Fuzzy Sets, Logic and Systems and Applications
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Lecture - 12
Properties of Fuzzy Sets

So, welcome to the lecture number 12 of Fuzzy Sets, Logic and Systems and Applications. This lecture will cover the Properties of Fuzzy Sets.

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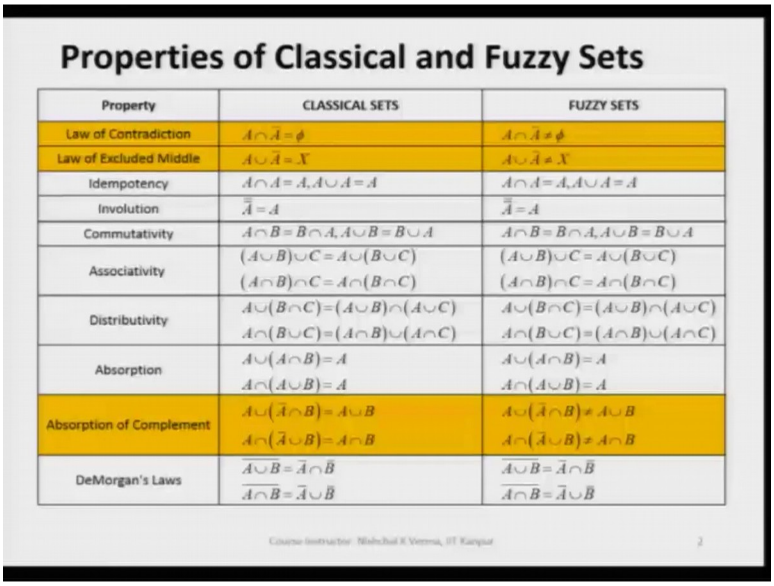

Property	CLASSICAL SETS	FUZZY SETS
Law of Contradiction	$A \cap \bar{A} = \phi$	$A \cap \bar{A} \neq \phi$
Law of Excluded Middle	$A \cup \bar{A} = X$	$A \cup \bar{A} \neq X$
Idempotency	$A \cap A = A, A \cup A = A$	$A \cap A = A, A \cup A = A$
Involution	$\bar{\bar{A}} = A$	$\bar{\bar{A}} = A$
Commutativity	$A \cap B = B \cap A, A \cup B = B \cup A$	$A \cap B = B \cap A, A \cup B = B \cup A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Absorption of Complement	$A \cup (\bar{A} \cap B) = A \cup B$ $A \cap (\bar{A} \cup B) = A \cap B$	$A \cup (\bar{A} \cap B) \neq A \cup B$ $A \cap (\bar{A} \cup B) \neq A \cap B$
DeMorgan's Laws	$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$	$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$

So, in a way here we have the set properties and we'll see whether fuzzy sets also follow these properties or not. And we already know that the classical sets follow these properties and these properties are Law of a Contradiction, Law of Excluded Middle, idempotency, involution, commutativity, associativity, distributivity, absorption, absorption of complement, DeMorgan's laws. So, these are the properties that we know that a classical sets follow, but let's see whether the fuzzy sets also follow these properties or not.

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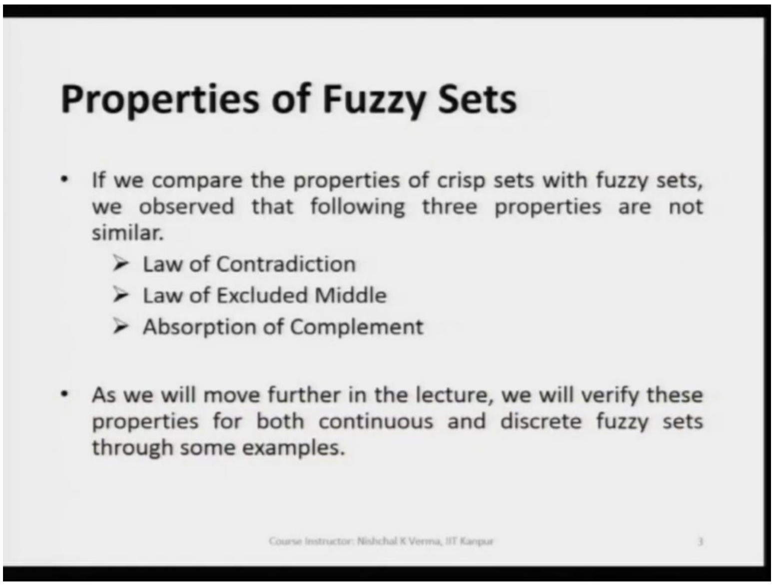

So, if we compare the crisp sets and fuzzy sets, we see that three properties are not followed by the fuzzy sets, like we have a law of contradiction. So, when we use fuzzy sets instead of crisp sets, we see that there is a contradiction and so that is why the law of contradiction comes into picture and similarly the law of excluded middle also comes.

And another one the third one the absorption of complement that is not followed by the fuzzy sets and as we move further in this lecture, we'll verify these properties for both continuous and discrete fuzzy sets through various examples.

