Fuzzy Sets, Logic and Systems and Applications
Prof. Nishchal K. Verma
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 24
T-norm Operators

So, welcome to lecture number 24 of Fuzzy Sets, Logic and Systems and Applications. So, in this lecture, we will discuss T-norm Operators. This is also called S-conorm operator.

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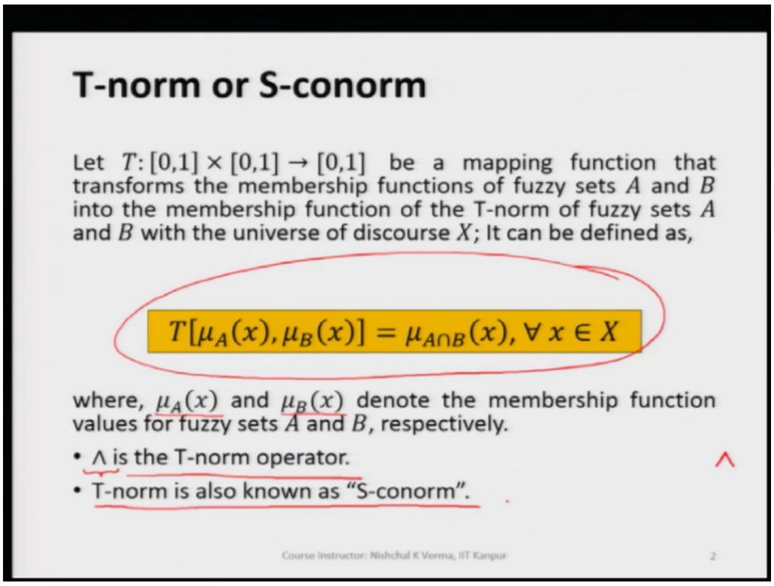

Now, understand what is a T-norm or S-conorm operator. So, let this be defined by capital T and this is nothing but mapping function that transforms the membership function of any fuzzy set. So, here we have taken let's say two fuzzy sets A and B into the membership function of the T-norm of fuzzy sets A and B with the universe of discourse capital X.

So, this can be defined by this expression here, and please read this as the operator T of $\mu_A(x)$, that is and $\mu_B(x)$. So, this means that if we apply T-norm or S-conorm here on pairs of membership values $\mu_A(x)$ and $\mu_B(x)$ this is going to be $\mu_{A \cap B}(x)$, for every X belongs to the universe of discourse. And it is very clear here that the $\mu_A(x)$ you can see here, and $\mu_B(x)$ these two as I already mentioned they denote the membership function values for fuzzy sets A and B respectively.

Please also understand that T-norm can be represented by a triangle sign, which is open triangle like this here, this a T-norm or S-conorm. So, wherever we want to write a T-norm we can either use capital T or we can simply use this open triangle sign or symbol for this thing.

So, we can say open triangle symbol is the T-norm operator which is mentioned over here and T-norm is also known as S-conorm which I have already mentioned. So, let us understand few more things related to T-norm or S-conorm.

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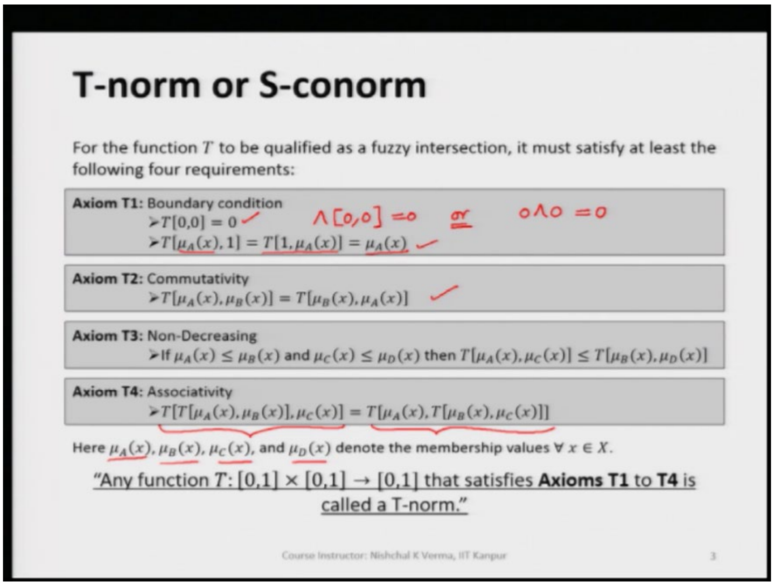

So, we have 4 axioms and these axioms are namely axiom T1, axiom T2, axiom T3, axiom T4. So, the axiom T1 is boundary condition, this means that if we take T-norm of 0 and 0, as I have already mentioned both the 0s are the membership values, so this is going to give us 0. So, if we apply the T-norm on two membership values, here these two membership values are 0, 0, means the lowest values and this is going to return as the 0 which is again a membership value. I can use the symbol open triangle symbol for T-norm or S-conorm.

So, I can write the same thing like this or this can also be written as the $0 \wedge 0$ which is going to give us 0. Now, if we apply the T-norm or we can say in other words, if we take T-norm of any membership value which is represented by $\mu_A(x)$ here and then we have 1. So, if we take these two membership values, first one is $\mu_A(x)$ and the second one is 1 and please note that this 1 is the highest value which a membership value can attain.

