that we have two discrete fuzzy sets; we already have done this thing that as to how we can quickly get from these two fuzzy sets A and B, how we can get a fuzzy relation set.

So, our relation set is here you can see. So, we will just have to get the Cartesian product of these two A and B fuzzy sets. And, here we write our relation fuzzy set R which is again this relation fuzzy set is basically is defined in the universe of discourse X and Y.

So, this can also be written as $R(x, y)$. Remember this R is of multidimensional fuzzy set. So, if we look at A fuzzy set we see that it is a one dimensional fuzzy set. Although, it is represented in two dimensional space because the we have the generic variable values x_1, x_2, x_3 and so on and then we have the corresponding membership values like $\mu(x_1), \mu(x_2), \mu(x_3)$ and so on.

So, we represent this in two-dimensional space, but so single dimensional fuzzy set. Similarly, the B also and here our relation fuzzy set that is $R(x, y)$ becomes this way the three-dimensional fuzzy set. Actually, it is a two-dimensional fuzzy set because only x and y are the generic variable values. But, the representation of this fuzzy set is in three-dimensional because one more dimension which is needed here is apart from x and y, we need $\mu(x, y)$ for the associated membership values.

So, as we see here that we have the $R(x, y)$ as the fuzzy relation set R here. Now, this can be represented in the form of fuzzy relation matrix, the same we know as to how we can manage to convert this into the matrix form.

You see here, the R we have the column and then we have rows. So, column is basically x_1, x_2, x_3 and rows are y_1, y_2, y_3, and then here these are the min values.

So, we know as to how we can get the relation matrix. We have already done this so I am not going to discuss this again. We now represent this whole thing this relation fuzzy set into the matrix form. So, we call this as the fuzzy relation matrix.

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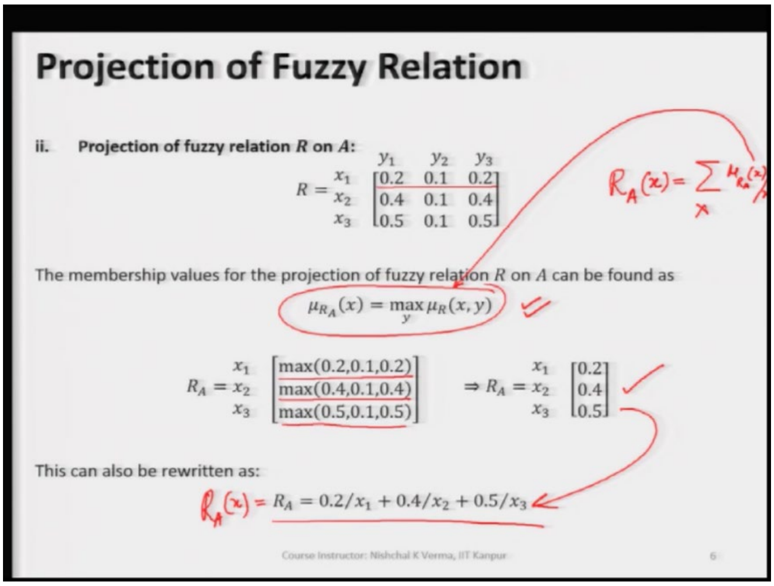

So, now when we have the fuzzy relation matrix R, let us now find out the projection of fuzzy relation R on A as we have already discussed. So, when we say projection of fuzzy relation R on A, it means that we have to retain only x generic variable and we have to eliminate y generic variable value. And when we apply the expression for doing this, this means that the R which is going to be the resulting fuzzy set will look like this.

So, we call when we say if projection of fuzzy relation R on A, we represent this by R_A and also this R_A now we will retain only the x as the generic variable and this R_A will be basically if we represent this in continue in discrete form this will be like this.

So, $\mu_{R_A}(x)$ and then x. So this will, the whole fuzzy set will look like this and now how to get this $\mu_{R_A}(x)$. So, this $\mu_{R_A}(x)$ can be, we can get from here. So, when we use this expression, the expression is $\mu_{R_A}(x) = \max_y \mu_R(x, y)$.

So, we see that we have a fuzzy relation matrix R. So, when we apply this we see that, for the first row corresponding to x_1 we see that we have to take the maximum of all the elements in the first row. So similarly, when we take maximum all the values, all the elements in x_2 row and x_3 row, we find this matrix which is the column matrix.

