Basic Electric Circuits
Professor Ankush Sharma
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Module 12 - Analogous Systems
Lecture 58 - Modelling of the Rotational Motion of Mechanical Systems

Namshkar. In last class we discussed about the mathematical modelling of translational motion of mechanical system. Today we will discuss about the mathematical modelling of rotational system. So, let us start the discussion of today's lecture.

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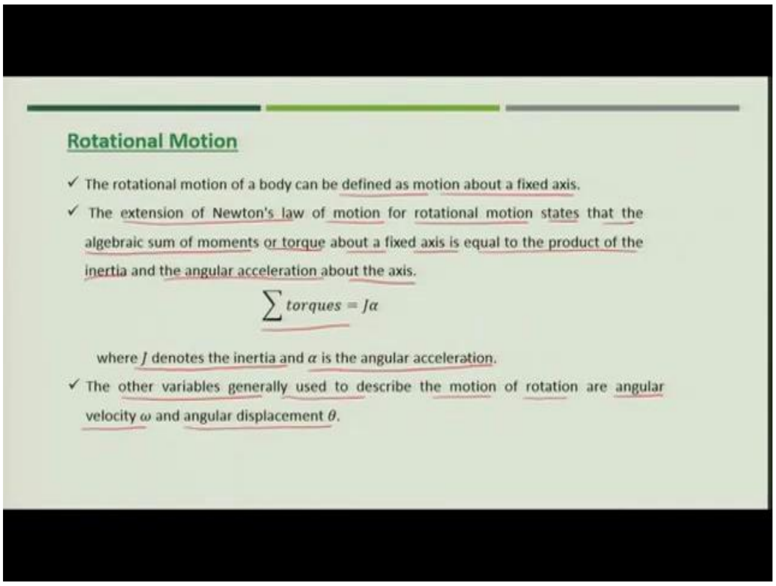

So, the rotational motion of a body can be defined as motion about a fixed axis. The extension of Newton's law of motion for rotational motion states that the algebraic sum of moments or torque about a fixed axis is equal to the product of the inertia and the angular acceleration about the axis. So, we define that total sum of torques which is applicable on a body, which is rotating is equal to $J\alpha$, J denotes the inertia and α is the angular acceleration. So, other variables which we generally use to describe the motion of rotation are angular velocity ω, and angular displacement θ. So, if you take $\frac{d\theta}{dt}$, you will get angular velocity ω, and $\frac{d^2\theta}{dt^2}$ gives the angular acceleration, α.

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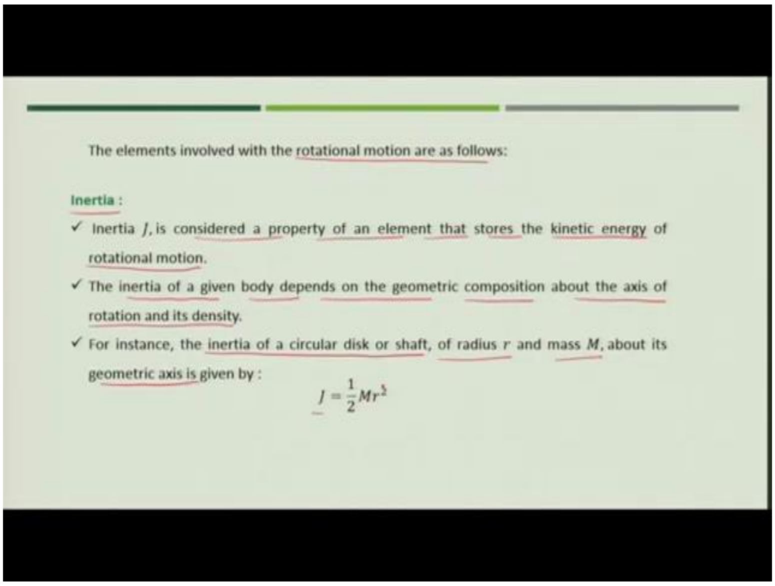

Now, the elements involved in the rotational motions are first inertia. So, inertia is considered as the property of an element that stores the kinetic energy of the rotational motion. So, you can correlate it with respect to the mass m which we discussed in case of translational motions. So, here also the inertia is the property of element which stores the kinetic energy of rotational motion.

Now, the inertia of a given body depends on the geometric composition about the axis of rotation and its density. So for instance, if the inertia of a circular disk or shaft of radius r and mass M, the value of inertia about its geometric axis is given as, $J = \frac{1}{2}Mr^2$.

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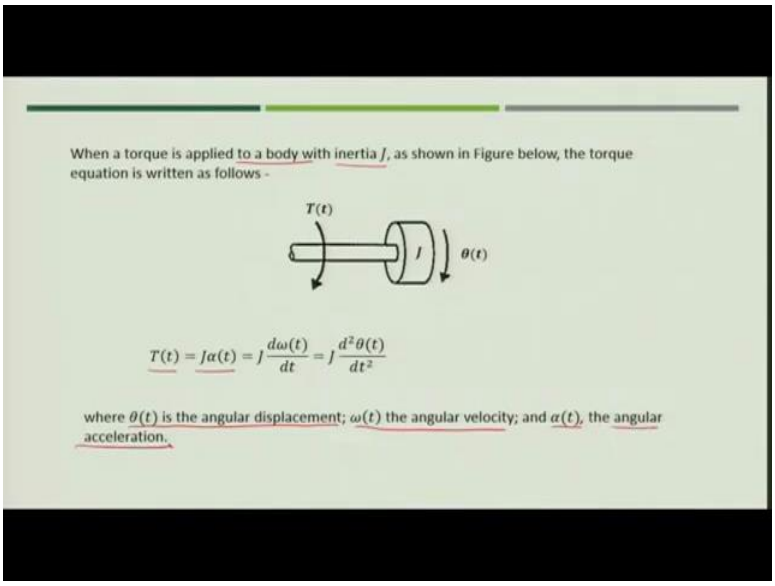

Now, when the torque is applied to a body with inertia J, if you see the figure, the body which is rotating having inertia J and it is having θ as the displacement and when the torque is applied, the governing equation which you can write is

$$T(t) = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

So, here $\theta(t)$ is the angular displacement; $\omega(t)$ the angular velocity; and $\alpha(t)$, the angular acceleration.