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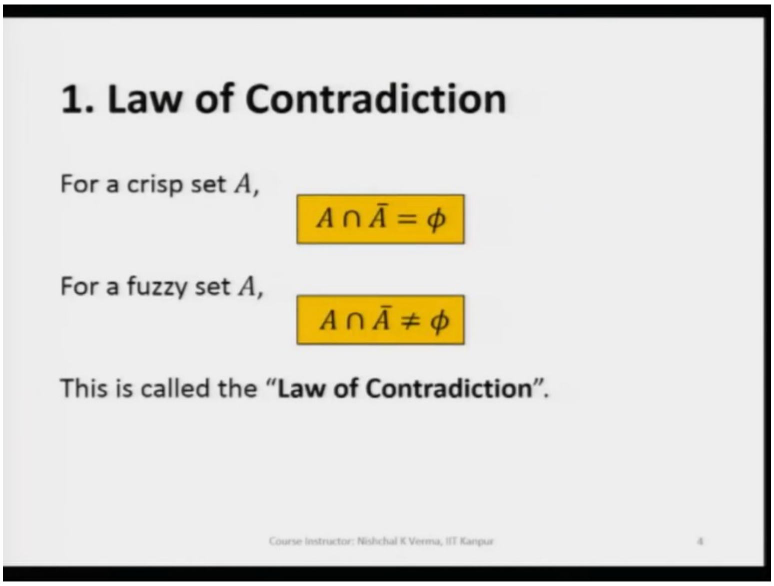

So, let us now discuss the set theoretic properties one by one and the first one is the law of contradiction. So, let us take first crisp set and see what is happening with the crisp set if we take $A \cap \acute{A}$. So, if we take any crisp set A, $A \cap \acute{A} = \phi$.

Now, if we take the case of a fuzzy set, so let's say we have a fuzzy set A and we do the same intersection, we take the same intersection on this fuzzy set A and its complement. So, this intersection is never a null set. So, we clearly see that crisp set, there is a difference in the intersection of crisp set and its complement and then intersection between fuzzy set and its complement. And this has been clearly mentioned here that if we have a crisp set A, intersection between its complement is a null set. And if A is a fuzzy set, $A \cap \acute{A} \neq \phi$; that means, a not a null set.

So, we'll take example here one example here and we'll see how if a fuzzy set we have and then how what happens when we take the intersection of fuzzy set and its complement.

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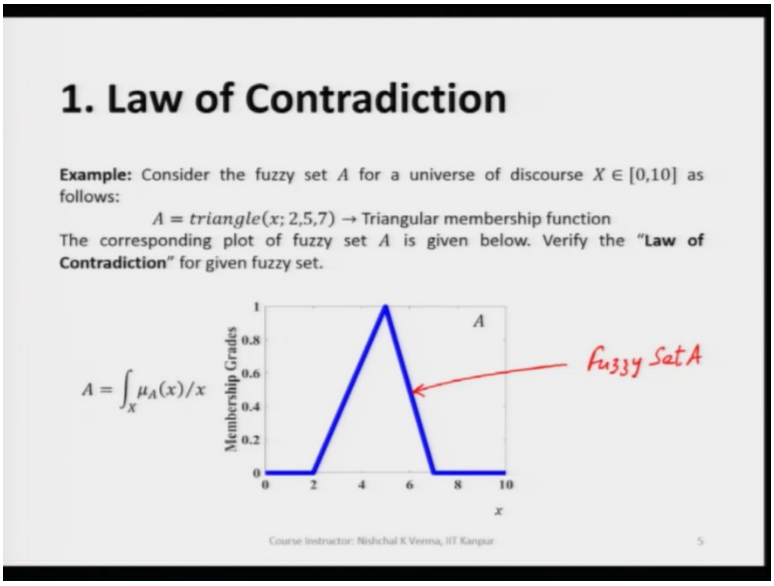

So, and of course, this is since this is not followed so, this contradicts right. So, there is a contradiction. So, contradiction means if we take a crisp set, we get null set as a result and if we take fuzzy sets we get a set which is not a null set as a result. So, that is why there is a contradiction and that is how this is called the Law of Contradiction.

So, if we take an example here, we take a fuzzy set A which is a triangular fuzzy set and this fuzzy set is defined by a triangular membership function. So, this is a triangular membership function here, but the whole thing is called the A fuzzy set fuzzy set A fuzzy set A.

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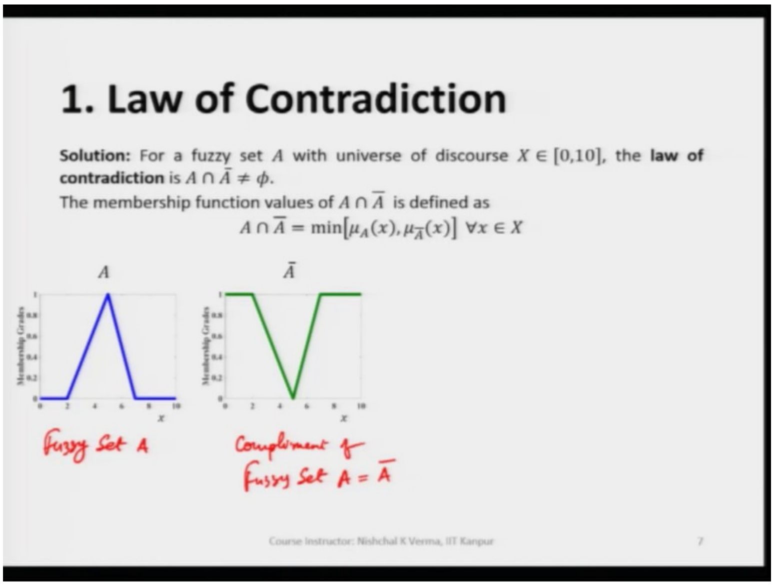

So, as we are trying to verify whether $A \cap \acute{A} = \phi$ or not so, let us now first have A here and then try to find $\acute{A}$. So, $\acute{A}$ is the complement of A. So, we see here is a complement of A fuzzy set which is represented by the green color. So, this is we can say the complement of fuzzy set A and this is this will be denoted by $\acute{A}$ which is complement of fuzzy set A and this is fuzzy set A fuzzy set A.

So, if we take intersection of these two fuzzy sets so, we already know if we are taking the intersection of any two fuzzy sets, we follow the criteria; we follow the condition as the min we take the minimum of the membership values of the corresponding generic variable values at each and every point. And this is needless to say that all of these generic variable values must belong to the universe of discourse.