So, if we have these two together and if we apply the T-norm or we take T-norm, this is going to give us the value which is a $\mu_A(x)$ which is here and this can also be written by $T[1, \mu_A(x)]$. So, this means that whatever membership value that $\mu_A(x)$ can attain right, this does not matter if we have 1. So, whatever value that $\mu_A(x)$ has will be returned if we take $T[\mu_A(x)]$ and 1. And please note, 1 is the highest value of any membership value which can be there in the in case of fuzzy sets, alright.

So, we see here that axiom T1 in boundary condition we have the lowest value and we have the highest value. The lowest value here we have taken 0s in case of highest value we have taken 1. So, this boundary condition needs to be clearly understood.

Now, let us understand axiom T2. This is commutativity property of the T-norm. So, when we have two membership values, T of $\mu_A(x)$ and $\mu_B(x)$ this can also be written as T of $\mu_B(x)$ and $\mu_A(x)$. Means, the order can be interchanged, ok.

So, next is the axiom T3, the third axiom. So, which is non decreasing property. We see here, if we have $\mu_A(x) \leq \mu_B(x)$; means that the value of $\mu_A(x) \leq \mu_B(x)$ $\mu_A(x)$ and $\mu_C(x) \leq \mu_D(x)$, then we have triangular norm or the T-norm of $\mu_A(x)$ and $\mu_C(x)$ will be less than or equal to triangular norm of.

So, I am saying triangular norm, so either T-norm or triangle, this both these terms are used interchangeably. So, we can either use T-norm or we can use triangular norm or we can use S-conorm. So, please understand all these 3 terms we will be using interchangeably.

So, this way I am repeating that if this is the condition which is satisfied, means if we have $\mu_A(x) \leq \mu_B(x)$ and $\mu_C(x) \leq \mu_D(x)$ then the triangular norm of $\mu_A(x)$ and $\mu_C(x)$ will be less than or equal to triangular norm of $\mu_B(x)$ and $\mu_D(x)$. So, this is called the non-decreasing property.

Now, the next axiom is the T 4 which is for associatively property. So, when we have 3 membership values say $\mu_A(x)$, $\mu_B(x)$, $\mu_C(x)$, these 3 membership values if we have, so we can apply the triangular norm like this that the triangular norm of, the triangular norm of $\mu_A(x)$ and $\mu_B(x)$ and $\mu_C(x)$ you can see here clearly, this is going to be equal to the triangular norm of $\mu_A(x)$ and the triangular norm of $\mu_B(x)$ and $\mu_C(x)$. Please understand that this $\mu_A(x)$, $\mu_B(x)$, $\mu_C(x)$, these 3 are the membership values of the respective fuzzy sets A, B and C, so which is mentioned here.

And we have one more membership value which is $\mu_D(x)$. So, D here signifies that we have the membership value, which is with respect to the D fuzzy set. And this is needless to say that all of these x the generic variable that has been included in all the fuzzy sets that we have just seen, the respective membership values that we have used the x here the generic variable here is belonging into the universe of discourse. So, now we can clearly say that any function $T: [0,1] \times [0,1] \rightarrow [0,1]$.

If it is a triangular norm this is going to satisfy all the axioms of the T-norm. So, we can say axioms T 1 to T; we can say that this operator is qualifying to be called as the T-norm or S-conorm. Once again we can also say this as the triangular norm. So, this way we understand that how these axioms needs to be satisfied before we call T as the triangular norm.

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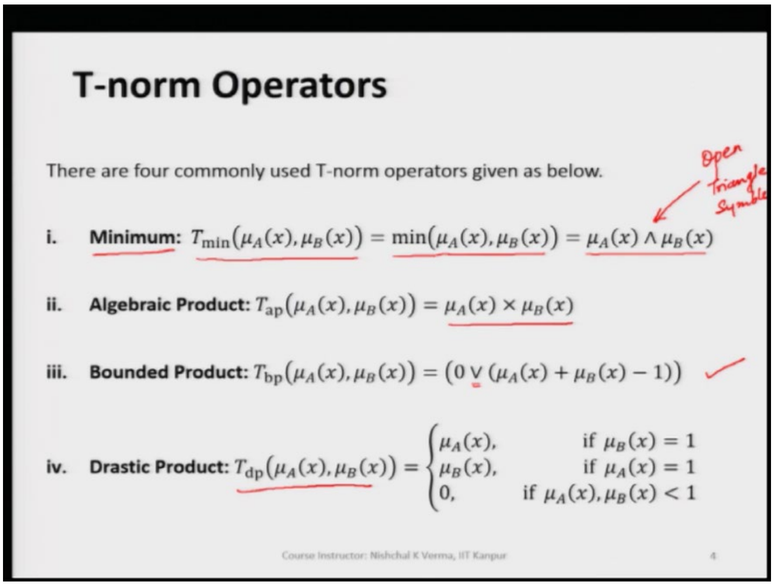

Now, there are 4 commonly used T-norm operators that we will be seeing here. So, we have the minimum and this is defined as the T min of $\mu_A(x)$ and $\mu_B(x)$ or can be written as min of $\mu_A(x)$ and $\mu_B(x)$. So, what we are seeing here is that, we have replaced T min by min. So, this is when we are dealing with the T-norm operator as the minimum case. So, this is there are 4 cases here, there are 4 commonly used T-norm operators. So, minimum is a one of the operators, if we are interested in a minimum T-norm operator then we will simply replace T min by min.

So, let us say if we are interested in finding the minimum T-norm operator. The minimum T norm operator $T_{min}\big(\mu_A(x), \mu_B(x)\big)$ is going to be $\min\big(\mu_A(x), \mu_B(x)\big) = \mu_A(x) \wedge \mu_B(x)$. And please understand that we can write this min by using the $\wedge$ symbol here. So, we can see here that this open triangle symbol for T-norm as min operator.

