Fuzzy Sets, Logic and Systems and Applications
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Lecture – 28
Fuzzy Relation

Welcome to lecture number 28 of Fuzzy Sets, Logic and Systems and Applications.

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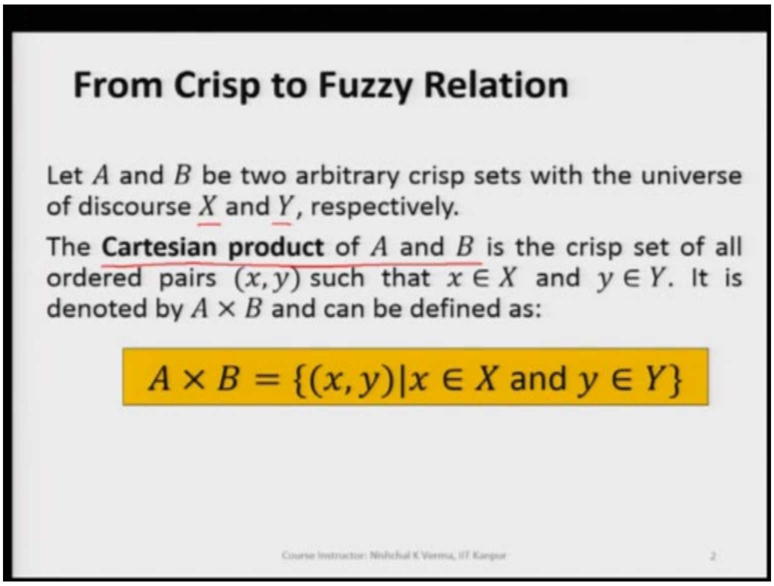

So in this lecture we will discuss a Fuzzy Relation. Before we move to fuzzy relation, let us first understand the crisp relation because this is very important for understanding fuzzy relation, and crisp relation also we need to know first the Cartesian product of crisp sets. So, if we have two arbitrary crisp sets with the universe of discourse capital X and capital Y, respectively.

The Cartesian product of A and B can be written as the A cross B and this will be equal to the set of all pairs of x and y, such that $x \in X$ and $y \in Y$.

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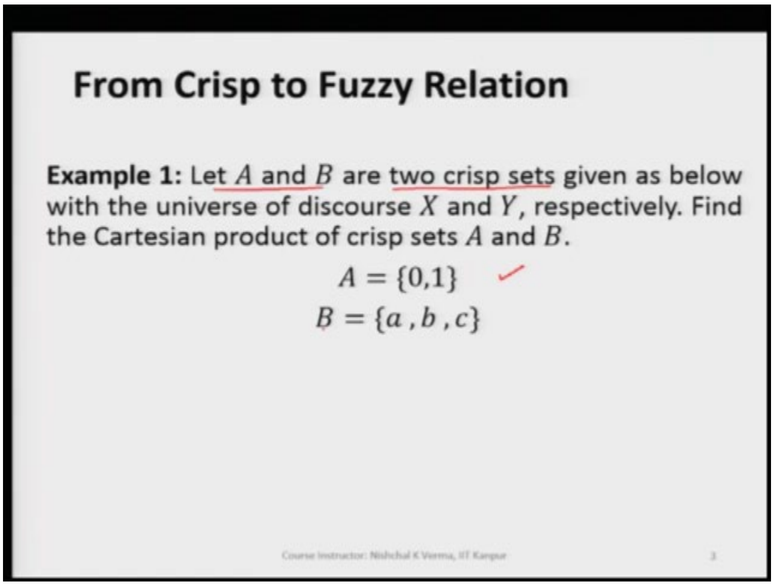

So, this can be understood better by this example of course, this is a very basic understanding, you must have done this in your earlier classes of mathematics. But here this becomes little important to understand before we move to the crisp relation and then the fuzzy relation. So, if we take an example here, where we have A and B two crisp sets. You can see here, A is a set is a crisp set where we have two elements; in $A\{0,1\}$ and in $B\{a, b, c\}$.

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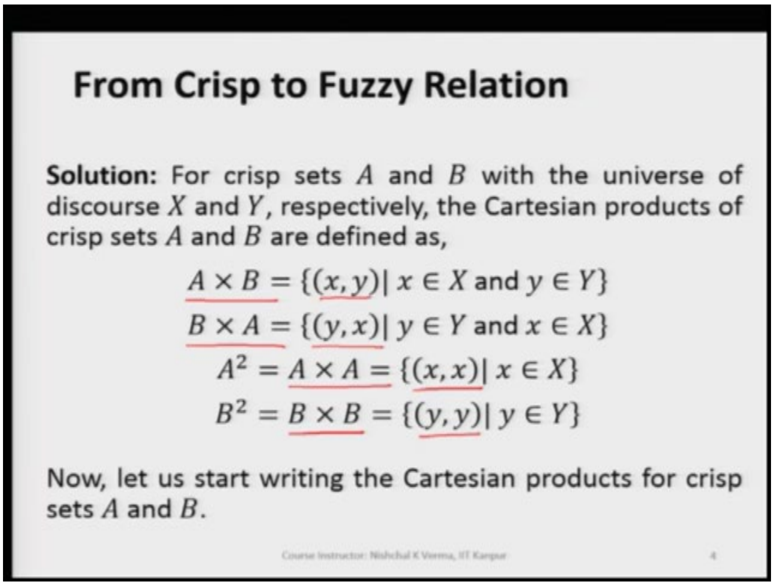

So, if we are interested in finding the Cartesian product, what do we do? We simply you know, we multiply or we take the cross product of the two sets A and B and this is going to give us the ordered pair of all the elements from A and B. So, but it should be within the universe of discourse.

