Basic Electric Circuits
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Module 12 - Analogous Systems
Lecture 57 - Modeling of Mechanical Systems

Namaskar, so today we will start our last topic of the course that is analogous system. Let us see what analogous system means.

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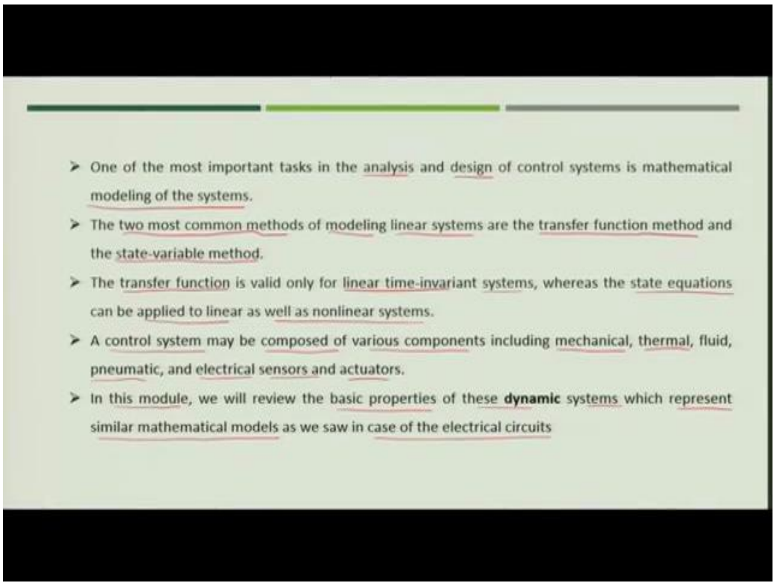

So, one of the most important tasks in the analysis and design of the control system is the mathematical modeling of the system. Now, we have studied two most common methods of modeling the linear system that were transfer function methods and the state variable method, so these two we have discussed.

Now, transfer function method is valid only for linear time invariant systems whereas State equations can be applied to linear as well as nonlinear system. Now, the control system may be composed of various components, not only the electrical but also mechanical, thermal, fluid, pneumatic and other electrical sensors and actuators as well. So, what we will do in this module? We will review the basic properties of these dynamic systems which represent the similar mathematical models as we saw in case of electrical circuits.

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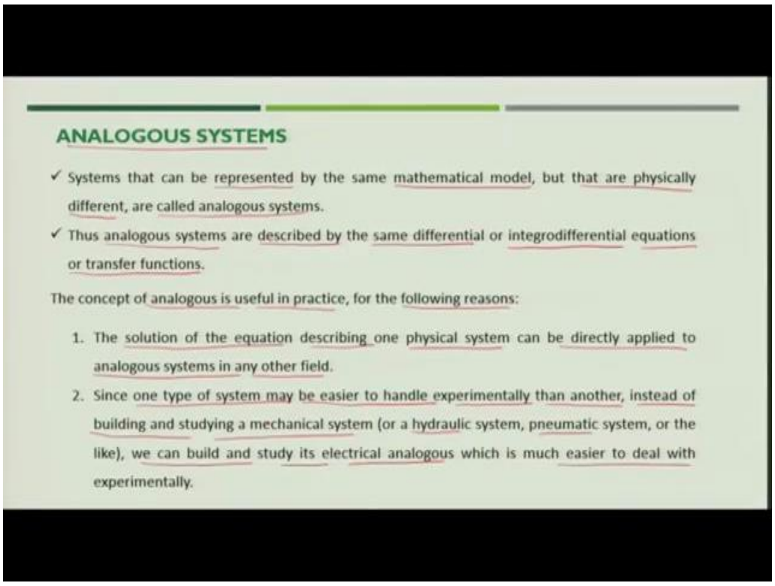

So, before proceeding to the mechanical modeling, let us understand what the analogous system means. Now, the systems can be represented by the same mathematical model as we saw in case of electrical circuits but that are physically different. So, that is why they are called as analogous system. So, the analogous systems are described by the same differential or maybe integrodifferential equations or the transfer functions.

So, analogous systems are those systems where you see the same mathematical representation but physically, they are different with each other. So, why the analogous system is important? So, the concept of analogous system is useful in practice because of the following 2 reasons. One is that the solution of this equation describing one physical system can be directly applied to the analogous system in another field. Now, second advantage is that since one type of system may be easier to handle experimentally than any other system instead of building and studying the mechanical or maybe hydraulic or pneumatic system.