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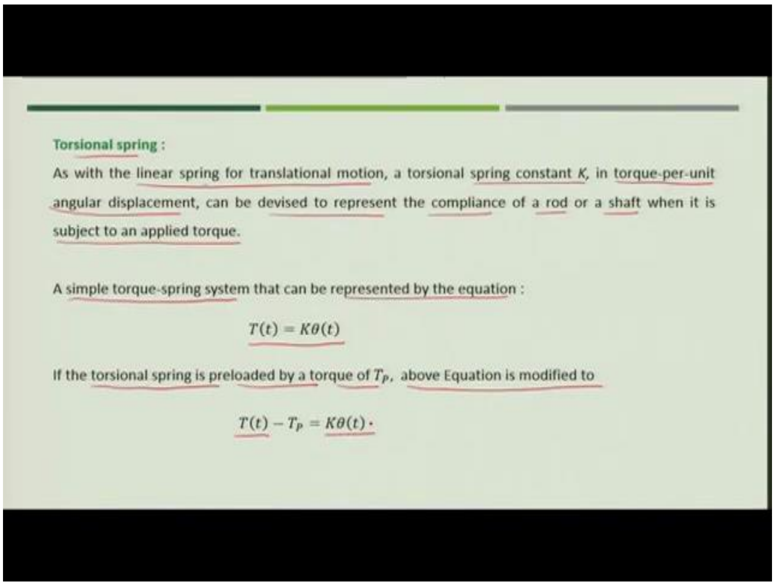

Now, like the spring, linear spring which we saw in case of translational motion, we can also define torsional spring. Torsional spring let us say we have a torsional spring constant K that is given in torque per unit angular displacement, so we can devised to represent the compliance of a rod or a shaft when it is subject to applied torque. So, torsional spring is the property of the shaft or the rod which is connected to your rotating body.

So, when you give torque it twists slightly which give the property of torsional spring. So, what you will write? Simple torque spring system can be represented by this equation that is T, $T(t) = K\theta(t)$. Now, if the torsional spring is preloaded by a torque say TP, the above equation can be modified as $T(t) - TP = K\theta(t)$.

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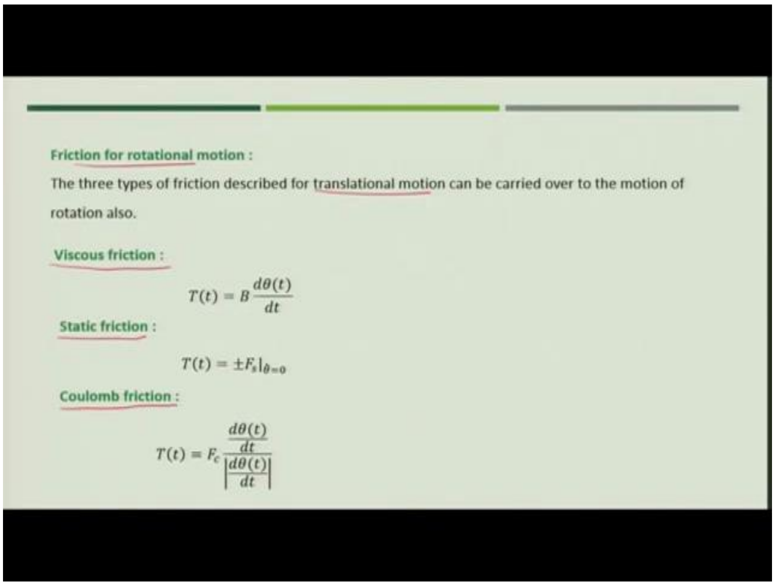

Now, similar to what we saw in case of translational motion, in this case also we will see various frictions. What are those frictions? First is viscous friction that is torque $T(t) = B\frac{d\theta(t)}{dt}$. This B is called a viscous friction coefficient. Then static friction, in that case the friction torque $T(t) = \pm F(s)|_{\dot{\theta}=0}$ that is the object is just about to move.

In that case the value of friction is given by $F(s)$, plus minus is basically used to show the direction of the friction which will be applicable in the body. Then Coulomb friction, Coulomb friction $T(t) = F_c\frac{\frac{d\theta(t)}{dt}}{\left|\frac{d\theta(t)}{dt}\right|}$. So, this is how we define viscous friction, static and Coulomb friction in the rotating body.

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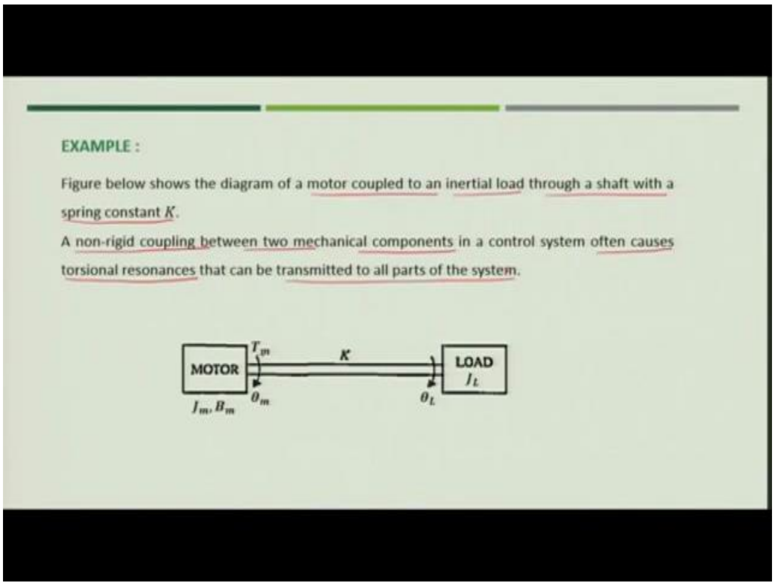

Now, let us take one example so that we can understand how we will create the mathematical model of mechanical system. So, in this mechanical system we have one motor connected to a load through shaft which has the friction coefficient as we will rather say stiffness coefficient as K which defines the torsional spring coefficient. Now, T_m is the mechanical torque applied by the motor and θ_L is the displacement at any time t for the load which is having inertia equal to J_L.