So, if I have to let us say find $A \times B$, this is going to be like this like A cross B is going to give us a set which will contain (x, y) means all the elements in the pair is coming from the set A and the y is coming from the set B. And of course, this x and y forms a pair here and small x is from the universe of discourse X, y from the universe of discourse Y. And we all know that in Cartesian product of two sets crisp sets, we follow the order of the elements while making the pairs here.

So, the first element will be from the set A and then the second element of the pair will be from the element B. And then we collect all the formed pairs in the set, and this is $A \times B$. Now when we are interested in finding $B \times A$, the this will be a set of all the pairs of y and x; means all the pairs here, where the first element is coming from the set B and then the second element will be from the set A. So, here this is to be noted that, the order is very important in the Cartesian product, the order in which the pairs are formed. Like in $A \times B$, we have x, y as a pair; but in $B \times A$, we have y, x as a pair, right.

So, similarly if we are interested in finding the $A \times A$, you see here that all the pairs will be like this, like both the elements will be from the set A. Similarly, if you are interested in finding the $B \times B$, so we will have the collection of pairs in this Cartesian product set, and all the elements will be from the same set B. And of course, it is needless to say that, they will have to follow this y, x and y will have to follow and the condition of universe of discourse; means x should be from the universe of discourse X and small y from the universe of discourse Y.

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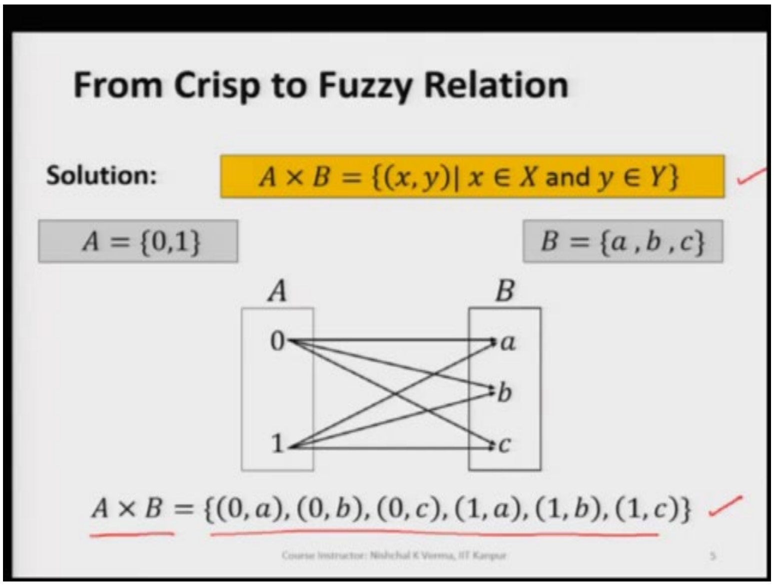

So, here in this example if we see that, $A \times B$ as we have defined here, if we take A set like this, like A has two elements $0 \; and \; 1$, B has three elements a, b, c. So, we see here that, if we take the cross product, if we take the Cartesian product here as $A \times B$ so Cartesian product of the set A. Please understand that this set is a crisp set. So, once again I am saying here that, the Cartesian product of crisp set A and crisp set B is going to be all the ordered pairs that are formed elements from A and B.

So, we see that $A \times B$ is going to give us first pair $(0, a)$; second pair $(0, b)$; third pair $(0, c)$ and then the fourth pair is $(1, a)$; fifth pair is $(1, b)$; sixth pair is $(1, c)$. So, we see that we have six elements, when we are taking the Cartesian product of capital A and capital B, where these capital A and capital B are two crisp sets. So, this way we get the Cartesian product of two crisps sets.

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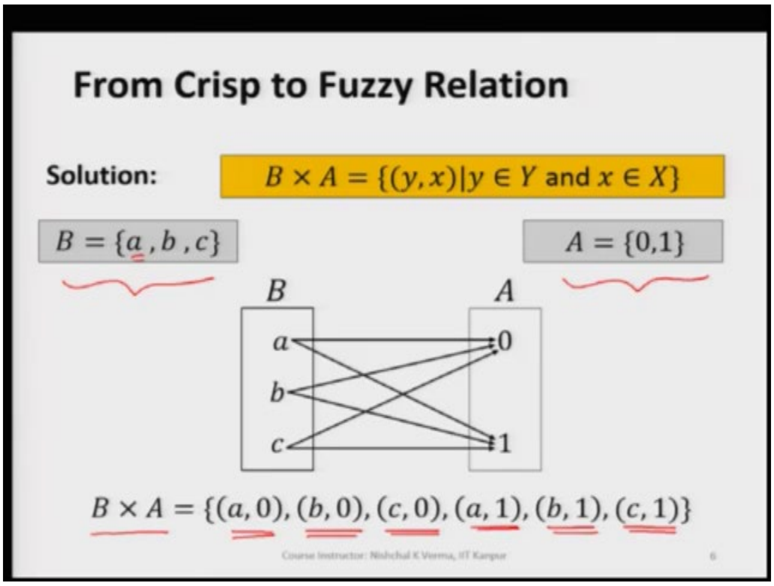

Now, let us change the order and let us find $B \times A$. So, we write B here first and then we write A. And as we know that B has three elements a, b and c, and A has two elements.

Let us now find the Cartesian product of crisp set B and crisp set A. So, here we will first start with this element a, and this element will combine with all the elements of set A. So, this is going to give us $(a, 0)$ as the first pair, and then the second pair will be $(a, 1)$ here. So, like that we will be getting other pairs. So, we will have then $(b, 0)$ and then $(b, 1)$; and then finally, with c we are going to get $(c, 0)$ and $(c, 1)$. So, this way we are getting six element in the set which is the Cartesian product of B and A.