We can build and study its electrical analogous which is much easier to deal with experimentally. So, in this case, the advantage which we will get that when we study the other systems we will come to know that how they can be modeled mathematically and we can create the electrical analogous of that system so that we can analyze the system without any physical building of that particular model.

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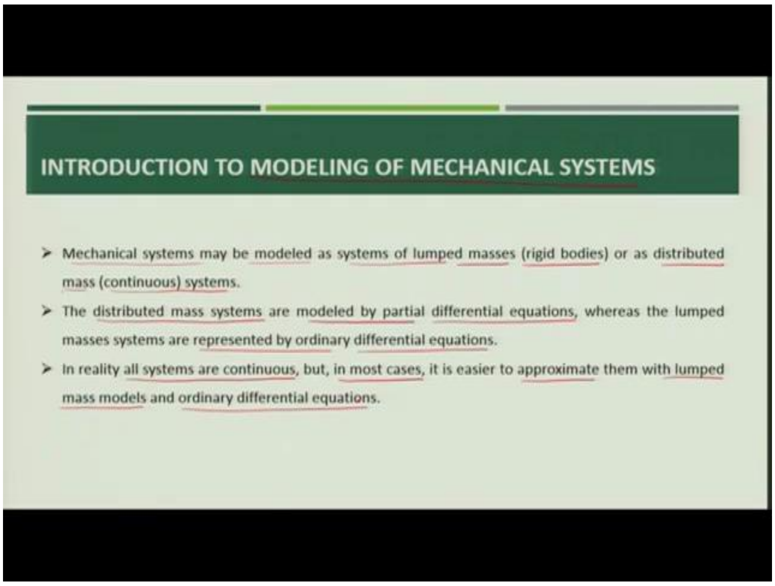

Now, let us first understand the modeling of mechanical systems. So, the mechanical systems may be modeled as systems of lumped masses or you can say as rigid bodies or the distributed masses or you can see the continuous mass system. The distributed mass systems are modeled by partial differential equations whereas the lumped masses are represented by ordinary differential equations. Now, almost all systems are continuous but in most of the cases it is easier to approximate them with lumped mass model. So, that is why we can use the ordinary differential equations to represent the models of those systems.

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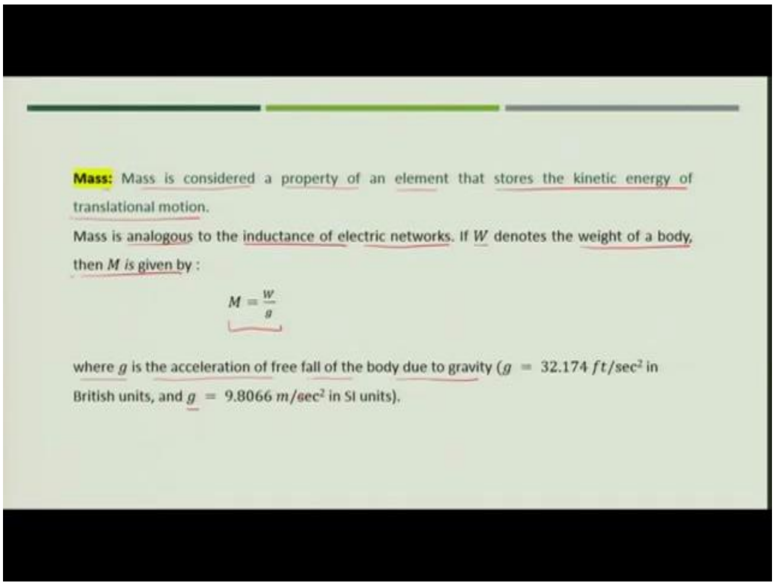

Now, first understand what is mass? Mass is considered a property of an element that stores the kinetic energy of translational motion. Now, we can also say that mass is analogous to inductance of electric network. If w is the weight of the body then mass M can be given as M is equal to w by g. What is g? g is the acceleration of free fall of the body due to gravity which is given as g is equal to 32.174 feet per second square or in SI unit if you are calculating g will be 9.8066 meter per second square.