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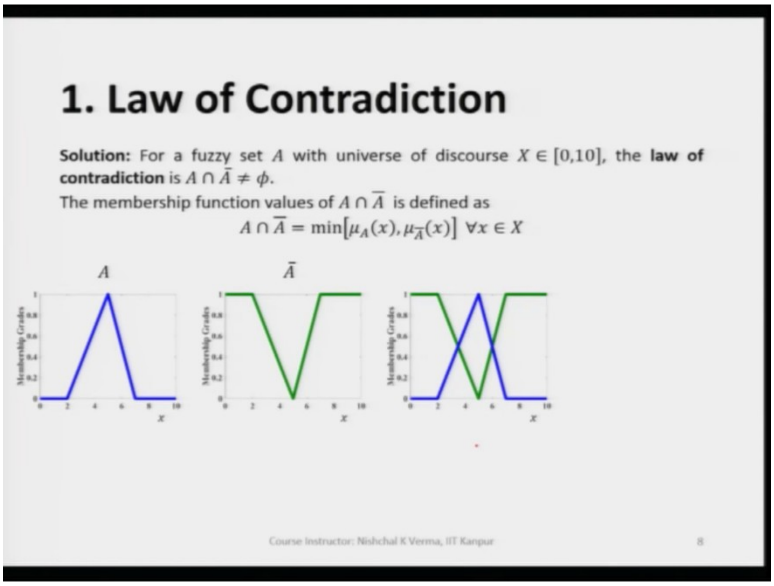

So, here we have the $A \cap \acute{A}$. So, these two A fuzzy set and $\acute{A}$ have been overlapped or super imposed to find the intersection of these two fuzzy sets.

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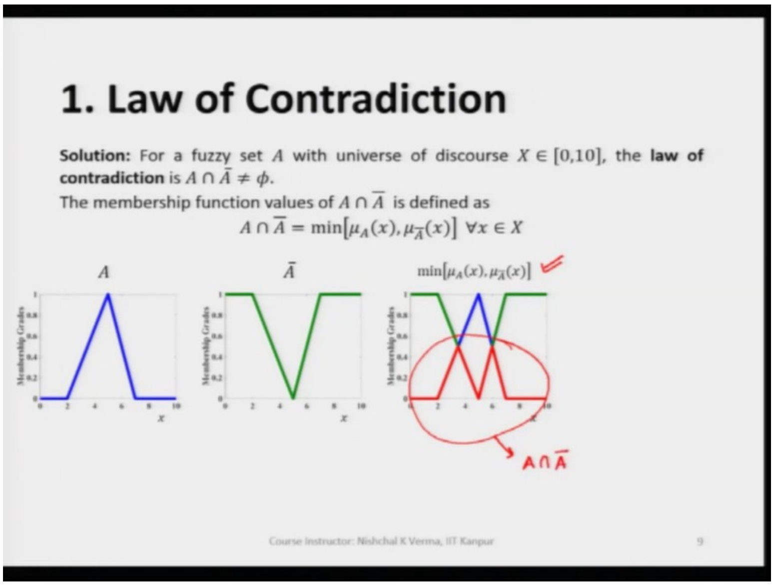

So, it's very clear here if we apply this criteria, this condition we see that we are getting this red the portion which is marked by red color. So, this portion is basically the $A \cap \acute{A}$ which is represented here by this separately by this membership function.

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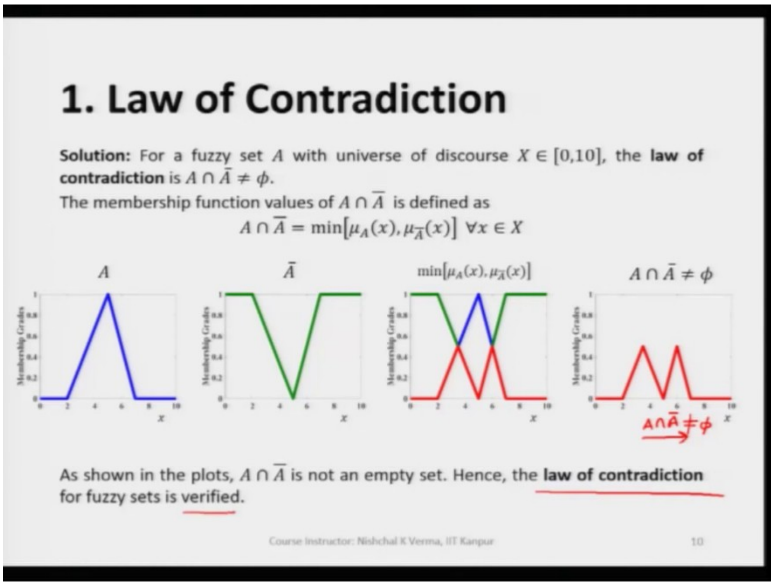

So, this is nothing but $A \cap \acute{A}$ and we clearly see that this is not a null set means, we are getting something here. We are getting a fuzzy set here, so obviously, this is not equal to ϕ means this not equal, this is not a null set. So, we can clearly say that the law of contradiction is verified.