Alright so, next is we have the T-norm as the algebraic product this is denoted by T_{ap}, small a, small p is here for algebraic product. So, $T_{ap}(\mu_A(x), \mu_B(x))$ is going to give us the membership values that are $\mu_A(x)$ and $\mu_B(x)$. So, this is simply here when we are interested in T-norm as algebraic product, so we will simply multiply the membership values.

Next is the bounded product. This represented by T_{bp}. So, T subscript bp, small b, small p, here we have used the triangular norm, but this is inverted triangle, $\vee$. So, here we can use this inverted triangle here which is used normally for max, like in the first case we use the open triangle symbol here, this is the inverted open triangle. So, we see that we take the $(0 \vee (\mu_A(x) + \mu_B(x) - 1))$. So, this way we can find the bounded product.

The fourth one is here the drastic product when we use this we write this as $T_{dp}(\mu_A(x), \mu_B(x))$, so this is going to be like this is equal to $\mu_A(x)$ when we have $\mu_B(x)$ equal to 1, and this is going to give us $\mu_B(x)$ when $\mu_A(x)$ is equal to 1. And this is going to give us 0, when we have $\mu_A(x)$ and $\mu_B(x)$ less than 1. So, this way we understand how these 4 commonly used T-norm operators are being defined.

Let us take an example here to understand T-norm operator better.

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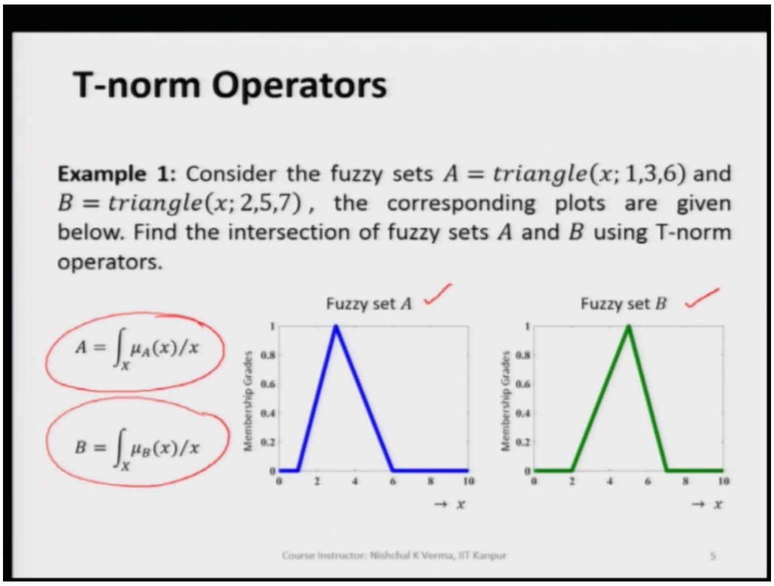

So, we have taken here fuzzy set A here and another fuzzy set B here. Both the fuzzy sets A B are triangular, continuous fuzzy sets. So, we see that these are defined by the fuzzy set A and B mathematically defined. And what we are interested here in is that we are interested in finding the intersection of fuzzy sets A and B using T-norm operator. So, basically when we say T-norm operator so if nothing has been set so we use min operator as T-norm.

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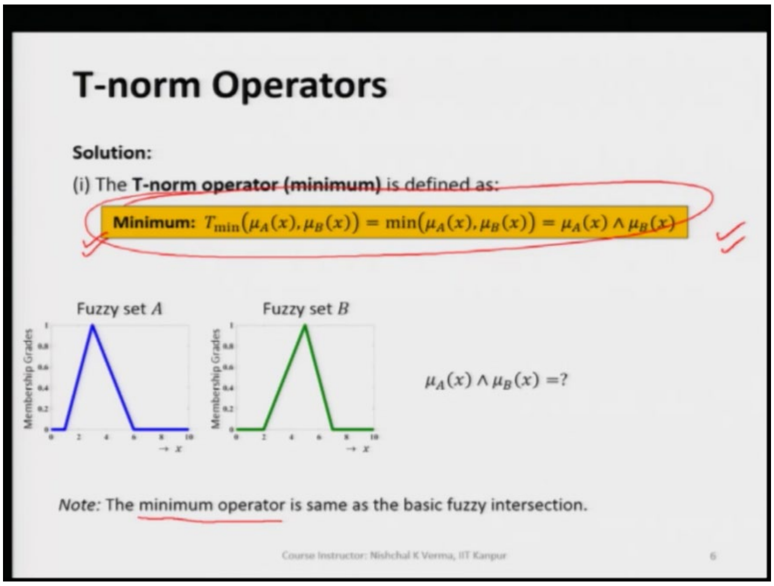

So, here we see that we have used minimum here because nothing has been set as the T-norm operator and which we can find here as the for corresponding every value of each and every value of $\mu_A(x)$ and $\mu_B(x)$ like this. So, we'll take the minimum of each and every corresponding values of $\mu_A(x)$ and $\mu_B(x)$ and you can see how it is found.

So, this way let us now proceed, and one more thing I would like to mention here that when we say the minimum T-norm operator or T-norm operator as the minimum operator. So, when we say this, this is also called a basic intersection operator. Whenever we are using the minimum T-norm operator, we normally get the intersection of the fuzzy sets.