So, what actually we are doing here is that we are forming the ordered pairs of the elements that are coming from the first set and then the second set. And please understand that, these pairs will belong to the universe of discourse that are created by the Cartesian product of the separate universe of discourses X and capital Y; or in this case if we take Y first then capital X, then Y cross X.

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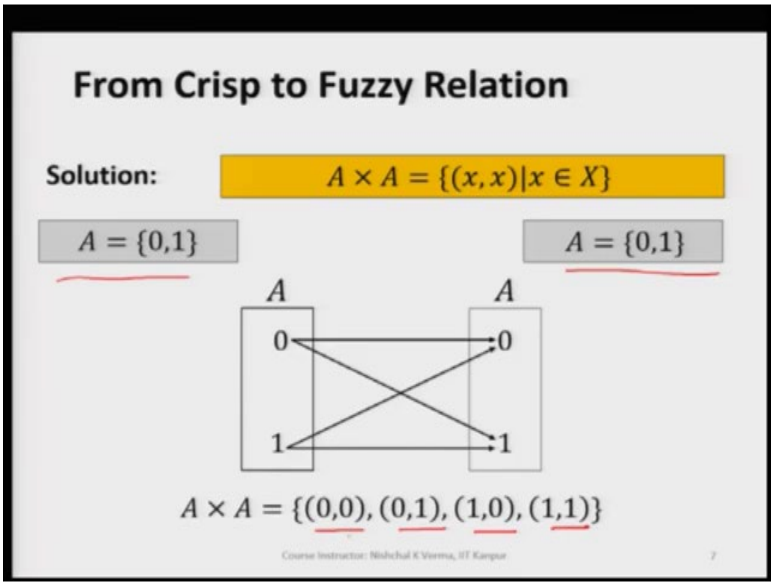

So, here this $B \times A$ is, if we see that we are getting a set of six pairs of elements. So, we can clearly say that, $B \times A$ that we have seen here is different from $A \times B$. The order is changed; so that is why we can say that, $A \times B \neq B \times A$. Now let us take let us find $A \times A$. So, in $A \times A$, on the same lines if we move we see that, $A \times A$ we are getting as; because in A we have two elements 0 and 1. So, we first take 0 and then we combine, we make pair of the elements that are written here in the other A. So, we have two sets A and A and when we take the Cartesian product, we are getting four elements.

And these four elements, basically we having elements from the same set A; so we get $(0, 0)$, then $(0, 1)$, then $(1, 0)$ and then $(1, 1)$. So, this way we get $A \times A$.

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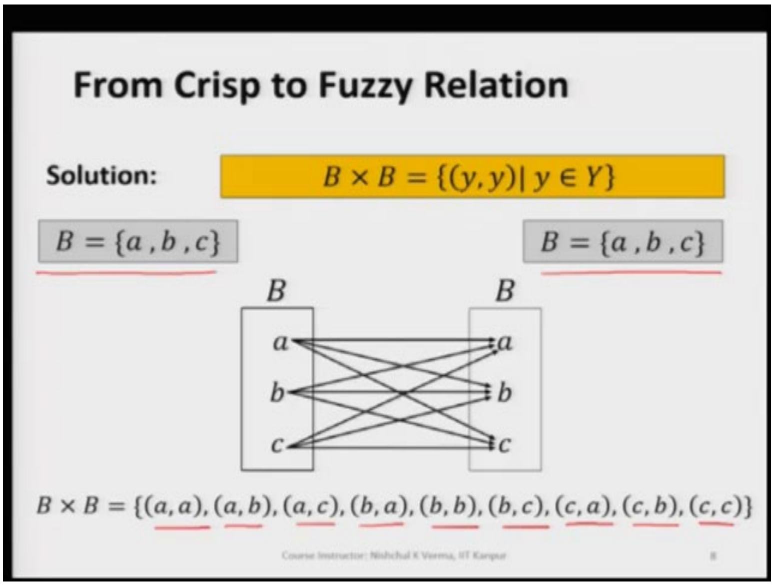

Now let us find $B \times B$. So, when we are interested in finding $B \times B$, means we are taking two sets both the sets are $B \times B$ only. So, when we make pairs of the elements from the first set as B and the second set also as B, so we see that, we are getting here is nine elements like this, so (a, a); (a, b); (a, c); (b, a); (b, b); (b, c); (c, a); (c, b); (c, c).

So, like that we have nine pairs of elements from B set only. So, this way we have seen that, we can get the Cartesian product of B and B. And as we have already seen that, the order is very important while we make pairs.

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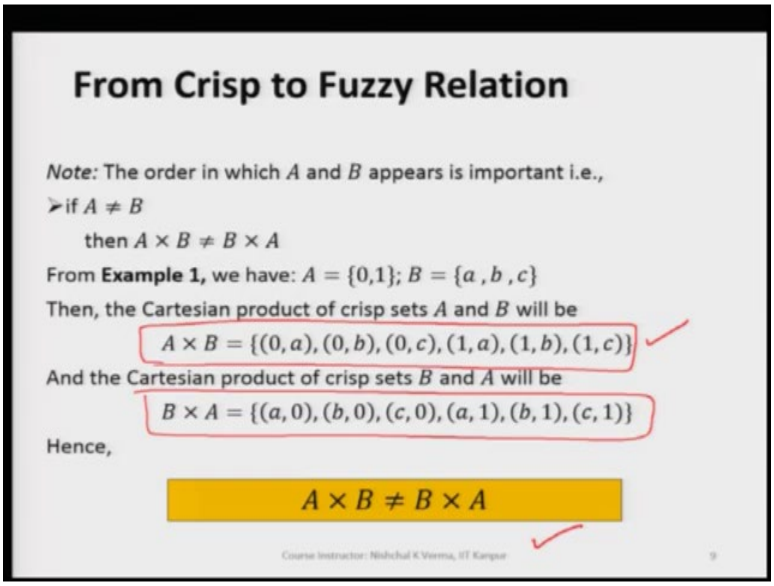

So, that is why the $A \times B \neq B \times A$. And please note that these two sets A and B are the crisp set here. So, this we have already seen that, $A \times B$ was coming out to be like this and then $B \times A$ is coming out to be this. And that is how we can with example we can see that, $A \times B \neq B \times A$. And it is because of the order in which the pair is made.