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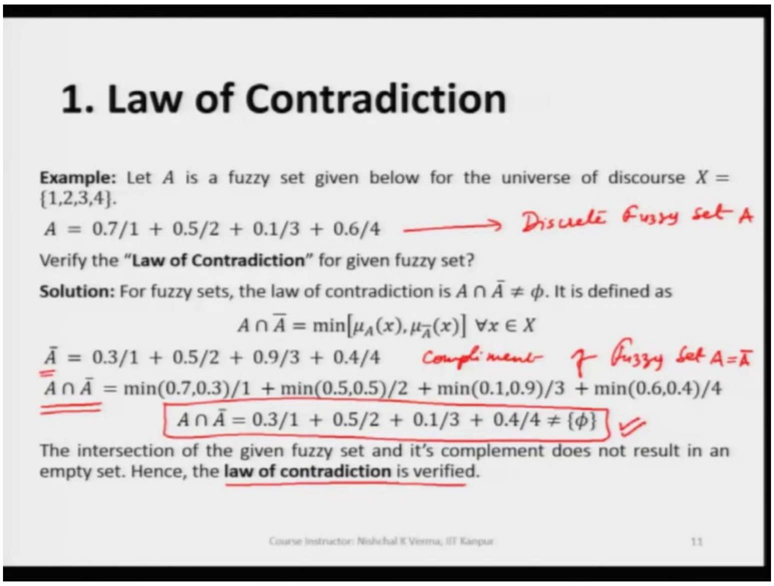

Now, if we take; so, this was the example when we took a triangular fuzzy set. So, obviously, it was a continuous fuzzy set. But if we take the discrete fuzzy set just to see whether this law of contradiction is verified for discrete fuzzy sets also. So, let us take an example here and this example is with discrete fuzzy set. So, this is a discrete fuzzy set A, discrete fuzzy set and if we try to find the $\acute{A}$ of it the complement of the fuzzy set A is complement of fuzzy set A and this is nothing but $\acute{A}$ as it is written here.

So, if you take the intersection of these two fuzzy sets, again we see that we get something here which is not equal to the null set. So, it means that here also we are getting a set which is not a null set. So, we can say that the law of contradiction here also is verified.

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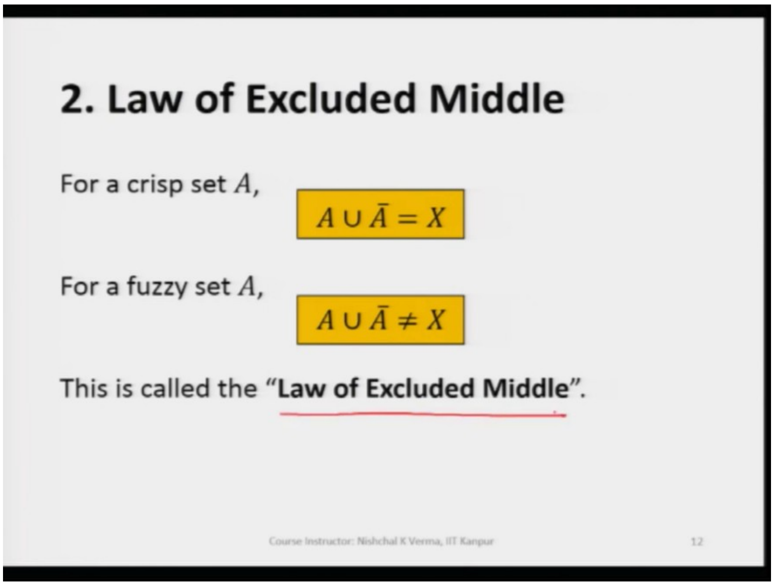

Then we have another set theoretic property and this is called the Law of Excluded Middle. So, if we take why is it called law of excluded middle, we will get to know in a moment. So, if we have crisp set A and if we take the $A \cup \acute{A}$. So, for crisp sets we always get the universe of discourse as a result, whereas if we take a fuzzy set A and if we do the same operation means we take the $A \cup \acute{A}$ this is never going to be the universe of discourse. So, here we have clear cut difference in between the crisp set the union of crisp sets and fuzzy sets. So, we call this discrepancy as this contradiction as the law of excluded middle.

So, here as I already mentioned that if we take crisp set A if we take union of $A \cup \acute{A}$, we should get the whole universe of discourse. Whereas, if we take fuzzy sets A, $A \cup \acute{A}$ is not going to give you the universe of discourse. So, that is what is the difference and this is called the law of excluded middle.

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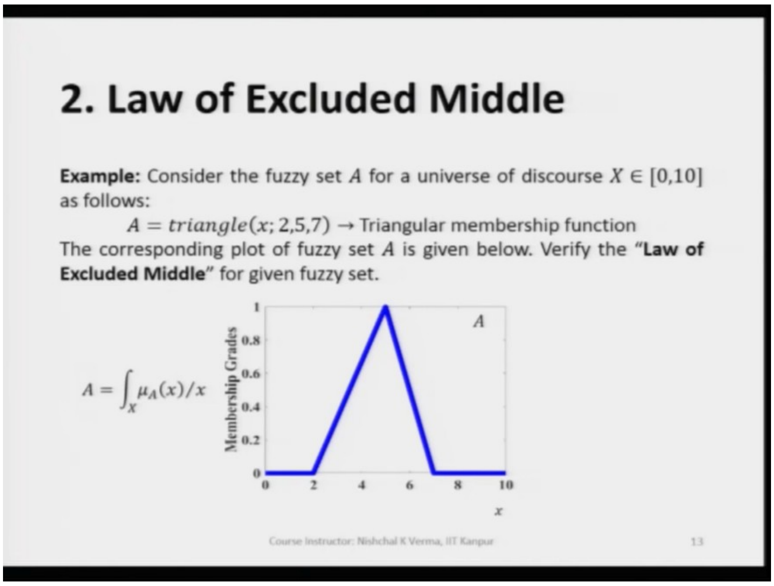

So, let us understand this law of excluded middle better by taking an example and here also we have taken a continuous fuzzy set which is triangular.