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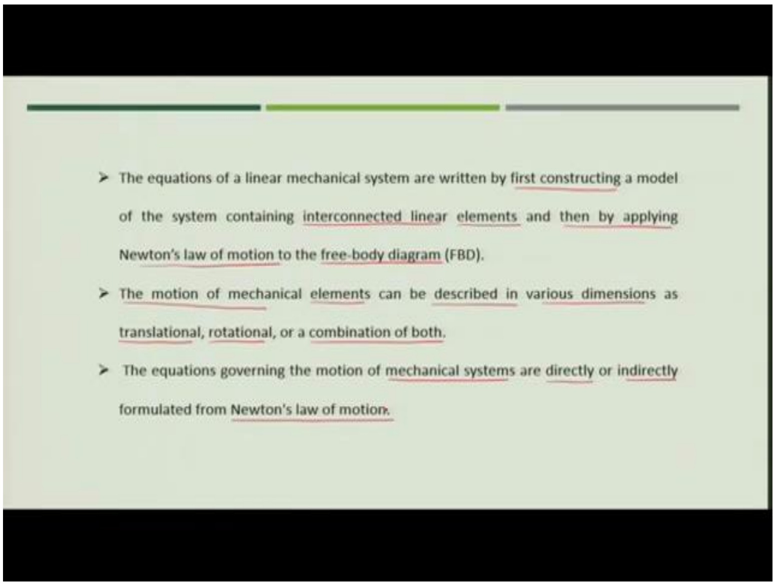

Now, the equations of linear mathematical system are written by first constructing model of the system containing interconnected linear elements and then by applying the Newton’s law of motion to the free body diagram. What is free body diagram? Free body diagram means you just take the main body of the element and show the forces applied on that body or mass. So, now the notion of mechanical elements can be described in various dimensions or we can say as translational, rotational or combination of both. The equations governing the motion of mechanical systems are directly or indirectly formulated from the Newton’s law of motion.

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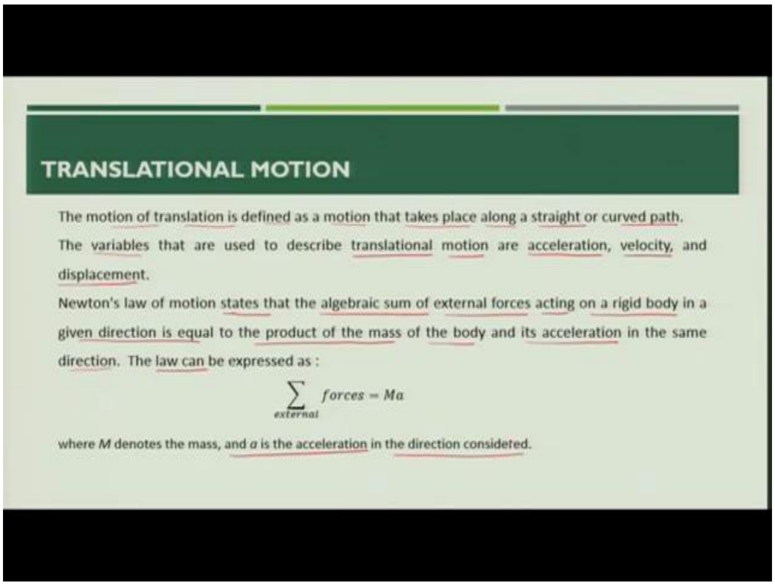

So, let us see how we describe it? So, the translational motion, we can say that the motion of translational is defined as the motion that takes place along a straight or curved path. The variables that are used to describe translational motion are acceleration, velocity, and displacement.

So, it governs the Newton's law of motion which states that the algebraic sum of external forces acting on a rigid body in each direction is equal to the product of the mass of the body and its acceleration in the same direction. So, we can represent the Newton's law of motion as we say that the total sum of forces applied on the body equal to Ma, where M is the mass and a is the acceleration which is in the direction considered.

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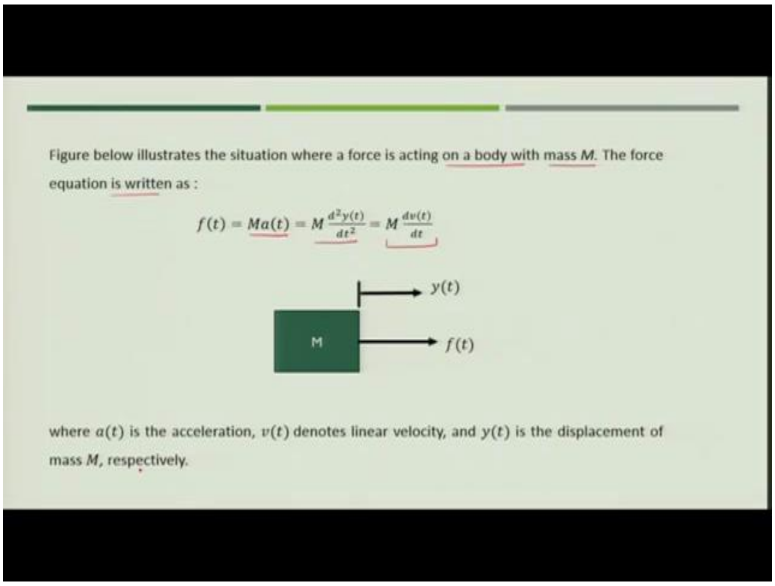