So, in $A \times B$, the first element of the pairs are coming from crisp set A, whereas in $B \times A$ the first element of the pair is coming from set B. So, that is why when we compare $A \times B$ and $B \times A$, these both of these are not equal to each other.

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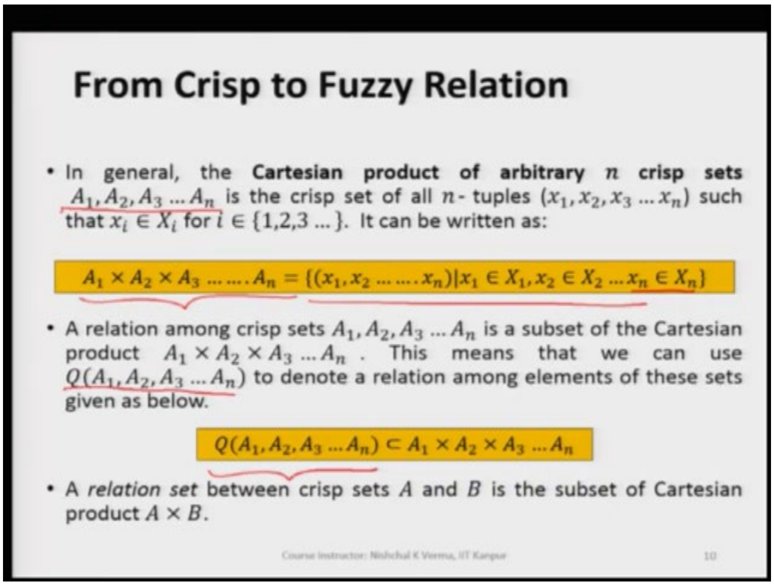

Now, let us move towards the Cartesian product of the n crisp sets. So, in general we can write the Cartesian product of the arbitrary n crisp sets like this like A_1, if we represent the n crisp sets like this like A_1, A_2, A_3 … …. A_n, so we can see here.

And then if we are interested in Cartesian product of all these n crisp sets. So, we can say this is the Cartesian product of A_1, A_2, A_3 …. A_n can be represented by the $A_1 \times A_2 \times A_3 \dots\dots A_n = \{(x_1, x_2, x_3 \dots\dots x_n)\} | x_1 \in X_1, x_2 \in X_2 \dots. x_n \in X_n$.

So, in other words we can say that, the cross product of n crisp sets A_1, A_2, A_3 …. A_n is nothing, but the collection of $n - tuples$; $x_1, x_2, x_3 \dots\dots x_n$. So, all such n tuples will be there in the Cartesian product, when we take the Cartesian product of n sets and crisp sets. So, when we represent a Cartesian product like this, now this Cartesian product contains all the n tuples that are formed from the elements that are coming from A_1, A_2, A_3 …. A_n. And please understand here that these n tuples will belong to the universe of discourse the $X_1 \times X_2 \times X_3 \dots\dots \times \ X_n$.

So, when we have the Cartesian product which is nothing, but the ordered n tuples here, in case of the Cartesian product of n crisp sets. Now let us understand the relation among the crisp sets, so here when we are interested in finding the relation, so normally we say the Cartesian product set is a relation set. But here we need to know that, relation set is based on certain conditions, so this I will be discussing in the next slide. But before that, I would like to define the relation set by $Q(A_1)$ here $Q(A_1, A_2, A_3 \dots\dots A_n)$.

So, this is here, this Q denotes the relation set. And this relation set will always be a subset of the Cartesian product of the A_1, A_2, A_3 .... A_n. So, relation set of A_1, A_2, A_3, A_n is a part of the overall Cartesian product of the sets A_1, A_2, A_3, A_n.

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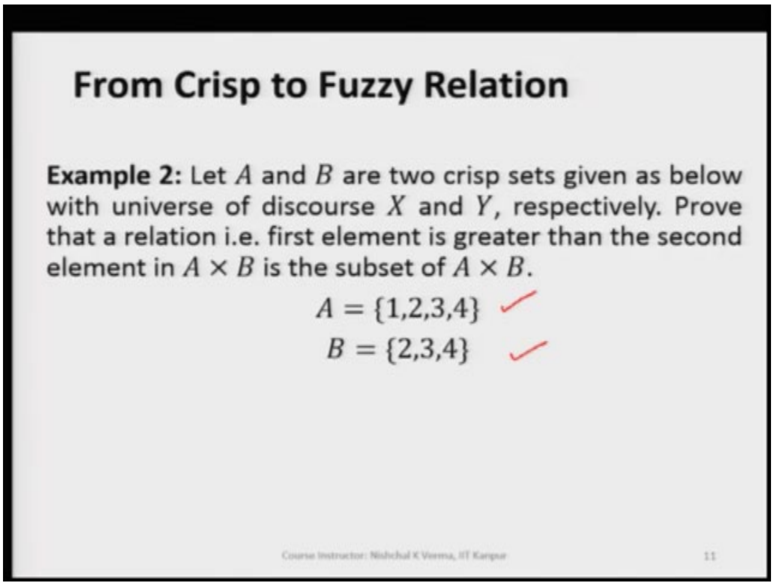

This can be understood by an example here. So, let us take an example to understand the crisp relation. So, here we are taking two crisp sets A and B and let us first find the Cartesian product of these two sets.