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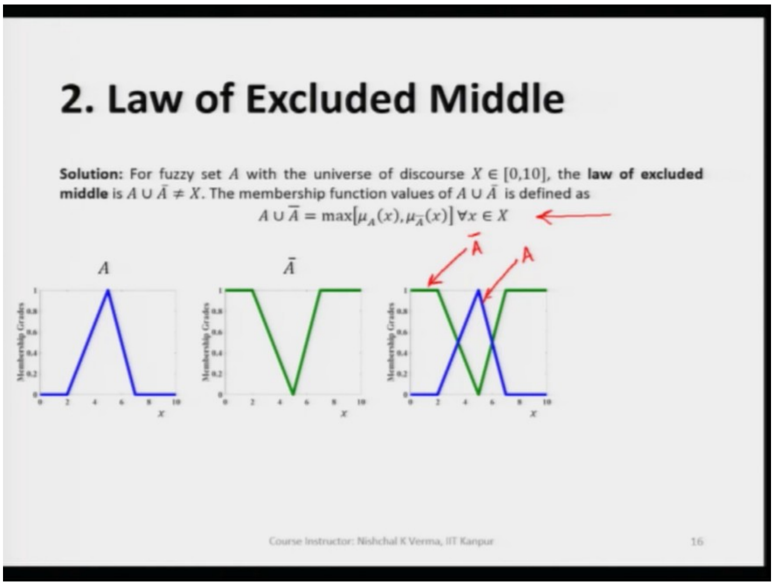

So, if we take a fuzzy set A here and the complement of this fuzzy set A is here and if we take the union of these two fuzzy sets so, here we have super imposed this fuzzy set which is $\acute{A}$ of this fuzzy set, right. So, if we apply the condition of the union the criteria which is here,

$$A \cup \acute{A} = max\left[\mu_A(x), \mu_{\acute{A}}(x)\right] \forall x \in X$$

So, what we get here is this which is represented by the red color.

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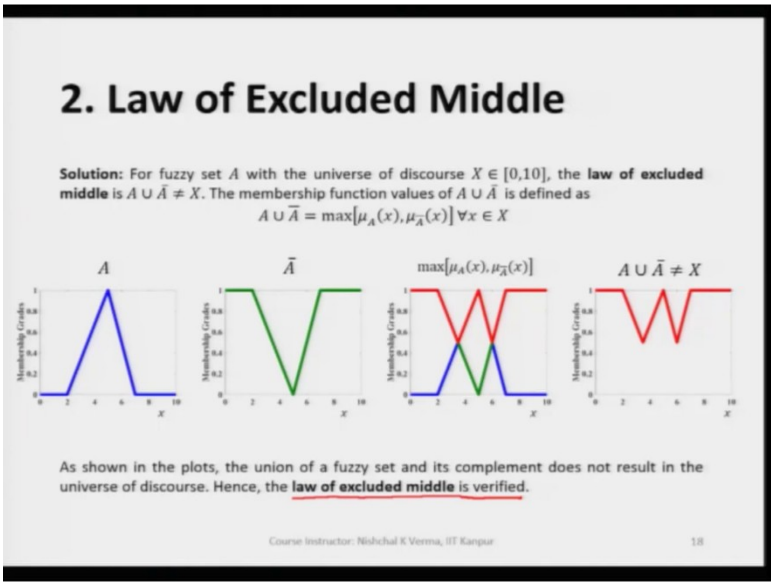

So, we get if we take the max, we get this portion right and here we apply the max of the corresponding membership values of the two fuzzy sets corresponding to the generic variable values within the universe of discourse. So, if we separately represent this, it looks like this. So, here is what we are going to get as result of fuzzy set $A \cup \acute{A}$ and which of course, is not the universe of discourse. So, what do we mean by this statement when we are saying that this is not equal to the universe of discourse? So, when this would have been the universe of discourse; so, when we would have got this as the straight line like this?

So, this portion should not have been there, then we would have said that the $A \cup \acute{A} = X$. But since here this is not exactly what I mean the straight line we have this portion also. So, we call this here as the $A \cup \acute{A} \neq X$.

So, this clearly gives us the idea as to how we check when we take the union of any fuzzy set and its complement. So, this is how the law of excluded middle is verified for a fuzzy set. Similarly we can take the example of a discrete fuzzy set here you see.

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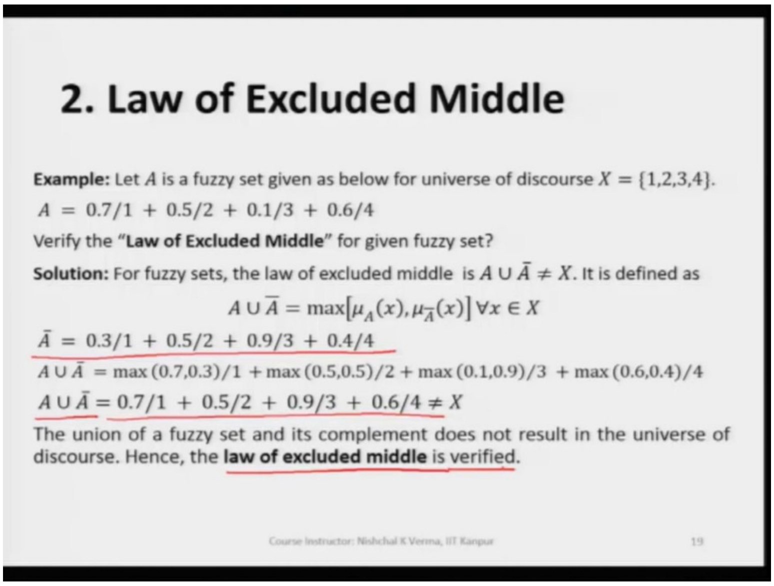

And then if we find the $\acute{A}$ which is the complement of fuzzy set A which was taken here which is taken here and we see that this is the complement of the fuzzy set A taken in this example which is also discrete. And if you take $A \cup \acute{A} \neq X$. So, here also this is true that the law of excluded middle is verified for any discrete fuzzy set.