Now, when the force is acting on the body with mass M the equation which you can write for the law which we just saw is supposed this is the mass of symbol M. We say that the direction of motion is in this direction that is $y(t)$ and force is $f(t)$. So, what you can write,

$$f(t) = Ma(t) = M\frac{d^2y(t)}{dt^2} = M\frac{dv(t)}{dt}$$

$a(t)$ is the acceleration, $v(t)$ denotes linear velocity, and $y(t)$ is the displacement of mass M, respectively.

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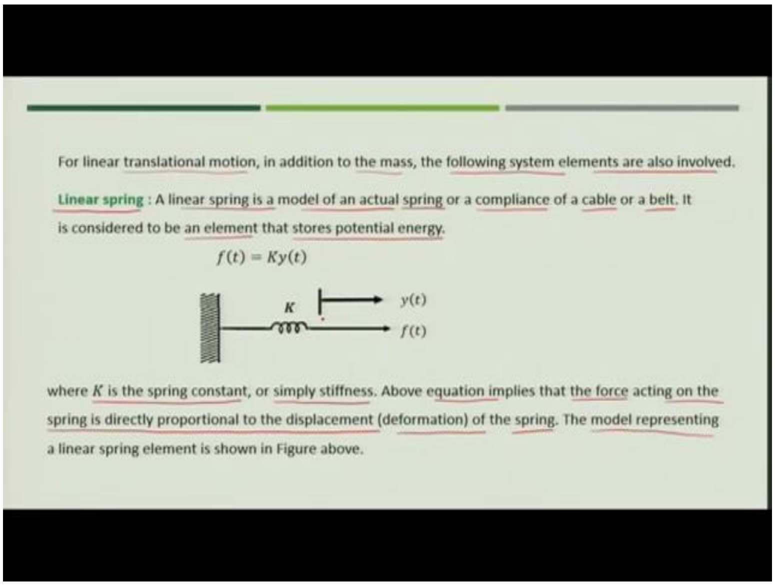

Now, for linear translational motion in addition to the mass, the following system elements are also involved. What are those elements? One is linear spring, so linear spring is the model of actual spring or a compliance of cable or belt. So, it is an element that stores potential energy. So, how we will represent? We will represent $f(t) = Ky(t)$. If you see in this figure, K is one constant which we generally call it a spring constant and then it is directly proportional to the displacement.

So, force $f(t) = Ky(t)$. So, the spring constant we also call it as a stiffness constant. So, in this above equation it implies that the force acting on the spring is directly proportional to displacement or we can say the deformation of the thing. The model representing a linear spring element is basically shown in the figure above.

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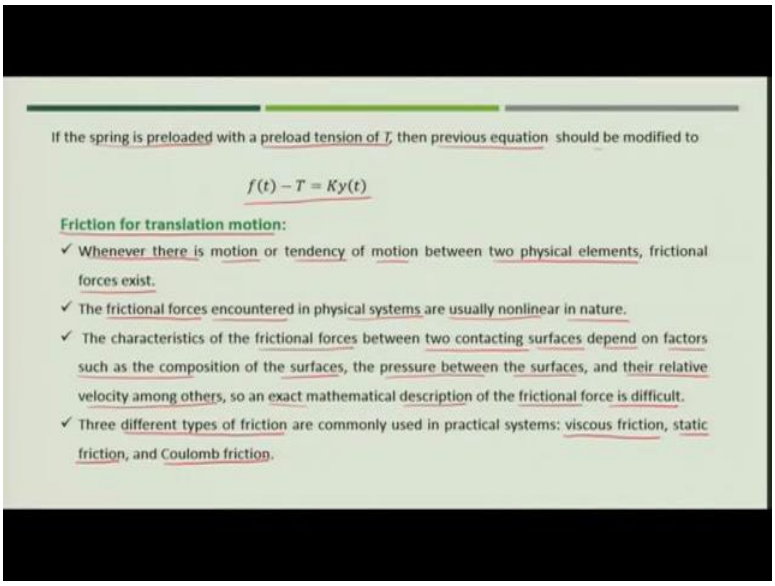

Now, if the spring is preloaded with some tension T then you can modify the previous equation while introducing the tension $f(t) - T = Ky(t)$. Now, there when you see that mass moving then you will see the friction also coming into the picture. So, what is the friction for translational motion? Whenever there is a motion or tendency of motion between two physical elements frictional forces will exist. Now, the frictional forces encountered in physical systems are usually non-linear in nature.