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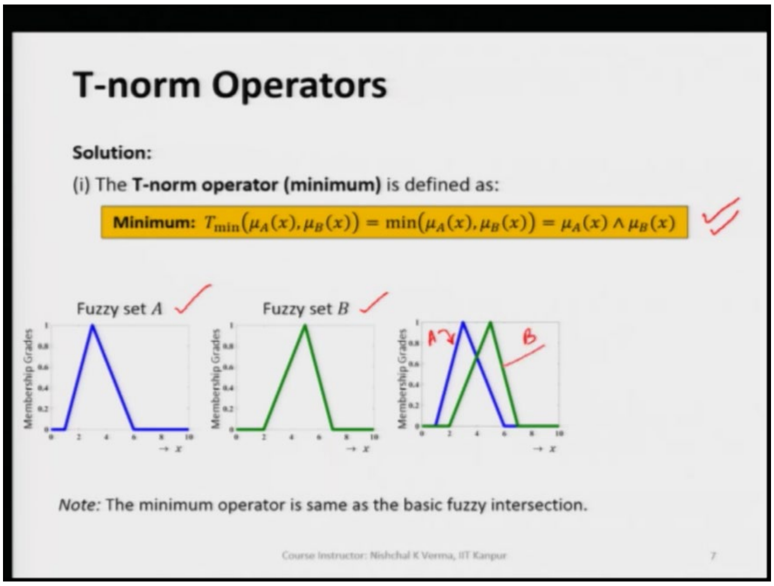

So, when we apply this condition of T-norm which is once again I would like to mention that the minimum class.

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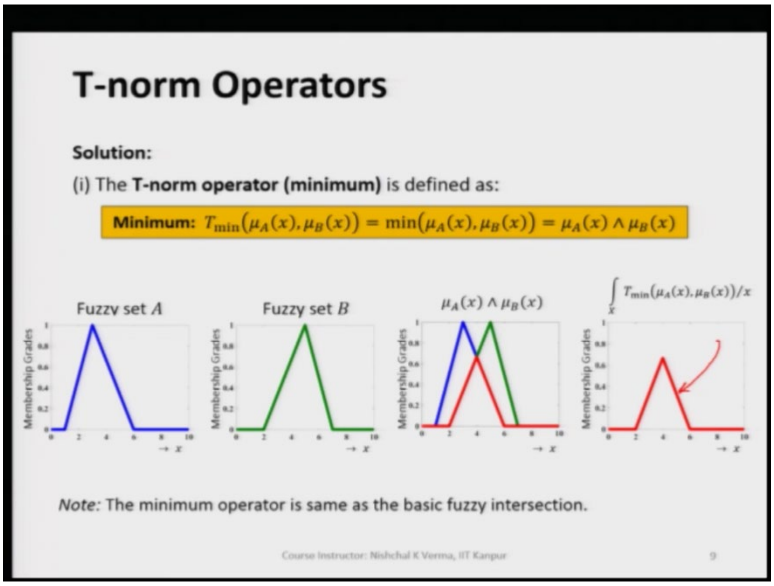

So, when we take the minimum T-norm operator we after applying the condition on fuzzy set A and fuzzy set B, and when we superimpose these two fuzzy sets here together we see that this is A, and this is B. So, when we apply this criteria we use the minimum criteria, minimum condition for each and every values of fuzzy membership. So, we see that we are getting this as the minimum, means the lower envelop we are getting as the outcome of T_{min}, which can be seen here that we are getting here as the output of intersection of fuzzy set A and fuzzy set B.

We are saying intersection, but since we are studying the fuzzy T-norm in this lecture we will use the term the T_{min}. When we use T_{min} it means we are getting the intersection of the two fuzzy sets.

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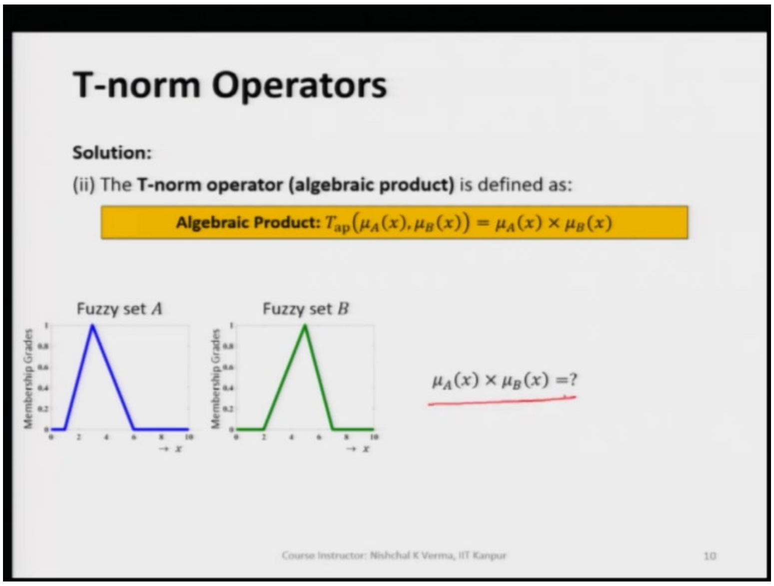

Now, on the same fuzzy sets let us see what happens when we use algebraic product. So, we have just seen that the T_{ap} we get by multiplying each and every membership values of fuzzy set A and fuzzy set B. So, there is how written here. And when we do that and we will multiply we are going to get this membership function I would say because here A and B are continuous fuzzy sets after multiplying the corresponding membership values of fuzzy set A and B we are getting a continuous membership function.