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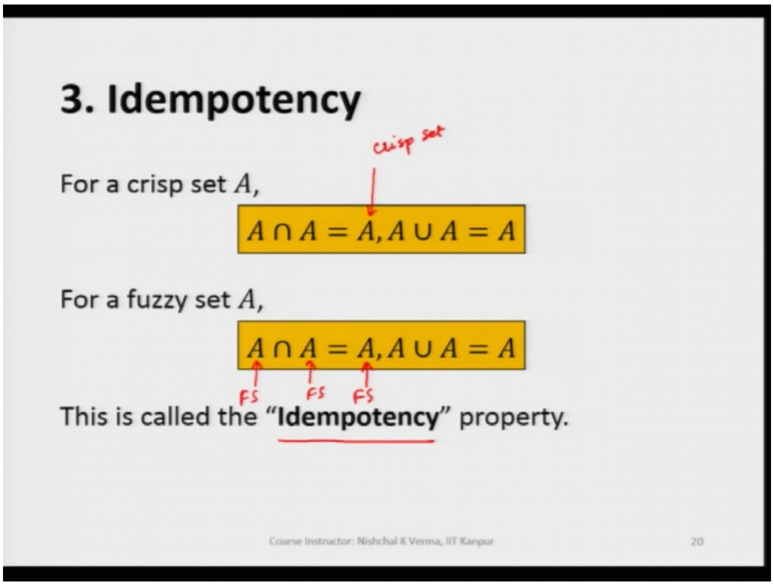

Now, we have another set theoretic property which is idempotency. So, let's now take a crisp set first and see how this idempotency property is verified for a crisp set. So, $A \cap A$ means if you take the intersection of the same set, we always get as a result the same set means if we take the intersection of $A \cap A = A$. So, which is here the same sets is a crisp set which was taken.

Then again if we take the $A \cup A = A$. So, this is known for crisp set which is you know which we have already done so many times in the past. But if we take fuzzy set let us see what happens. So, if we take fuzzy set here A. So, this is fuzzy set A and this is also fuzzy set A fuzzy set and this is also fuzzy set. So, if we take intersection of the same fuzzy set, we are going to get the fuzzy set same fuzzy set means $A \cap A$ or we say $A \cap A = A$. Similarly we take union it is going to return the same set and this is called the idempotency property.

So, what we see here is that we get the same set whether we take a crisp set or fuzzy set. So, that's how the idempotency property is verified for both the kinds of sets.

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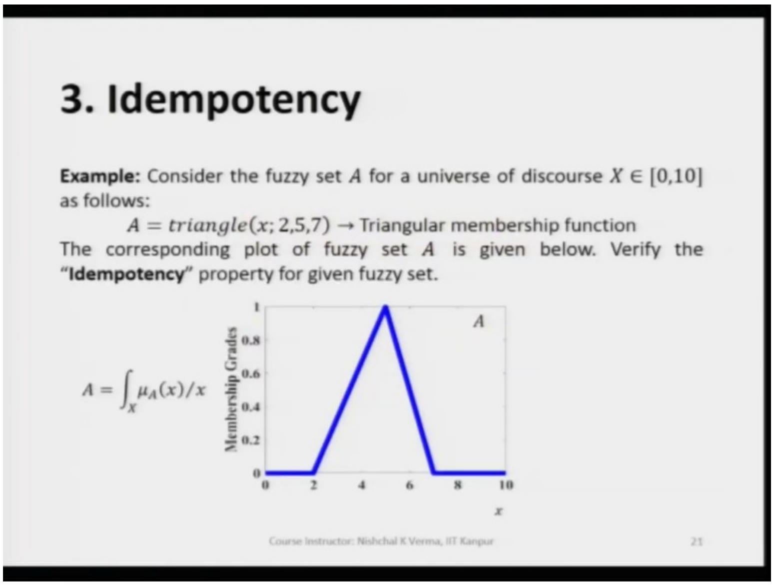

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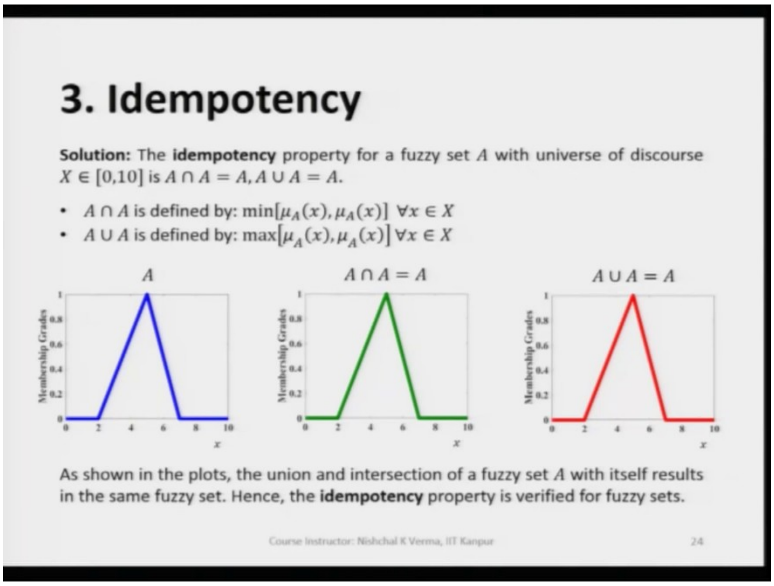

Let us take an example to better understand this property. So, if you take a triangular fuzzy set if we take a fuzzy set here as A, so since we would like to verify the idempotency property. So, we have to have the $A \cup A$and $A \cap A$ and we see what we are getting as a result of it. So, here we have A and then here we have $A \cap A = A$. So, if we intersect these two fuzzy sets, we are going to get the same set. Similarly if we are doing here if we are taking $A \cup A$, we are going to get the same set. So, this way we can say the idempotency property is verified for fuzzy sets.