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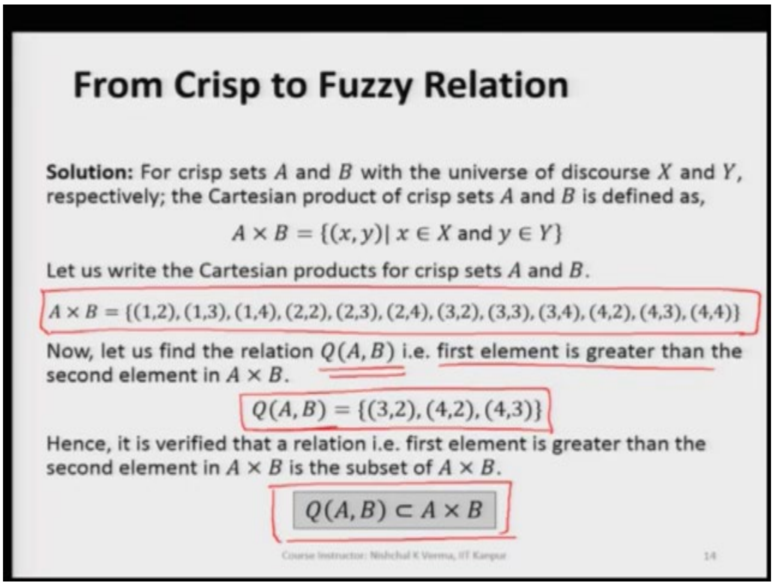

So, $A \times B$ can be written as the collection of all the pairs of the elements from A and B sets and this way $A \times B$ is going to be a set here, the Cartesian product of A and B. So, we see that, we have here, since we have four elements in A and three elements in B crisp set, so, we are going to get 12 pairs here; like this, here the first pair is $(1, 2)$; second pair is $(1, 3)$; third pair is $(1, 4)$; fourth pair is $(2, 2)$ and like that we get the twelfth pair is $(4, 4)$. So, this set contains all the ordered pairs which are possible in, possible out of the Cartesian product of $A \times B$.

So, we see that this is the overall space, overall population that is possible here as the Cartesian product of A and B. Now let us see what is a relation, if we are interested by putting some condition. So, let's say we are interested in a particular relation and the relation here is that, we are interested in a set let us say $Q(A, B)$; so this is a relation set and this defines some relation based on some condition. So, $Q(A, B)$ is based on the condition here is that, first element is greater than, so first element of the pair that are formed is greater than the second element in the Cartesian product of A and B.

So, we have already seen that we have Cartesian product here, where we have twelve pairs. Now we are putting the condition that, the first element in all the pairs should be greater than the second element. So, if we put this condition, will find few pairs where this is true. So, if we are putting this condition and based on that, we pick those pairs for which this condition is satisfied. And $Q(A, B)$ relation is based on that condition, if that is true, then we can write $Q(A, B)$ is equal to a set of these three pairs for which the first element is greater than the second element.

So, we can see here that we have three pairs that is $(3, 2)$; $(4, 2)$; $(4, 3)$ where the first element of the pair is greater than the second element in all the three pairs. So, $Q(A, B)$ represent here a relation set. And we see that, this $Q(A, B)$ is coming from the total population, which we have got from the Cartesian product of A and B. So, Cartesian product of A and B is here, and $Q(A, B)$ is here and we can clearly see that $(3, 2)$; $(4, 2)$; $(4, 3)$; all these three pairs are drawn from the Cartesian product of A and B after applying the condition, the first element is greater than the second element.

So, since $Q(A, B)$ is a set and it has some elements which are drawn from the Cartesian product of crisps set A and B, so we can write here that $Q(A, B)$ is the subset of the Cartesian product of A and B, where A and B both are the crisps set. So, this way the statement that I made in the previous slide that the relation you can say here, the relation $Q(A, B)$ is the subset of here the relation set is a subset of the Cartesian product of the sets.

So, this way we can write that relation set, the relation set is always a subset of the Cartesian product. Or in other words if we say the relation set in between A and B here in this case a subset of the Cartesian product of set A and B.

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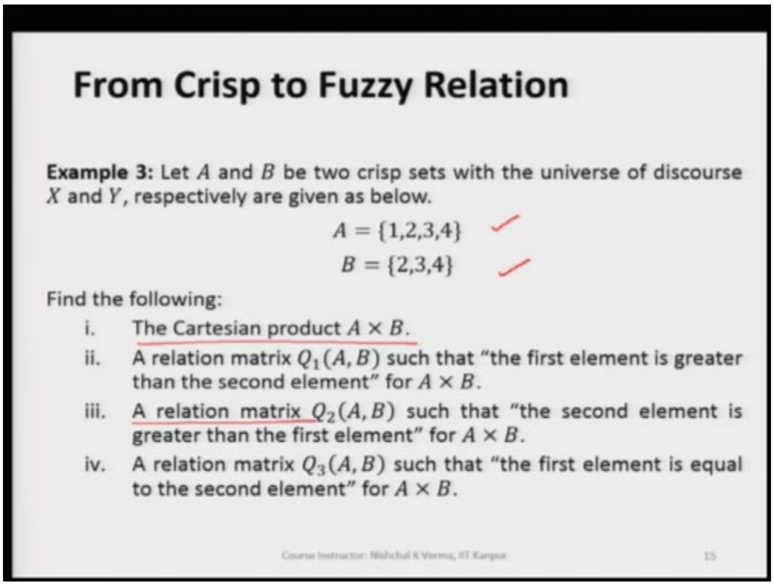

Now, let us take another example here, where we take crisp set A and crisp set B here. And we have the universe of discourse here as for A we have X and for B we have capital Y. So, in this example we are supposed to first find the Cartesian product of crisp set A and crisp set B, that is $A \times A$. And then we are supposed to find the relation matrix $Q_1(A, B)$, such that the first element is greater than the second element as we have seen in the previous example.