Motor inertia is J_m and viscous friction coefficient is B_m. Now, in this case the motor is coupled to an inertial load through a shaft with a spring constant K. The non-rigid coupling between two mechanical components which we see here often causes torsional resonance that can be transmitted to all parts of the system.

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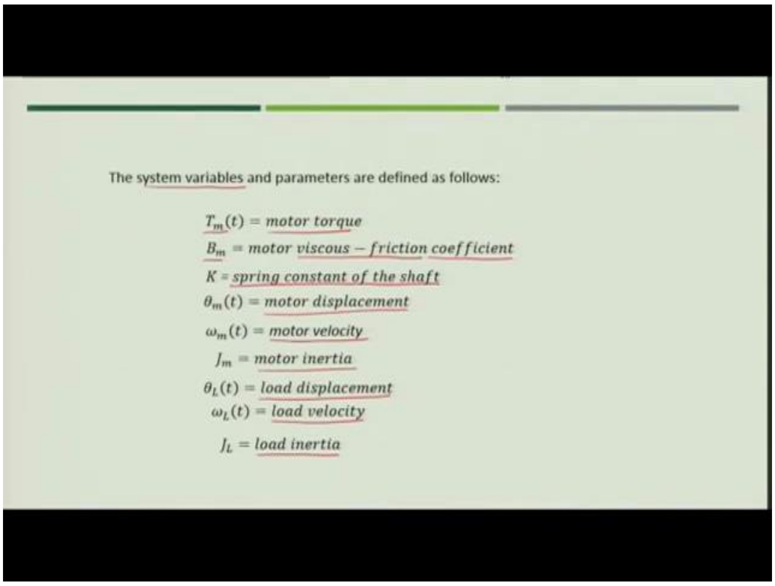

Now, let us see how we can create the free body diagram. The variable which we will use to define free body diagram of this mechanical system. The variables are T_m, T_m is nothing but motor torque, B_m is viscous-friction of the motor, so this is viscous friction coefficient related to motor, K is spring constant of the shaft, θ_m is motor displacement, ω_m is given as motor velocity, J_m is motor inertia, θ_L we consider it as load displacement, ω_L is load velocity and J_L is the load inertia. So, we will use these variables to define the free body diagram of the system.

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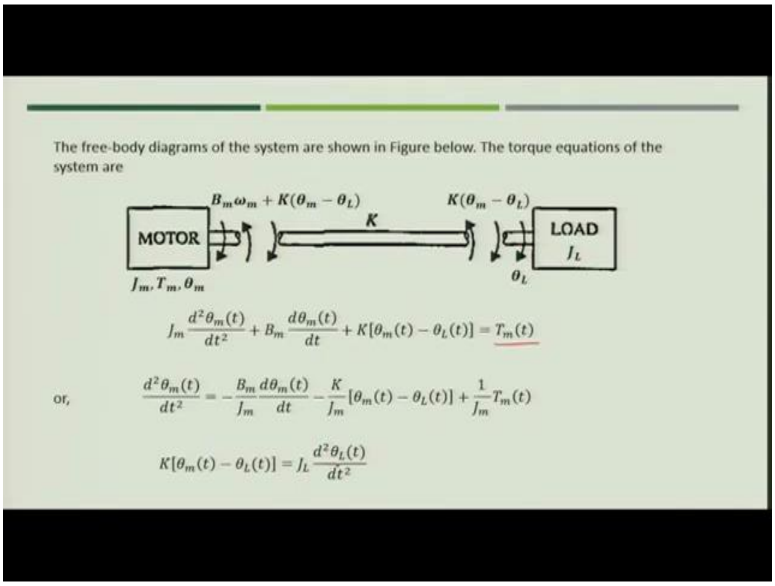

So, from the motor side, if you see the free body diagram you will see that the torque T_m when it is applied, you will have the viscous friction part plus spring torque part applicable in the opposite direction of the motor torque applied. The torque equations of the system are

$$\frac{d^2\theta_m(t)}{dt^2} = -\frac{B_m}{J_m}\frac{d\theta_m(t)}{dt} - \frac{K}{J_m}[\theta_m(t) - \theta_L(t)] + \frac{1}{J_m}T_m(t)$$

$$K[\theta_m(t) - \theta_L(t)] = J_L\frac{d^2\theta_L(t)}{dt^2}$$

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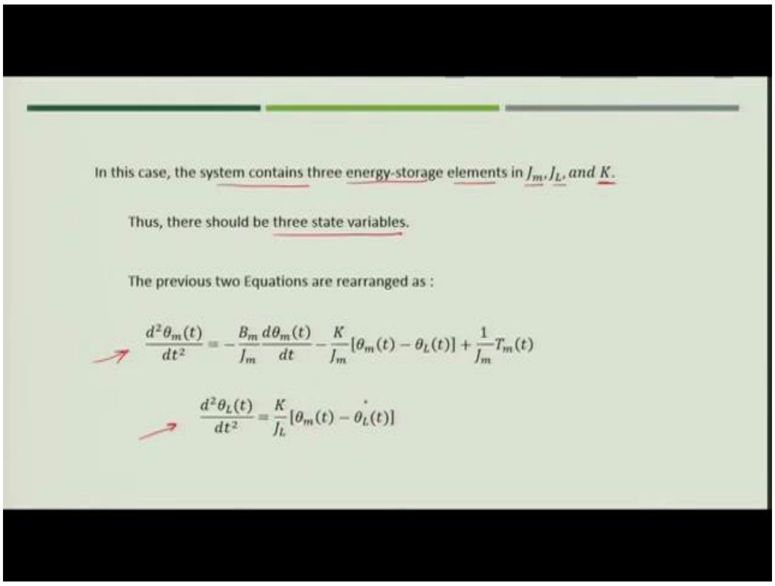