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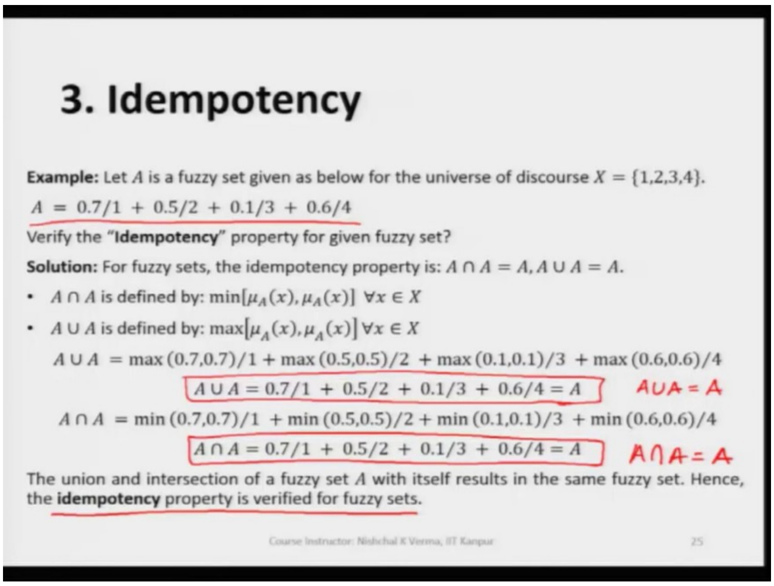

So, if we take this as the fuzzy set here which is discrete and if we try to find the $A \cup A$. So, we see that $A \cup A = A$. Similarly if we do this operation, we take $A \cap A = A$. So, this way we can say the idempotency property is satisfied or verified for fuzzy sets also because the crisp set this property is satisfied.

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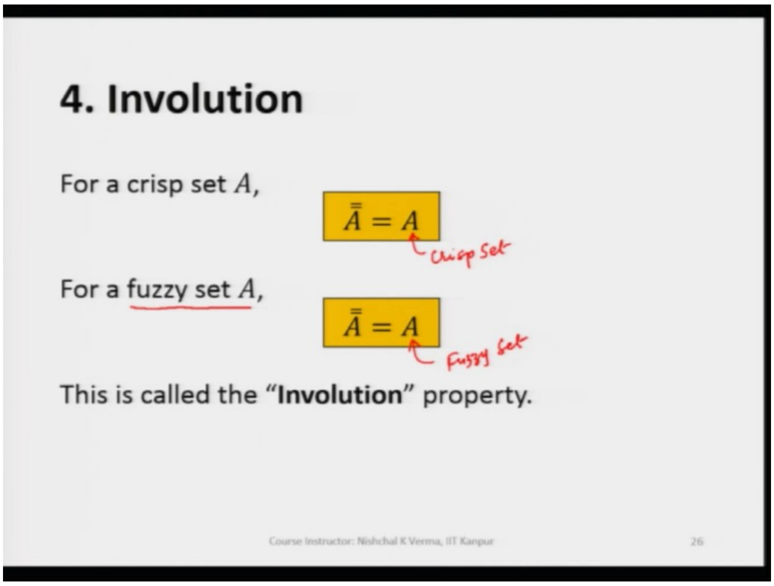

So, here also for fuzzy sets, this property holds good. Then comes the involution property. So, this means that if you take any $\acute{A}$. So, what does this mean is that if we take a crisp set A we take the complement of it and then we further take the complement of it and this is going to return us the crisp set which was originally taken. So, this is the crisp set, this is crisp set which was originally taken. So, this is going this is going to return us the same set.

So, let us see this involution property holds good for fuzzy sets also. So, if we take a fuzzy set A and if we take the $\acute{A}$, so it means that if we take the complement of fuzzy set and then we take further complement of it here also this is going to return us the same fuzzy set same fuzzy set which was used for taking the complement. So, it means $\acute{A} = A$ and this is true for both the cases means the, if we take crisp set or the fuzzy set both the cases, we are going to get the same set.

So, but we are in here interested more on fuzzy sets. So, we can clearly say that if we take fuzzy set A and then if we take the double complement of it, we are going to get the same fuzzy set as a result.

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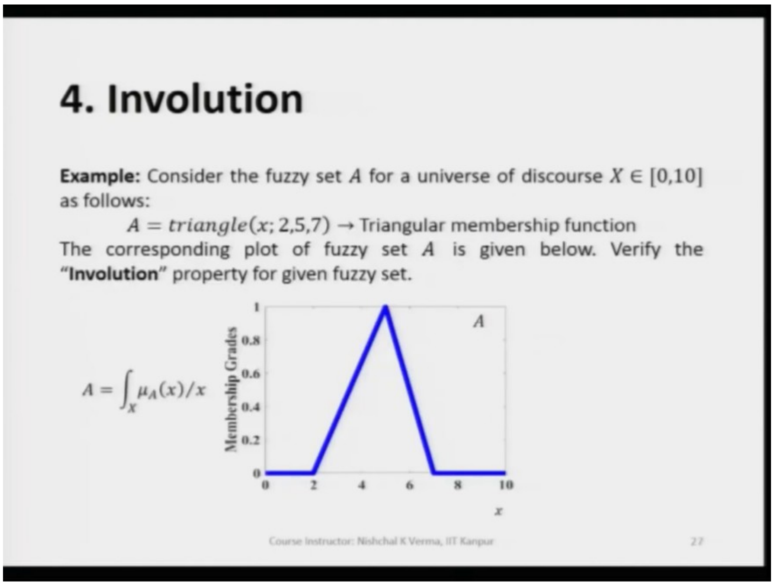

Let us know understand better this involution property fuzzy set. So, let us take an example here and if we take a fuzzy set A.