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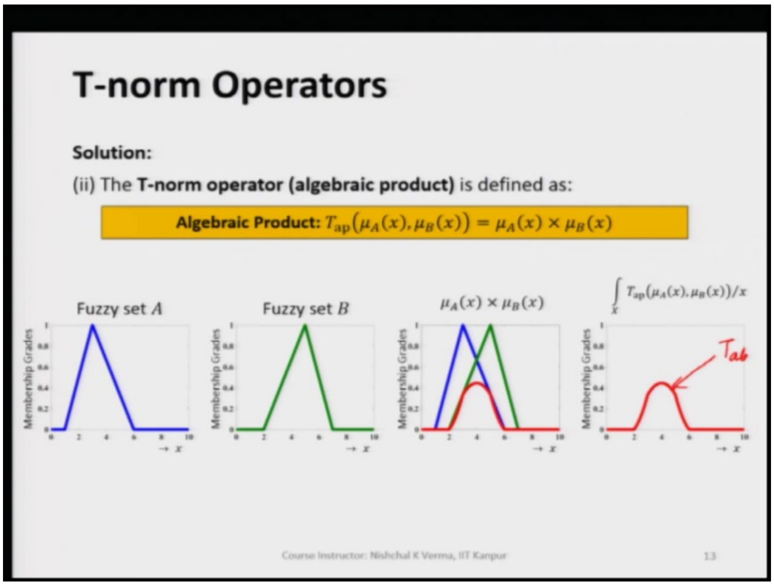

So, it has been shown by the red color and we can see here that this is we are getting as $T_{ap}(\mu_A(x), \mu_B(x))$.

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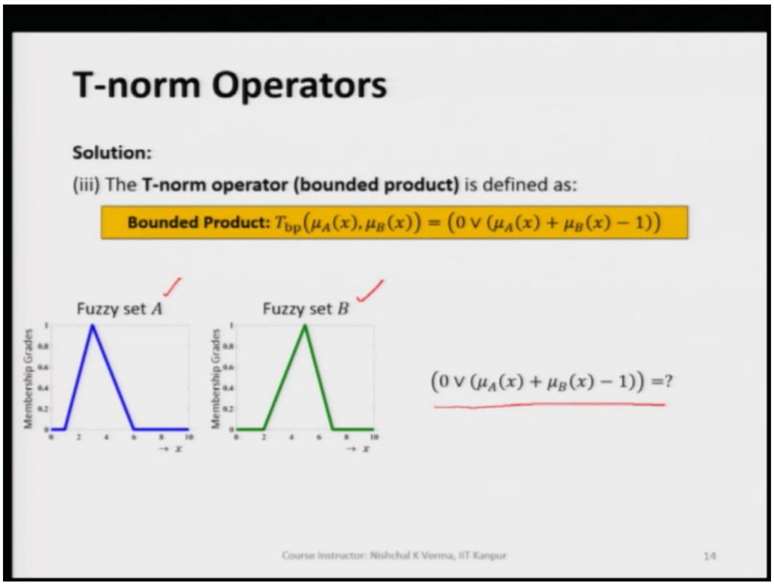

Now, let us quickly understand the bounded product as the T-norm operator. So, keeping these fuzzy sets same A and B, when we apply the bounded product let us see what we are getting.

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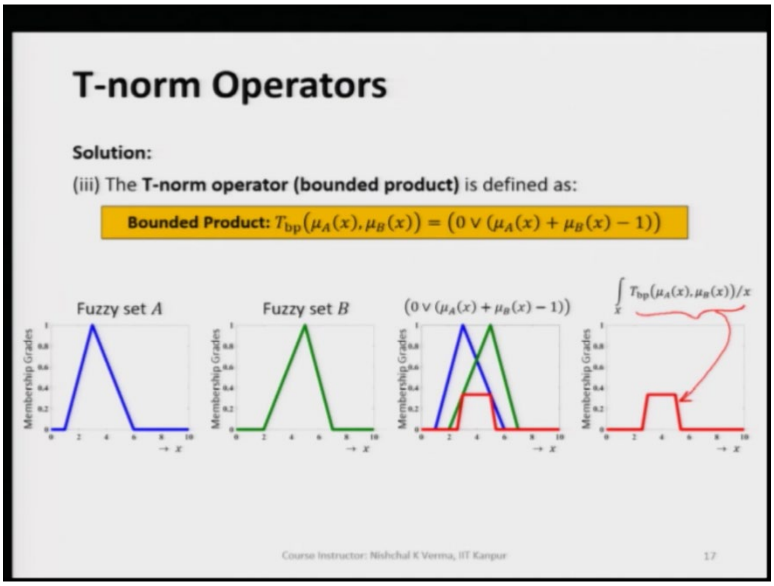

So, we are getting here, a continuous curve which is the membership curve, which is the resultant of the bounded product of membership values of A and B. So, I would say here that each and every values of membership for A and B fuzzy sets. So, when we write this separately, we see that we are getting here bounded product $T_{bp}(\mu_A(x), \mu_B(x))$.