So, this very easy, we can first find the Cartesian product, so that we have the space, the whole population of the Cartesian product and then from that we pick the elements based on the condition that is given and this will be our relation matrix $Q_1(A, B)$. And then next in the third case we are supposed to find relation matrix $Q_2(A, B)$, such that second element is greater than the first element. So, here based on another condition, we are supposed to find another relation set $Q_2(A, B)$ and based on that we can find a relation matrix.

In the fourth case we are supposed to find $Q_3(A, B)$, such that first element is equal to the second element for $A \times B$. So, let's now quickly move ahead and first find the Cartesian product of set A and set B, here both the sets are crisp sets.

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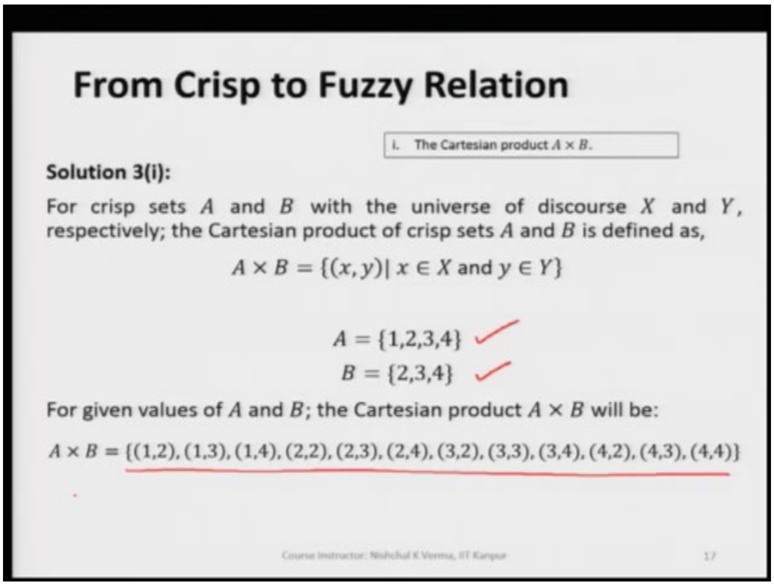

So, $A \times B$ as we have already done, we can quickly find here a set like this and this represents the Cartesian product of this set A, where we have four elements $1, 2, 3, 4$ and then we have set B, where we have three elements $2, 3$ and 4. And $A \times B$ the Cartesian product of A and B here is a collection of all the elements which are ordered pairs; so collection of all the ordered pairs of the elements coming from set A and set B, respectively. So, we are getting here 4×3, that is 12 ordered pairs as elements of the Cartesian product of set A and set B.

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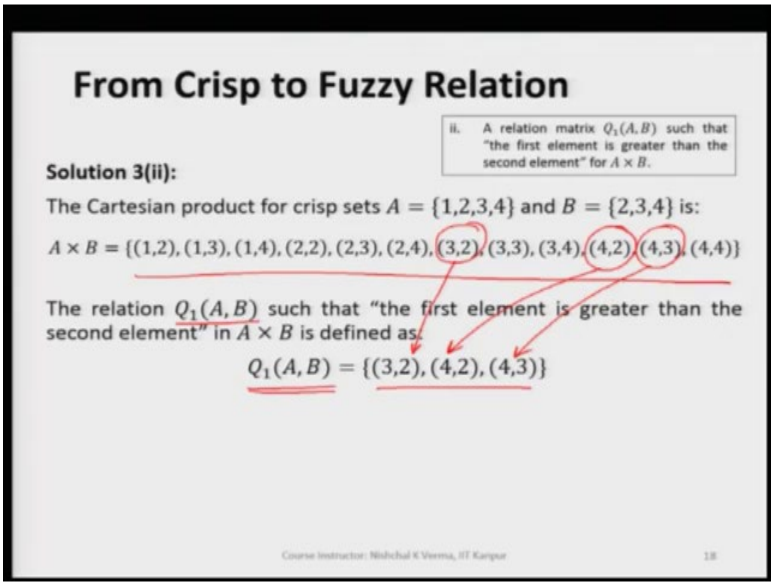

Then when we have this as the whole population here of the order pairs that are possible. Now let us go ahead and apply the condition for the first relation. So, for the first relation$Q_1(A, B)$ here, $Q_1(A, B)$ let us apply the first condition; first condition here is that, the first element is greater than the second element. So, first element is greater than the second element means the we have to look into all the ordered pairs for the first element, which should be greater than the second element in the same pair.

So, we find here three pairs as elements of this set, $Q_1(A, B)$. So, $Q_1(A, B)$ is giving us 3 ordered pairs, where we see that the first element is greater than the second element, no other pair in $A \times\ B$ in the Cartesian product of A and B is giving any pair which satisfies this condition.

We can see here one by one, so we see we have the first pair $(1, 2)$, where $1 \leq 2$, so the first element is less than the second element, so this is not qualified. Then if we keep moving ahead we see that, here 3 and 2 satisfies, the pair which is consisting of 3 and 2 which gives us this condition satisfied. So, we have collected this element here.