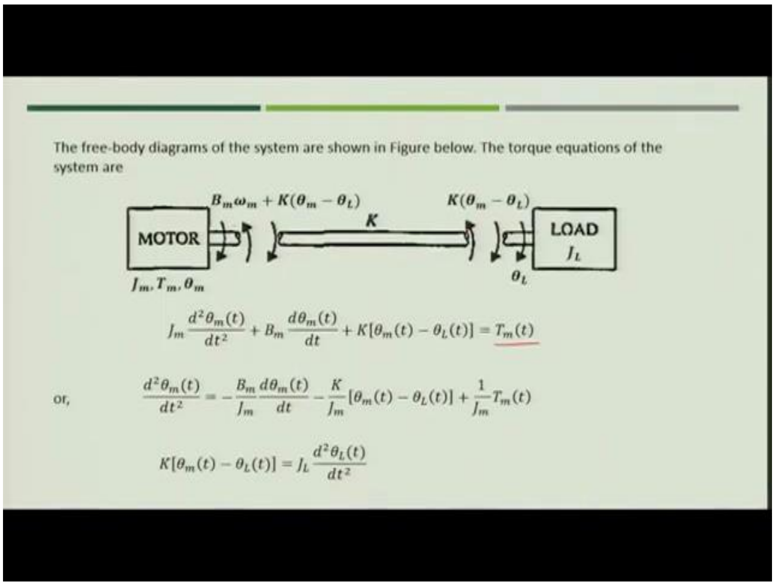

Now, in this case the system contains 3 energy storage elements that is J_m, J_L and K. So, what we will do, we will find 3 state variables in this case. The equations which we just saw in case of the motor side and the load side free body diagram, we came with these 2 equations that is

$$\frac{d^2\theta_m(t)}{dt^2} = -\frac{B_m}{J_m}\frac{d\theta_m(t)}{dt} - \frac{K}{J_m}[\theta_m(t) - \theta_L(t)] + \frac{1}{J_m}T_m(t)$$

$$K[\theta_m(t) - \theta_L(t)] = J_L \frac{d^2\theta_L(t)}{dt^2}$$

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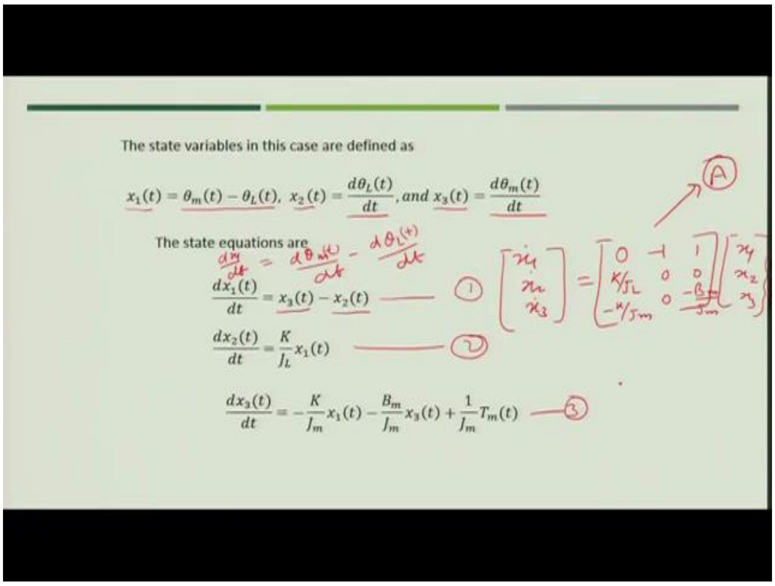

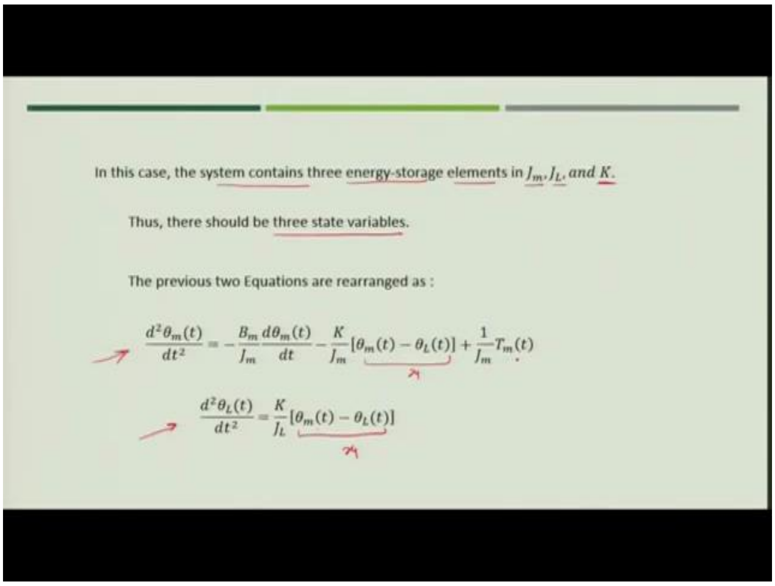

Now, let us assume that we have 3 state variables $x_1(t) = \theta_m(t) - \theta_L(t),\ \ x_2(t) = \frac{d\theta_L(t)}{dt},\ and\ x_3(t) = \frac{d\theta_m(t)}{dt}$. So, the state equations which you can write, if you differentiate,

$$\frac{dx_1(t)}{dt} = x_3(t) - x_2(t)$$

$$\frac{dx_2(t)}{dt} = \frac{K}{J_L}x_1(t)$$

$$\frac{dx_3(t)}{dt} = -\frac{K}{J_m}x_1(t) - \frac{B_m}{J_m}x_3(t) + \frac{1}{J_m}T_m(t)$$