So, this is called R_A and this is equal to the column matrix 0.2, 0.4, 0.5, and this can be written as in the normal form normal form is these summation form. So, when we write the summation form we can write it like this like R_A is nothing but this R A is equal to $0.2 \,/\, x_1 + \; 0.4 \,/\, x_2 + \; 0.5 \,/x_3$.

So, this is how we get the projection of fuzzy relation set on A and this is represented by the $R_A(x)$.

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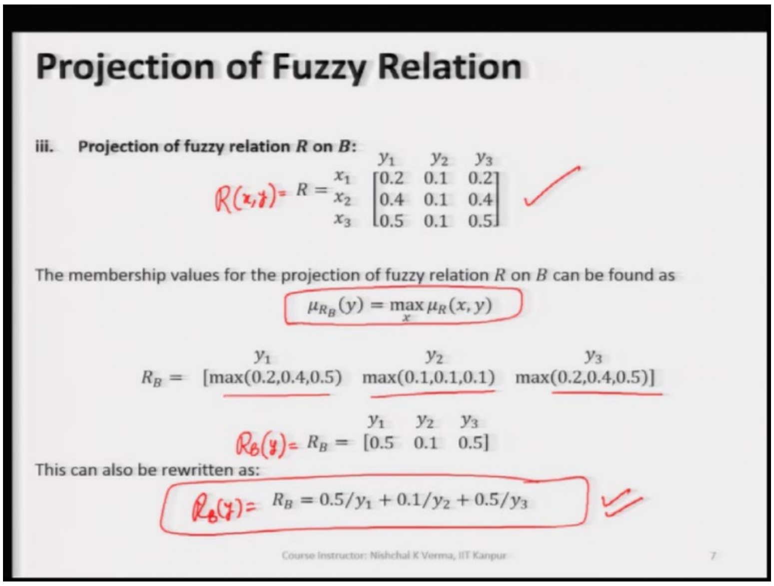

Now similarly, when it comes to projecting the same fuzzy relation capital R on the same lines we move ahead. So, we have here the relation fuzzy set which is again the defined on 2 generic variables. So as I have discussed, for the projection of fuzzy relation R on B on the same lines when we apply this expression, this criteria on the relation fuzzy set we get a row matrix.

Here, since we are projecting the relation fuzzy set on B, we are eliminating x we are retaining only y. So, that is why we see that we are taking the max for all the column values, max of all the column values means over all x values.

So, that is why all the 3 rows are reduced to only single row. So, we see that we get these 3 elements only and this is nothing but R_B and which is defined on only y. And this can be represented by the normal fuzzy representation form and which is actually $0.5 \,/y_1 + \; 0.1 \,/\, y_2 + \; 0.5 \,/\, y_3$.

So, this way we have got the projection of fuzzy relation R on A and on B. And in both the cases we have seen that the original fuzzy relation set R was the multidimensional, that means both the R was on x and y, but here when we have projected this either A or B we saw that we have reduced the dimension by 1.

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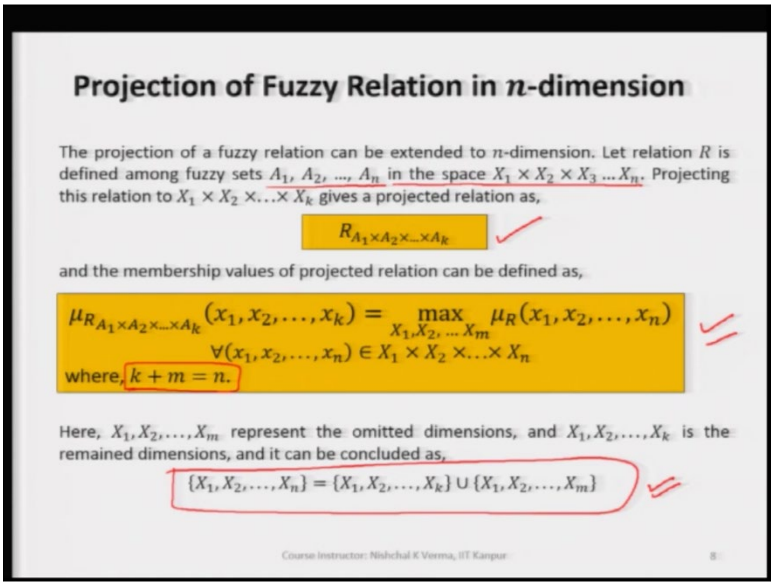

So, this way we can generalize the projection of fuzzy relation R, so, R in n-dimension. So, let us understand this here better. So, the projection of a fuzzy relation let us say R can be extended to n-dimension. So, let relation R is defined among fuzzy sets A_1, A_2, A_3, $A_4 \ldots\ldots A_n$, where small n is the total number of generic variables.