And then $(4, 2)$ is another pair which satisfies this condition and then we have $(4, 3)$ also which satisfies this condition. So, this way we get three elements here, in this set $Q_1(A, B)$ which is relation set, where the first element is greater than the second element. So, this is how the relation set is formed by the crisp sets.

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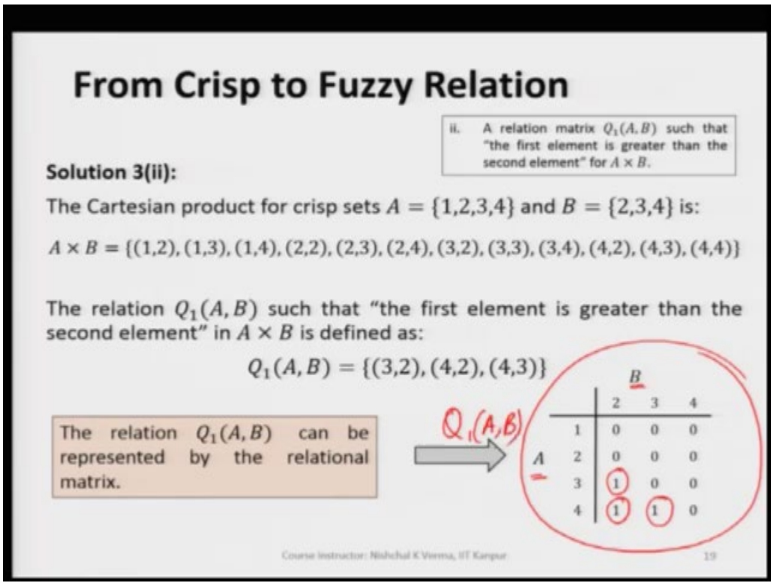

Now, let us form a matrix here. So, relation sets can also be represented by the relational matrix. So, if we have A like this and B like this, like we have rows and columns and based on that the relation we can write in form of matrix.

So, we see $Q_1(A,B)$, so we have $(3,2)$, $(4,2)$, $(4,3)$. So, if we write $1,2,3,4$ in as a column and then $2,3$ in the row. So, we see that, we have 1 here, we can write here 1, because $(3,2)$ is existing as pair in $Q_1(A,B)$ and then we have $(4,2)$ and then we have $(4,3)$. So, in this Q_1 relation set, we have only three elements, so we have only for this we write once, otherwise all other elements will be 0. Here this has to be noted that, why are we writing 1, because the logic is Boolean logic. So, since this pair is completely present, so that is why we are writing 1 for all the existing pairs and 0 for non-existing pairs.

So, that is why we have this matrix here as the relational matrix. So, this is $Q_1(A,B)$.

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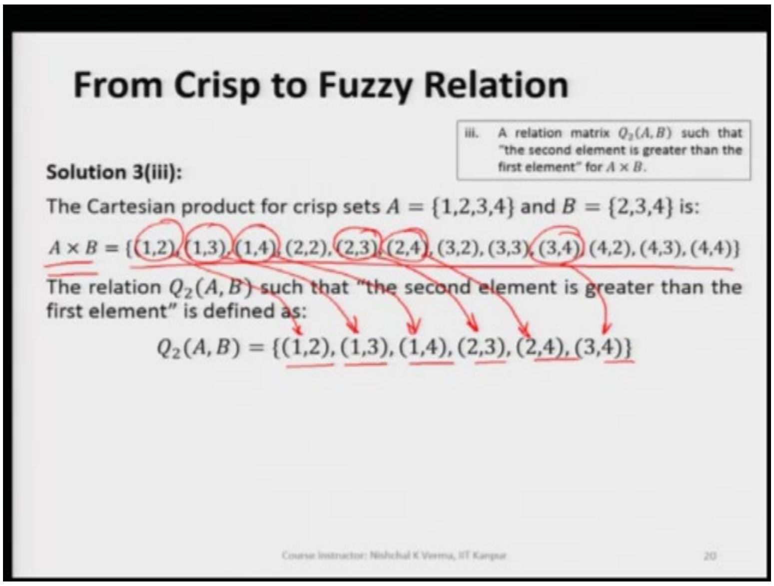

Next is when we apply the second condition that was given and this relation set is represented by $Q_1(A, B)$. So, we already have the set of the Cartesian product of set A and B here. And now we look for the pairs which satisfy this condition; what is this condition? Condition is that the second element is greater than that of the first element. So, we see that here, we have got $(1, 2)$ as the first pair which satisfies this condition.

So, first pair in $Q_2(A, B)$. So, $Q_2(A, B)$ is the set of all such pairs which satisfy the condition of the second element is greater than the first element. So, $(1, 2)$ is the first element of Q_2 and then we have $(1, 3)$ also this pair which satisfies this condition, then we have $(1, 4)$ which also satisfies this condition. And then we have $(2, 2)$ which is not satisfying this condition, because 2 and 2 both are equal. So, here the second element is not greater than the first element, so this is not satisfying.

And then $(2, 3)$ is satisfying the condition, $(2, 4)$ is satisfying this condition. Similarly we see that here $(3, 4)$ is satisfying the condition. So, this way we see that we have 6 elements or I would say the six pairs in $Q_2(A, B)$, means the relation set Q_2has 6 elements which satisfy this the given condition that is the second element is greater than the first element.