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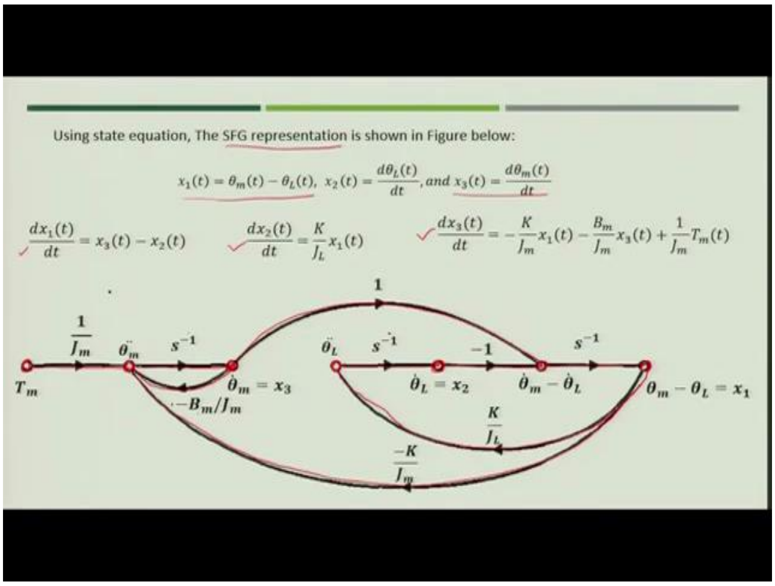

We can represent the state equations which we have just got using the signal flow graph. So, how we will represent? We have defined the state variables as $x_1(t) = \theta_m(t) - \theta_L(t)$,, $x_2(t) = \frac{d\theta_L(t)}{dt}$, *and* $x_3(t) = \frac{d\theta_m(t)}{dt}$.

Now, for the first equation what you are getting, $\frac{dx_1(t)}{dt} = x_3(t) - x_2(t)$, so here you have x3, you have x2, you have x3 minus x2. So, x3 will be having gain of 1, x2 will be having gain of minus 1, when you sum up you get $\frac{d\theta_m(t)}{dt} - \frac{d\theta_L(t)}{dt}$ that is, nothing but equal to $\frac{dx_1(t)}{dt}$, So, from first equation what you have got is these two links.

Now, from $\frac{dx_2(t)}{dt} = \frac{K}{J_L}x_1(t)$. So, we will take the loop from x_1 with gain $\frac{K}{J_L}$ and connect it to the $\ddot{\theta}_L$. Now, $\ddot{\theta}_L$ if you integrate you will get $\dot{\theta}_L$, that is nothing but x_2, so you have 1 integrator link with the s^{-1} that is nothing but $1/s$.

Then when you again integrate $\frac{d\theta_m(t)}{dt} - \frac{d\theta_L(t)}{dt}$, you will get x_1 so again you create 1 integrator block and you get the x_1. Now, next is the equation that is $\frac{dx_3(t)}{dt} = -\frac{K}{J_m}x_1(t) - \frac{B_m}{J_m}x_3(t) + \frac{1}{J_m}T_m(t)$, so we put T_m as a input note and then we put $\ddot{\theta}_m = \frac{dx_3(t)}{dt}$.

So, we use this equation and find out the links. So, first is $-\frac{K}{J_m}x_1(t)$, so from x_1 we get $-\frac{K}{J_m}$, then from x_3 we get $-\frac{B_m}{J_m}$, so from x_3 we connect the link again with gain of $-\frac{B_m}{J_m}$ and then you have $\frac{1}{J_m}T_m$. So, T_m is connected to $\ddot{\theta}_m$ that is with the gain $\frac{1}{J_m}$. So, with this you complete the summation at $\ddot{\theta}_m$. Now, $\ddot{\theta}_m$ is connected to x_3 with integrator with integration. If you take the integral of $\ddot{\theta}_m$, you will get x_3. So, using these 3 equations, you have now created the signal flow graph presentation of the system.

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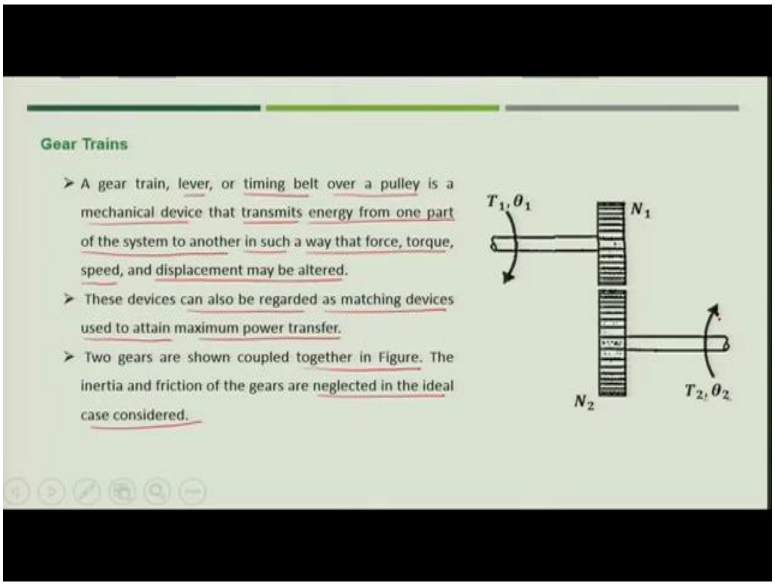

Now, let us talk about the gear train. So, gear train or lever or the timing belt over a pulley is a mechanical device that transmits energy from one part of the system to another in such a way that force, torque, speed and displacement may be altered. So, these devices can also be regarded as matching devices used to attain maximum power transfer. Now, we see these in the figure, we see 2 gears are coupled together. The inertia and friction of the gears are neglected when you consider the ideal case. In this case T_1 through these gears is transferred to the second gear that is having torque T_2.