So accordingly, we will have a space here. So, the universe of discourse space will be $X_1 \times X_2 \times X_3 \ldots\ldots X_n$. So, we can project this relation to $X_1 \times X_2 \times X_3 \ldots\ldots X_k$. So, we are taking some multiple dimensions where let us say up to k is the dimension where we are projecting.

So, when we project the n-dimensional fuzzy relation set into k-dimension k-dimensional a space. So, what we are getting is here. So, we represent this the resulting relation fuzzy set by capital $R_{A_1 \times A_2 \times A_3 \ldots\ldots A_k}$. It means here we are retaining only this universe of discourse means k universe of discourse.

So, the membership values of this projected relation can also be computed and this is computed by this relation here. So, this is self-explanatory, so, we define the membership value of the resulting fuzzy set and this resulting fuzzy set of course will have k universe of discourse.

I should not say k universe of discourse, but I will say the universe of discourse will be the Cartesian product of k generic variable. So, here $\mu_{R_{A_1 \times A_2 \times A_3 \ldots\ldots A_k}}(x_1, x_2, \ldots\ldots, x_k) = \max_{X_1, X_2, \ldots X_m} \mu_R(x_1, x_2, \ldots., x_n) \, \forall (x_1, x_2, \ldots\ldots, x_n \in X_1 \times X_2 \times \ldots\ldots \times X_n$.

So this way, it is defined and here this k plus m. So, here it is to be noted that the total number of dimensions are the generic variables that are there, the total number is here n.

So, we have a small n, total number of generic variables, the relation fuzzy set and this relation fuzzy set is projected in the k universe of discourse or k generic variable based fuzzy set. So, what is the elimination here is the m, the m generic variable values are eliminated.

So, what does this mean? It means that the k plus m is going to be equal to m, means the total generic variables that we had initially is n, and then the resulting fuzzy set after projection we have small k and then the small m here is the generic variable values that are eliminated by this projection.

So this way, we have understood as to how this projection of fuzzy relation in n dimension is done and this is also to be noted here that the relation that I have just mentioned among the universe of discourse or the generic variable values here.

So, this can be understood by the union. So, here the total universe of discourse R, I would say the generic variable values is equal to the union of the k generic variables and then the m generic variables.

So this way, we have understood as to how we can project fuzzy relation in the reduced dimension. So, as I have already mentioned that the projection of fuzzy relation always gives a resulting fuzzy set, which will have the dimension less than the dimension of the fuzzy relation set.

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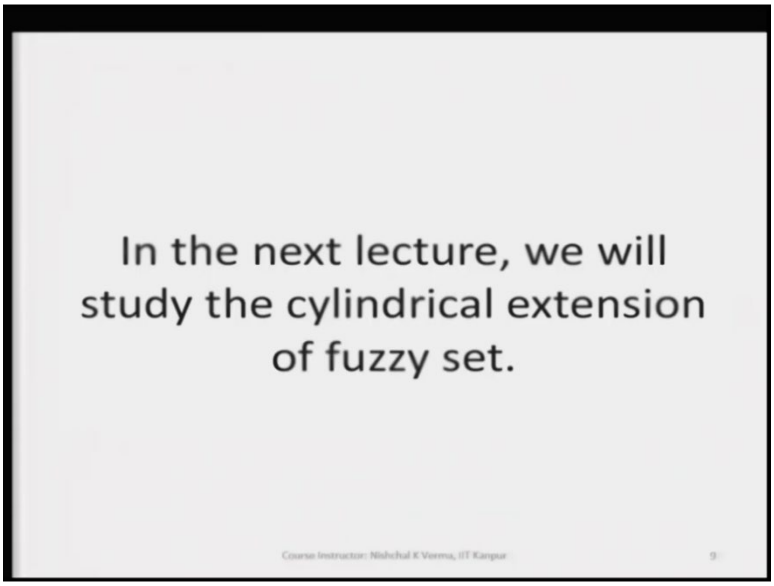

So, this way we have understood the projection of fuzzy relation very clearly and the remaining part, that is the cylindrical extension of fuzzy set will be a covered in the next lecture.

Thank you.