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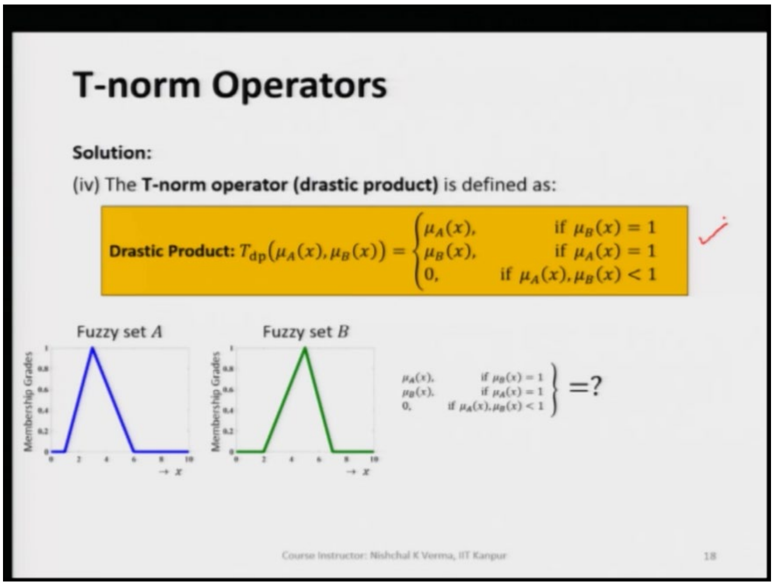

Similarly, when we see that drastic product, if we are interested and we apply the drastic product conditions, the drastic product formula that we have just covered, so we see that we are getting here this as the outcome of the drastic product of the membership function of fuzzy set A and the membership function of fuzzy set B.

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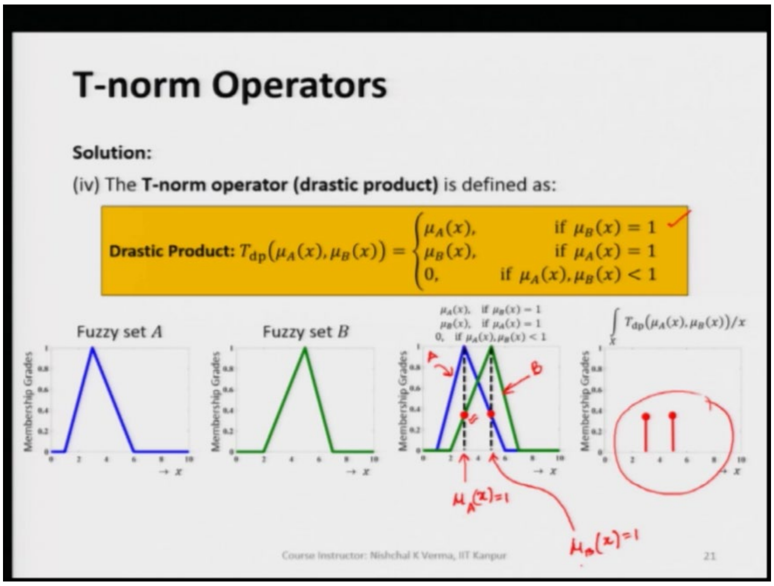

And please understand, here we are getting only two values. So, here we are getting the discreet values or I would say you are getting only two membership values as the outcome of the drastic product of the continuous fuzzy sets after applying this condition. So, here we can clearly see that we are getting drastic product as the mu A the value of $\mu_A(x)$ is equal to 1, if $\mu_B(x)$ is equal to 1. So, we can see here this is our membership function for fuzzy set A and this is the membership function for the fuzzy set B.

So, we can see here that we are going to get only this value here as the outcome at this we have $\mu_A(x)$ which is equal to 1, and at this condition here we have $\mu_B(x)$ is equal to 1. So, at both the points we have here we have $\mu_A(x)$ is equal to 1. So, at this point we will have $\mu_B(x)$ whatever value of $\mu_B(x)$ is and then here we will get at a $\mu_B(x)$ is equal to 1 and for all other values we are going to get 0. So, that is why we are getting here only two membership values out of the drastic product of the membership functions of continuous fuzzy set A and continues fuzzy set B respectively.

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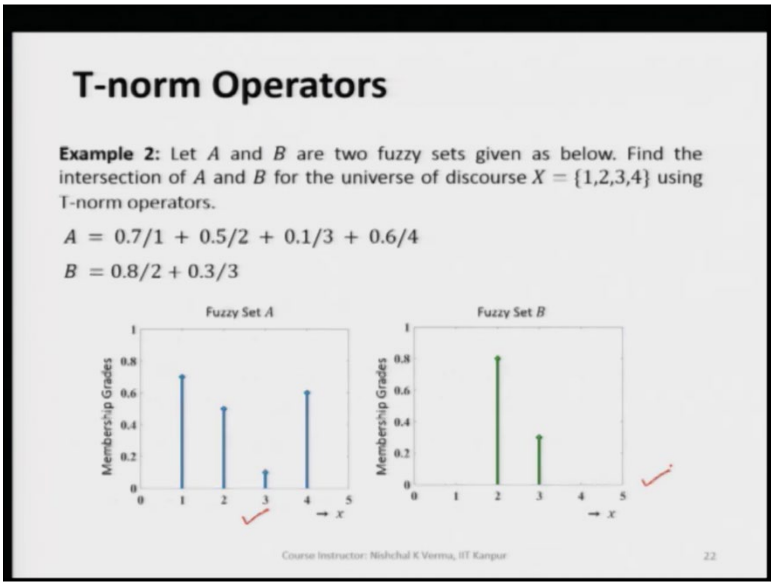

Let us take another example where we are taking the discreet fuzzy sets A and B, and these are represented by here the discrete points. So, we can see here fuzzy set A and fuzzy B.