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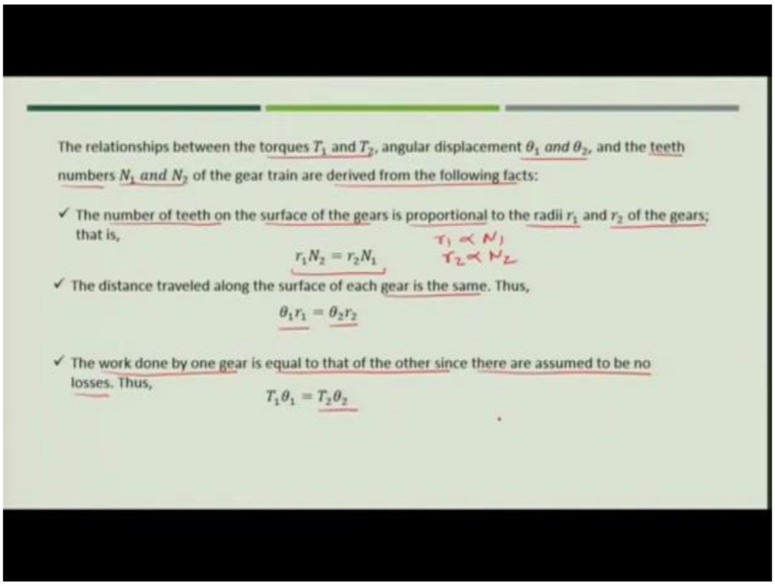

Now, how we will create the relationship between T_1 and T_2? So, the relationship between torque T_1 and T_2 and then angular displacement θ_1 *and* θ_2 and the teeth numbers in the gear that is N_1 *and* N_2 can be correlated with the following facts. First is the number of teeth on the surface of the gear is proportional to the radius r_1 and r_2 of the gears, so in that case we can define $r_1 N_2 = r_2 N_1$ because r_1 is proportional to N_1 and r_2 is proportional to N_2.

So, using this you can correlate that $r_1 N_2 = r_2 N_1$. Now, the distance travelled along the surface of each gear is same. Therefore, you can write $\theta_1 r_1 = \theta_2 r_2$. Now, third one is that the work done by 1 gear is equal to that of others, since there are assumed to be no losses. In that case $T_1 \theta_1 = T_2 \theta_2$.

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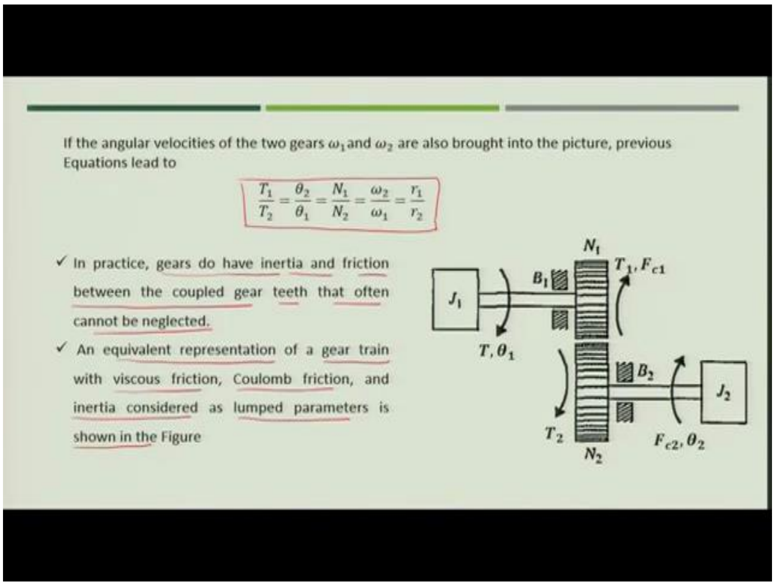

So, using these 3 correlations, we can write that

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$

So, this is the most important correlation which we use in case of gear system to correlate the various parameters from one side to other side. Now, in practice the gears also have inertia and friction between the coupled gear teeth and that often cannot be neglected.

So, in that case the equivalent representation of the gear train with viscous friction, Coulomb friction and inertia as lumped parameters is considered, which is shown in the figure. So, here you have the Coulomb frictions, then B_1 and B_2 are the viscous frictions, T is the torque applied from the gear 1, so the opposite torque which is T1 applied on the gear N1 and translated to gear 2 is T2 and the energy transferred from gear 2 the object which is having inertia equal to J2.

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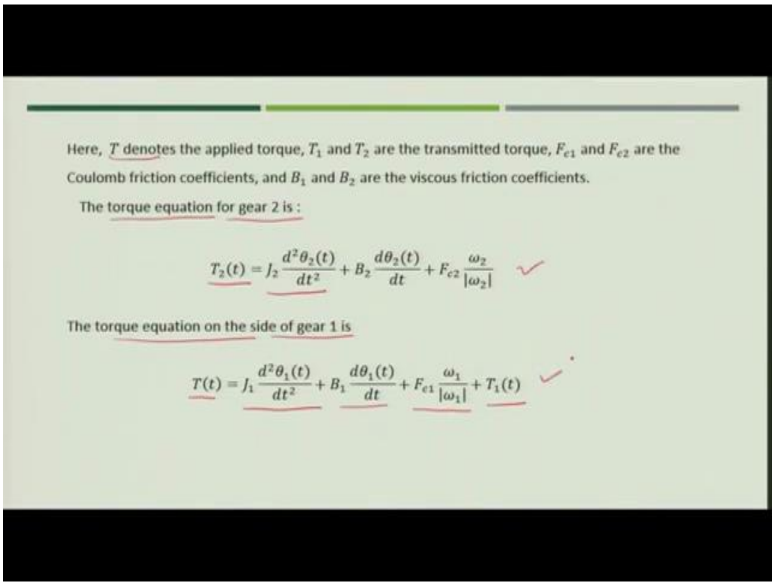