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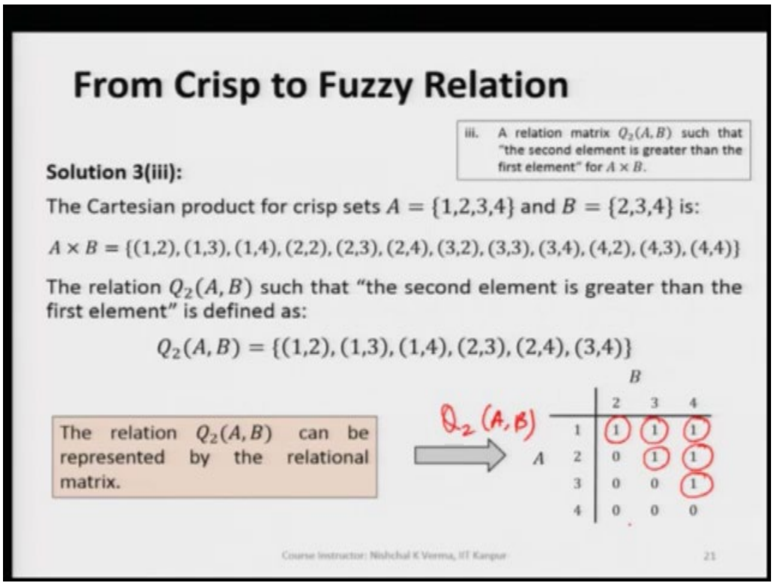

So, let us form a relational matrix. So, as we have done this in the previous case, so we can write a relational matrix like this, like we have $(1, 2)$ existing in $Q_2(A, B)$.

So, we take $(1, 2)$ here and we see that, for $(1, 2)$ we have written 1. And please understand that the column here is coming as the set A and the row as the set B. So, since we have 1, 2 pair already existing in Q_2, so we write here 1. And similarly for $(1, 3)$ we write here 1, and then $(1, 4)$ we write 1 and then $(2, 3)$ we write 1; $(2, 4)$ we write 1; and $(3, 4)$ we write 1. So, all this six elements now have been represented in a form of a matrix and this way you know all these elements have been included in the $Q_2(A, B)$ set.

And since no other element exists apart from this six, so for all other elements, for all other combinations we are writing 0. And this way we have written the relational matrix $Q_2(A, B)$.

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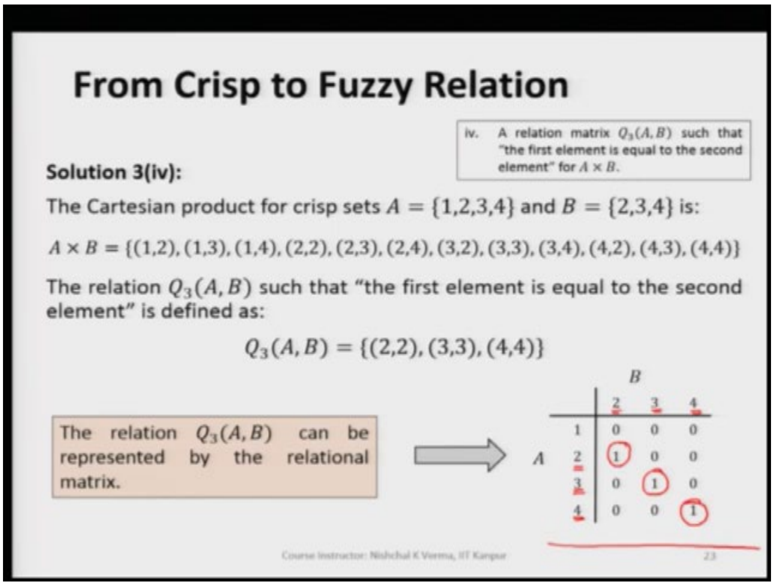

And now we have to find the $Q_3(A, B)$ based on the condition, the first element is equal to the second element. So, since we already have the Cartesian product of set A and set B and we apply this condition, the condition that is given which is first element is equal to the second element, if we apply this. So, we get $(2, 2)$; $(3, 3)$; $(4, 4)$ you know the pairs as the elements of $Q_3(A, B)$ here and, which satisfies the condition that first element of this pairs is equal to the second element.

So, $Q_3(A, B)$ has three elements only $(2, 2)$; $(3, 3)$ and $(4, 4)$. So, first element, second element and then the third element; and this $(2, 2)$ is coming from here, $(3, 3)$ coming from here and $(4, 4)$ is coming from here. So, this way we have another relation set, Q_3 based on the condition that was given, and now further we can write this in form of a relational matrix. So, this relational matrix can be again on the same lines we can write.

So, we see that since we have $(2, 2)$ presents, so for 2 here and 2 here, since $(2, 2)$ is present in Q_3, so these two are forming one pair, so we are writing here 1. And then we have $(3, 3)$; $(3, 3)$; means 3 of set A and 3 of set B these two also are forming one of the elements of Q_3, so we are writing 1 and then $(4, 4)$ also, so we see that here writing 1. Now since we have only 3 elements no other elements are present in Q_3, so all other elements will be put as 0 of this matrix and this is called a relational representation of $Q_3(A, B)$ based on the condition that is the first element is equal to the second element.

So, this way we have seen that as to how we can find the Cartesian product of two crisp sets. And from the Cartesian product of the two sets based on certain conditions, we find the relation matrix, and based on the certain conditions we find the relation set, and this can be represented in the matrix form, so we call this as the relational matrix. And also the relation set that we get here is always the subset of the set which is coming out by the Cartesian product of the sets.

So, in today's lecture we have understood that, how do we find the Cartesian product of two crisp sets and then how do we get the relation set based on certain conditions. And we have seen this by taking couple of examples. And this way we have understood the Cartesian product and relation and then relational matrix. And in the next lecture, we will continue with the fuzzy relations.

Thank you.