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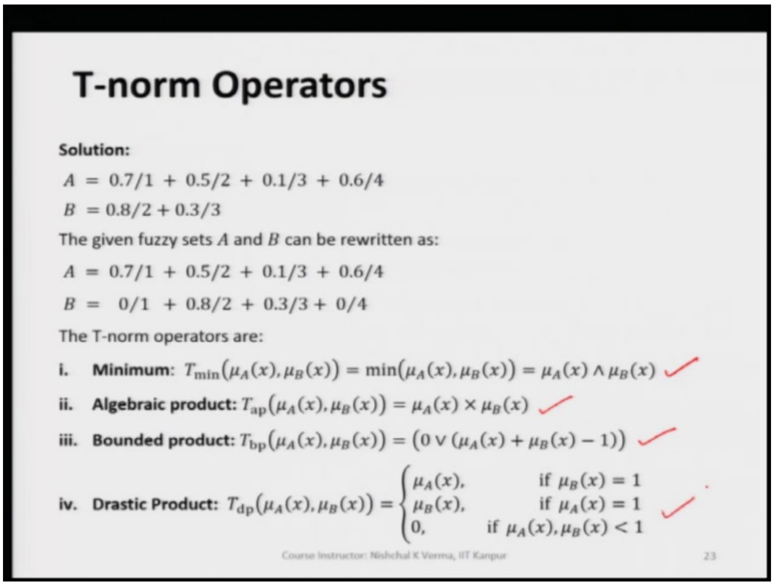

And when we apply the T_{min} criteria, like when we are interested in T-norm of the $\mu_A(x)$ and $\mu_B(x)$ as the minimum. So, then we apply this criteria, similarly for T_{ap} we applied this criteria and T_{bp} we apply this criteria, T_{dp} we apply this criteria.

So, when we apply this for T_{min} when we apply we are going to get this as the outcome.

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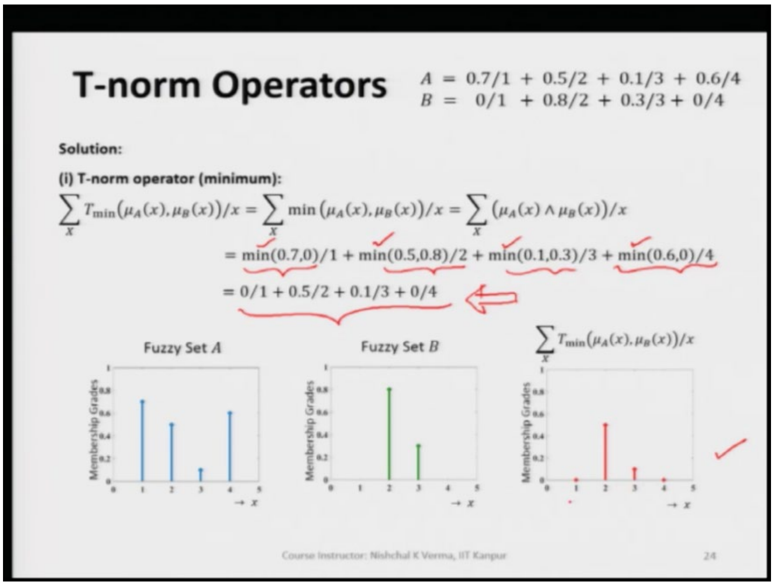

You can see here for each and every values of membership for the fuzzy set A and B we are taking min. So, you can clearly see here we are using min here. And when we take this, this is the outcome that we are getting. And when we show it here as the fuzzy set you can see that these values are plotted here and we see here that we get the T_{min} of these two discrete fuzzy set has shown here.

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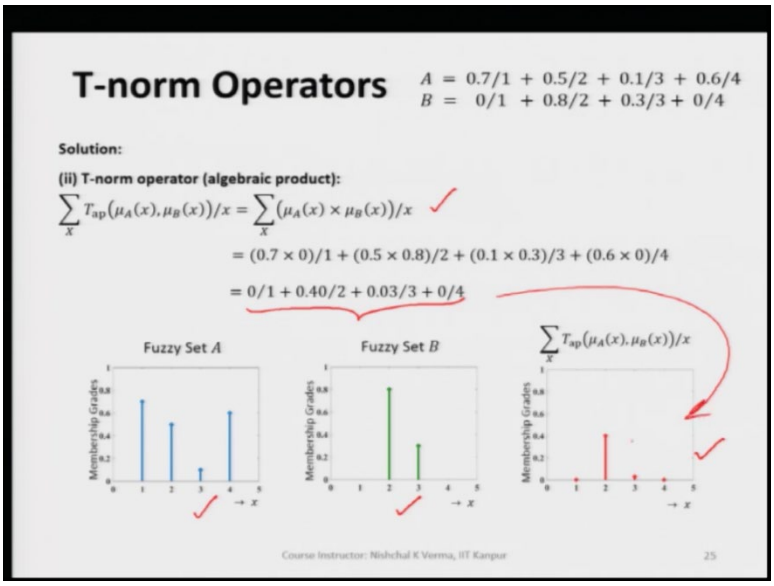

So, next is the T_{ap} of the same discreet fuzzy sets A and B. So, as you have already seen these two fuzzy sets A and B, now when we take the algebraic product when we apply the this criteria, the multiplication and when we multiply the respective membership values for the A fuzzy set and B fuzzy set we are getting this as the outcome which has come like this and when we plot this we are getting.

So, now, the next is for the same fuzzy set we have bounded product and when we apply this criteria, this formula forT_{bp}, we are going to get here this as the outcome $0/1 + 0.3/2 + 0/3 + 0/4$.