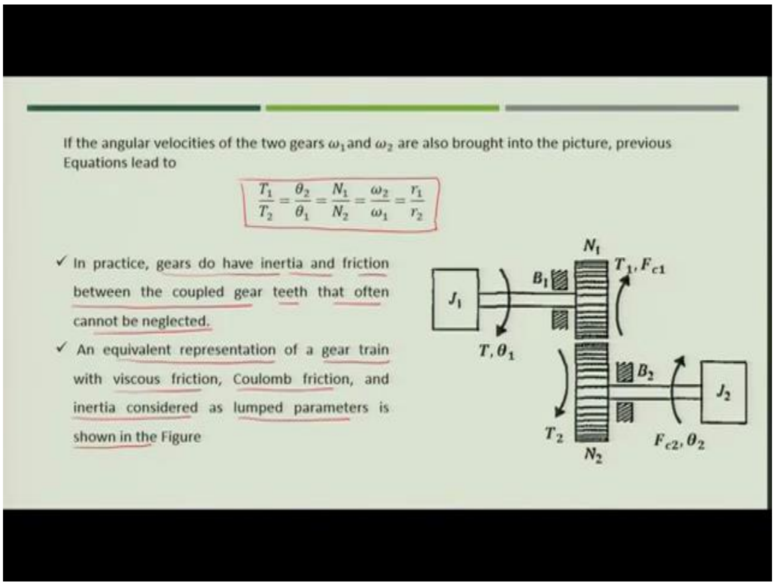

Now, since we have 2 gears, so we need 2 torque equations, so for gear 2 what we can write? That is

$$T_2(t) = J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + F_{c2} \frac{\omega_2}{|\omega_2|}$$

Now, the torque equation on the first side what you can write, the applied torque

$$T(t) = J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + F_{c1} \frac{\omega_1}{|\omega_1|} + T_1(t)$$

Now, we have got these 2 equations related to gear 1 and gear 2.

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Using previous Equations, we get -

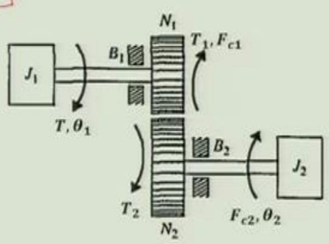
$$T_1(t) = \frac{N_1}{N_2}T_2(t) = \left(\frac{N_1}{N_2}\right)^2 J_2 \frac{d^2\theta_1(t)}{dt^2} + \left(\frac{N_1}{N_2}\right)^2 B_2 \frac{d\theta_1(t)}{dt} + \frac{N_1}{N_2}F_{c2}\frac{\omega_2}{|\omega_2|} \quad (9)$$

This indicates that it is possible to reflect inertia, friction, compliance, torque, speed, and displacement from one side of a gear train to the other. The following quantities are obtained when reflecting from gear 2 to gear 1:

If the angular velocities of the two gears ω_1 and ω_2 are also brought into the picture, previous Equations lead to

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$

- ✓ In practice, gears do have inertia and friction between the coupled gear teeth that often cannot be neglected.
- ✓ An equivalent representation of a gear train with viscous friction, Coulomb friction, and inertia considered as lumped parameters is shown in the Figure



Here, T denotes the applied torque, T_1 and T_2 are the transmitted torque, F_{c1} and F_{c2} are the Coulomb friction coefficients, and B_1 and B_2 are the viscous friction coefficients.

The torque equation for gear 2 is :

$$T_2(t) = J_2\frac{d^2\theta_2(t)}{dt^2} + B_2\frac{d\theta_2(t)}{dt} + F_{c2}\frac{\omega_2}{|\omega_2|}$$

The torque equation on the side of gear 1 is

$$T(t) = J_1\frac{d^2\theta_1(t)}{dt^2} + B_1\frac{d\theta_1(t)}{dt} + F_{c1}\frac{\omega_1}{|\omega_1|} + T_1(t)$$

Using the previous correlation,

$$T_1(t) = \frac{N_1}{N_2} T_2(t) = \left(\frac{N_1}{N_2}\right)^2 J_2 \frac{d^2\theta_1(t)}{dt^2} + \left(\frac{N_1}{N_2}\right)^2 B_2 \frac{d\theta_1(t)}{dt} + \frac{N_1}{N_2} F_{c2} \frac{\omega_2}{|\omega_2|}$$

So, this indicates that it is possible to reflect inertia, friction, compliance, torque, speed and displacement from one side of the gear train to other.

So, when you see this equation, you can correlate these equations with respect to the transformer equations, which we derived in earlier discussions so you can also see that as we saw in case of transformer, we transfer the power from one side to other side, similarly the gear is the mechanical arrangement which transfers power from one side to other side. So, in this way you can see that how closely the mechanical system of gears is analogous to the electrical system of the transformers.

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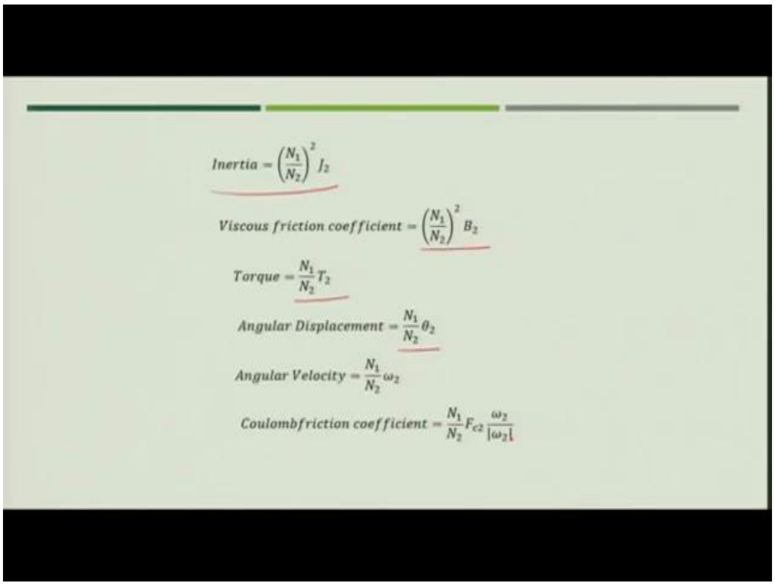

Now, the following quantities we obtained when reflecting from gear 2 to gear 1 we define them as the inertia that is $\left(\frac{N_1}{N_2}\right)^2 J_2$, viscous friction coefficient we say as $\left(\frac{N_1}{N_2}\right)^2 B_2$, torque $\frac{N_1}{N_2} T_2$, angular displacement we say with respect to the gear 1, that is $\frac{N_1}{N_2} \theta_2$. Similarly, angular velocity we convert into the gear one side is $\frac{N_1}{N_2} \omega_2$, Coulomb friction coefficient is $\frac{N_1}{N_2} F_{c2} \frac{\omega_2}{|\omega_2|}$.