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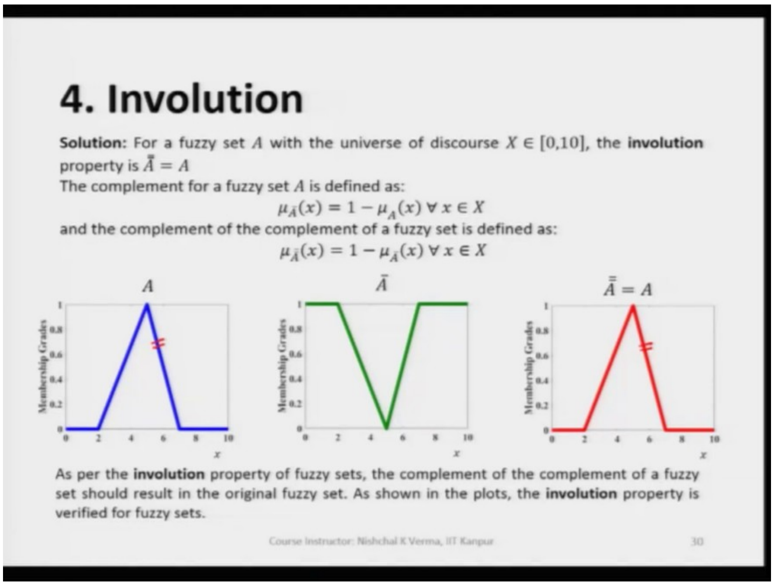

Let us now take the complement of it. So, here we have the fuzzy set, the original fuzzy set which we have taken and then the complement is here. Now if we take the double complement see the double complement here and which is nothing, but the same set. If we see here we are going to get the A, this set and this set is coming out to be the same. So, double complement of any fuzzy set is going to return us the same fuzzy set. So, this way we can say that the involution property is verified for continuous fuzzy sets. So, let us now understand this for discrete fuzzy sets also.

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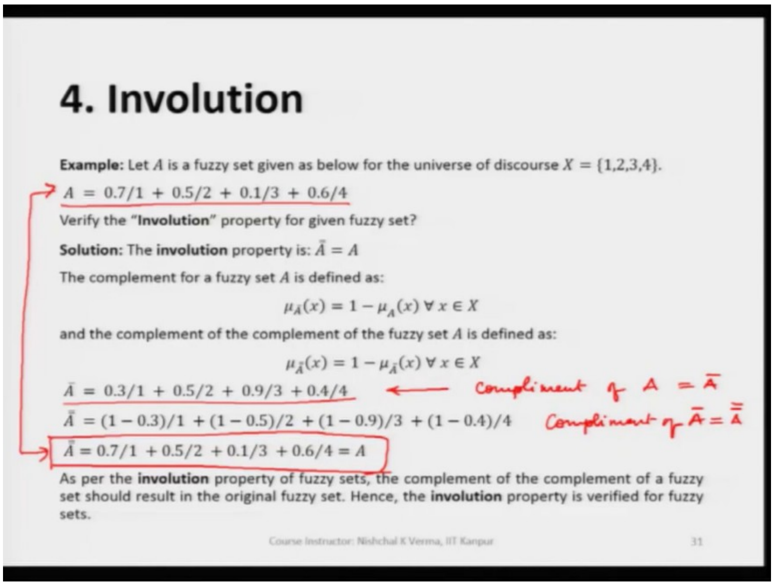

So, if we take here an example of discreet fuzzy set which is here. So, let us take the complement of it. So, this gives us the complement of complement of A that is $\acute{A}$. So, this is represented by this discrete fuzzy set and if we take the double complement of is double complement means further complement of; so, complement of $\acute{A}$ and this is going to give us $\acute{A}$ and this is the $\acute{A}$ and which if we see is the same set which we have taken, we have started with.

So, we see that these two sets remain the same. So, we can clearly say that we are going to get the same set if we are taking the double complement of it and this property is nothing, but the involution property. So, this is verified.

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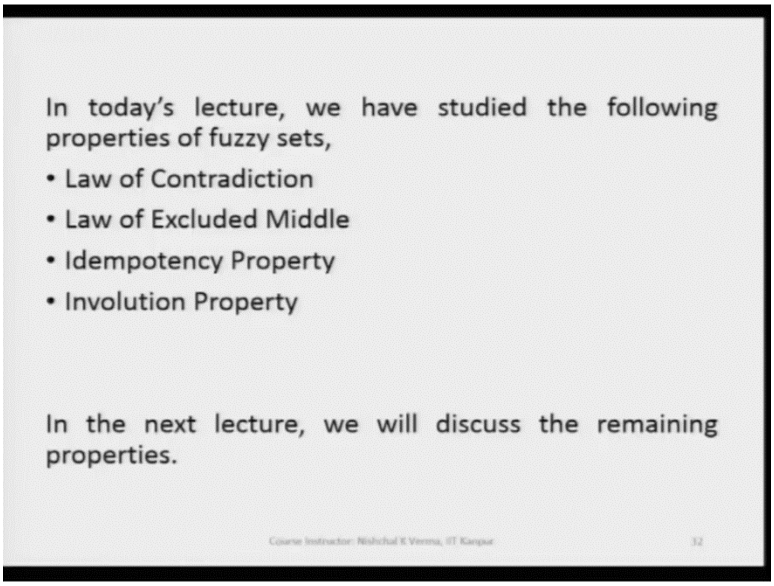

So, in this lecture today we have studied the following properties of fuzzy sets and these properties are law of contradiction, law of excluded middle, idempotency property, involution property. So, but here we have so many other properties left which we will be discussing in the next lecture.