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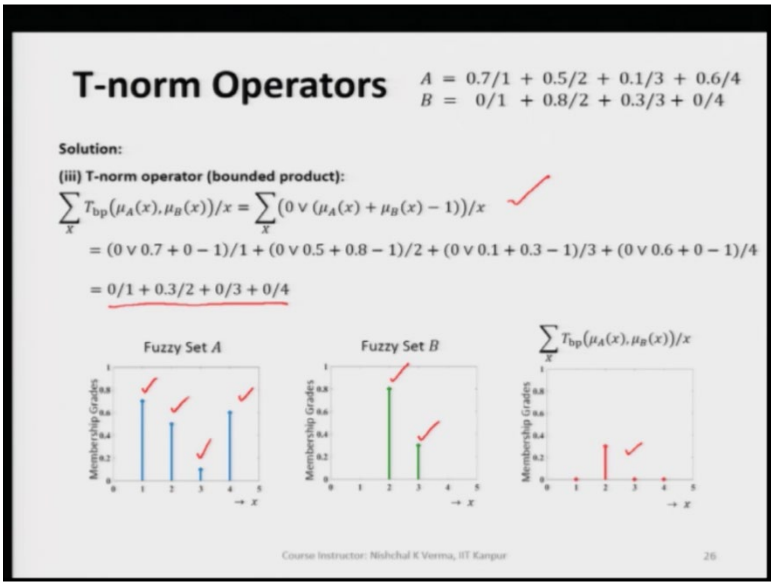

So, we can clearly see here that we are getting only one membership value out of many membership values here, where we have in fuzzy set we have 1, 2, 3, 4, 4 membership values and in fuzzy set we have two membership values, but as the outcome we are getting only one membership value. So, when we are interested in the bounded product of these two fuzzy sets, I would say that T_{bp} you are going to get this as the outcome.

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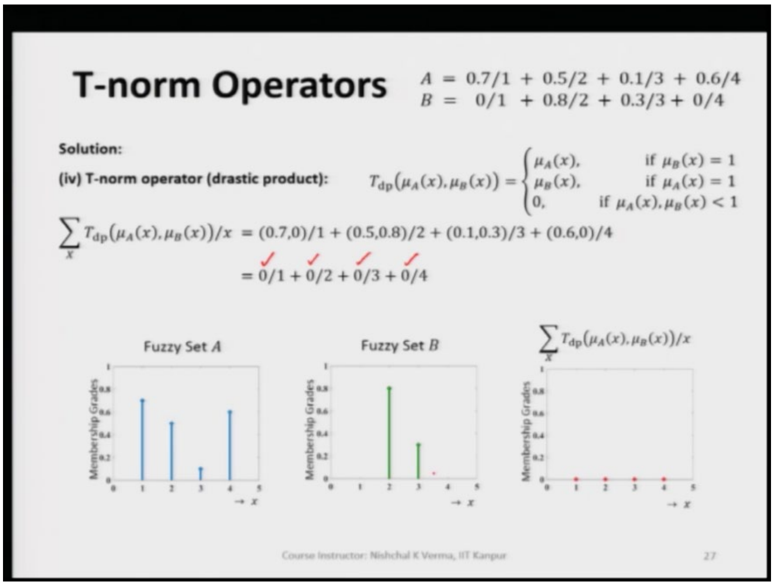

Similarly, when we find the T_{dp} of the same fuzzy set, same discreet fuzzy sets A and B we are getting here null fuzzy set. So, means, we are getting for the same fuzzy sets, we are not going to get any output. You can clearly see here that we have all the elements of this fuzzy sets discrete fuzzy set as the outcome, so each and every element has its membership value 0, which we never account as the element of the fuzzy set. We only account only those elements, which has the membership value more than 0. So, here we can say we are getting a null fuzzy set out of these two when we are taking the T_{dp} of these two discrete fuzzy sets.

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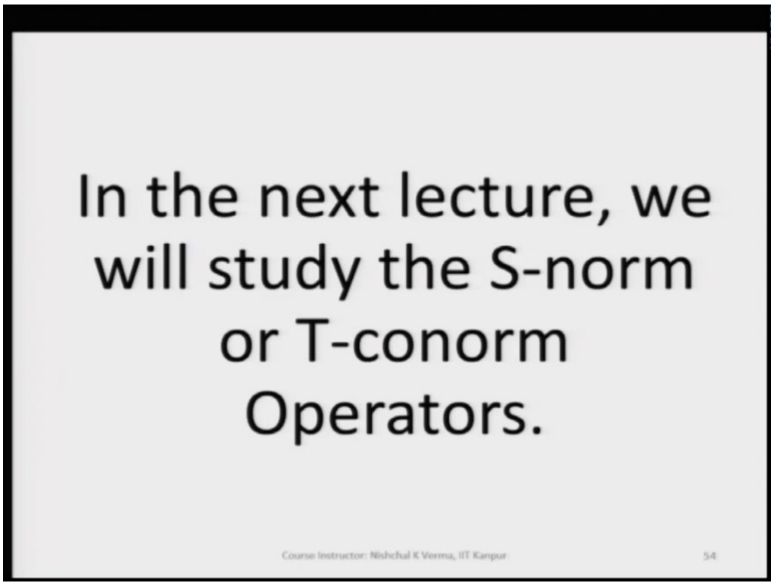

In the next lecture, we will study the S-norm which is also known as T-conorm.

Thank you very much.