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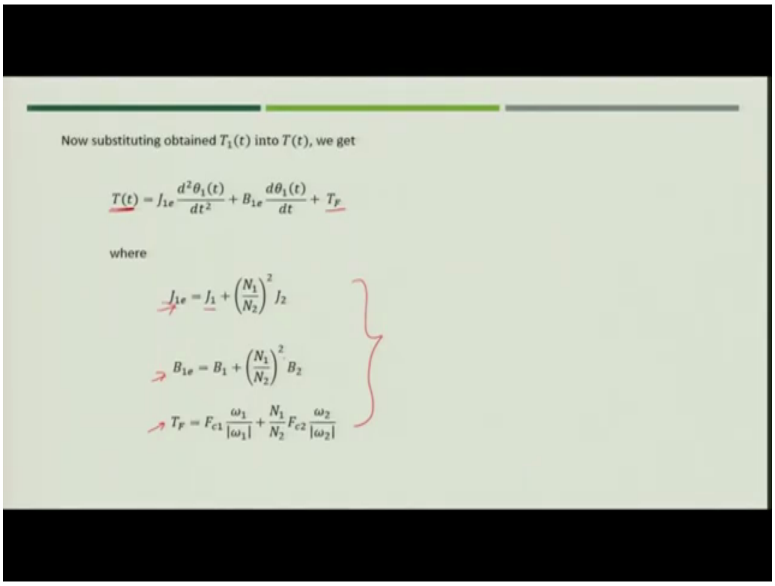

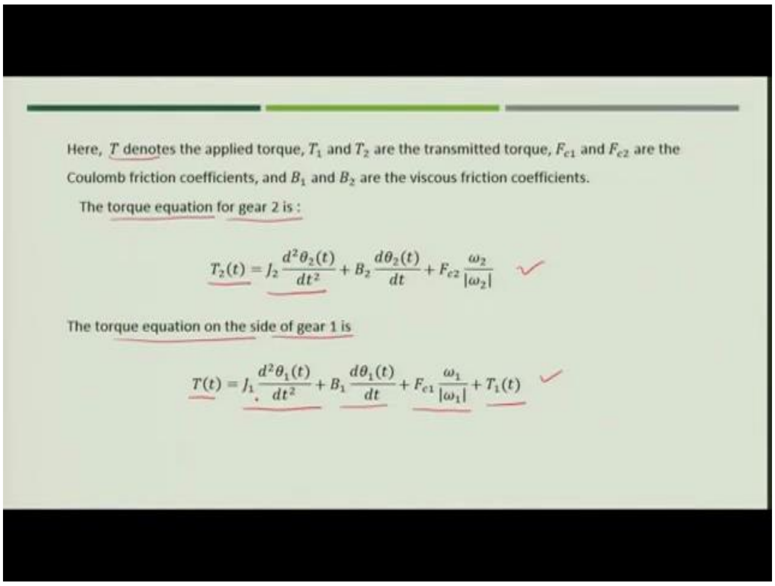

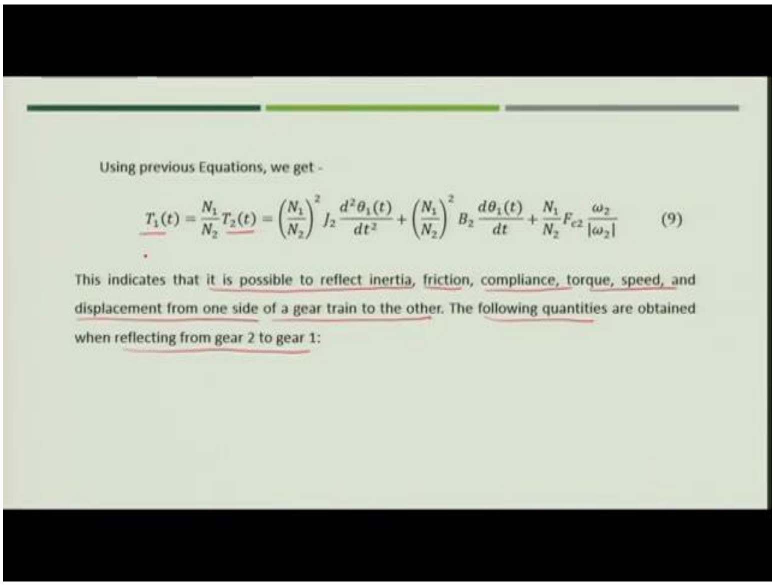

So, if you consider the equivalent of all those components what we can write total torque T you can just put the value of

$$T(t) = J_{1e}\frac{d^2\theta_1(t)}{dt^2} + B_{1e}\frac{d\theta_1(t)}{dt} + T_F$$

T_F is equivalent Coulomb torque.

So, $J_{1e} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2$, $B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2$ and $T_F = F_{c1}\frac{\omega_1}{|\omega_1|} + \frac{N_1}{N_2}F_{c2}\frac{\omega_2}{|\omega_2|}$. So, if you see these equations you can easily correlate, in case of transformer also when you transfer the resistance and reactance value from secondary side to primary side you will see similar kind of relationship, as we see in this case of gear, the reflection of the various properties from gear 2 to gear 1. So, in this way you can say that the gear arrangement is closely an`alogous to the electrical system in case of the transformer.

So, with this we can close our today's discussion. In this discussion we mainly focussed about the rotational motion, so from next lecture onwards we will discuss about the correlation between electrical and the mechanical or other than the mechanical system so that you can understand how the mechanical system can be equivalently represented with the help of electrical circuit. Thank you.