The characteristic of frictional forces between two contacting surfaces depends on the factors such as the composition of the surfaces and the pressure between the surfaces and their relative velocity among the others. Therefore, the exact mathematical description of the frictional force is difficult. So, what we do? We generally approximate with 3 different types of friction. What we call them? We call them viscous friction, static friction and Coulomb friction.

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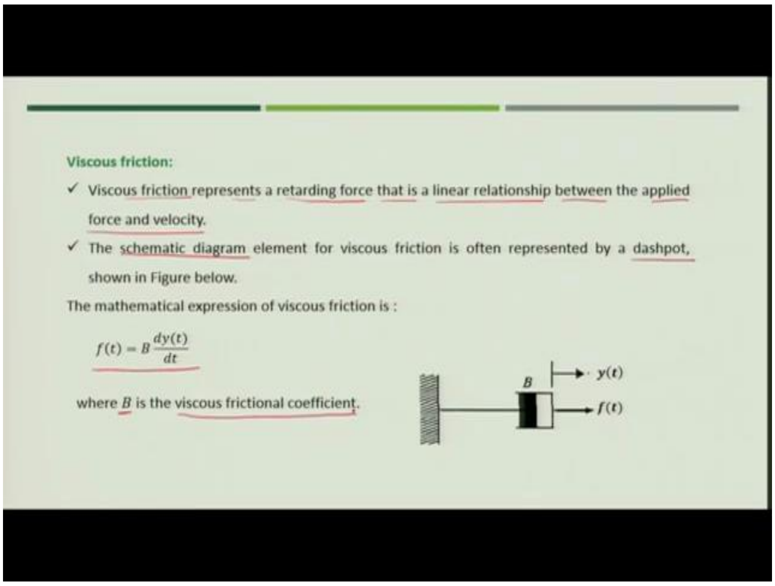

Now, what is viscous friction? Viscous friction represents retarding force that is a linear relationship between the applied force and velocity. The schematic diagram will contain dashpot in the figure. So, if you see this figure, we represent the viscous friction with the dashpot and the constant value is B. Then, force

$$f(t) = B\frac{dy(t)}{dt}$$

B is the viscous frictional coefficient.

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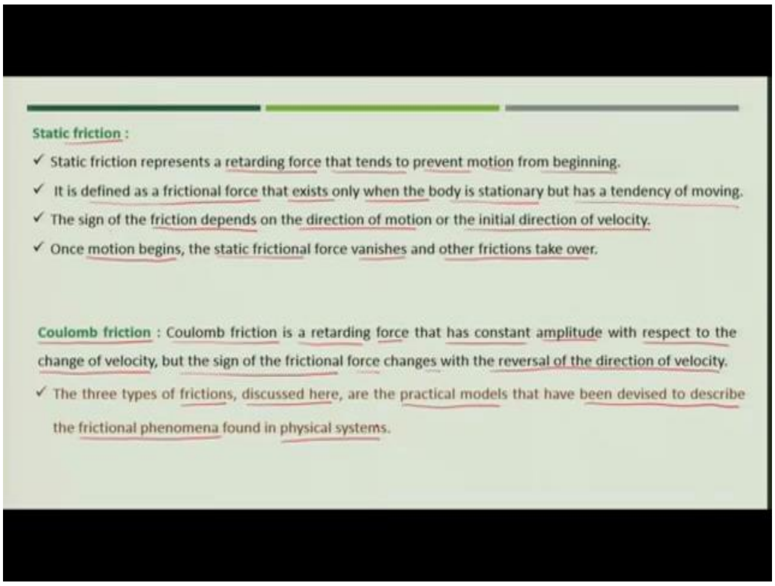

Now, next static friction, static friction represents retarding force that tends to prevent motion from beginning. So, it is defined as a frictional force that exist only when the body is stationary but has a tendency of moving. The sign of friction depends on the direction of motion or the initial direction of the velocity. Once this motion begins, the static frictional force vanishes, and other frictions will take over.

Now, the third friction type is Coulomb friction. The Coulomb friction is a retarding force that has constant amplitude with respect to the change of velocity but the sign of frictional force changes with the reversal direction of the velocity. Now, these three types of frictions which we have just discussed are the practical model that has been devised to describe the frictional phenomena which we find in the physical systems.

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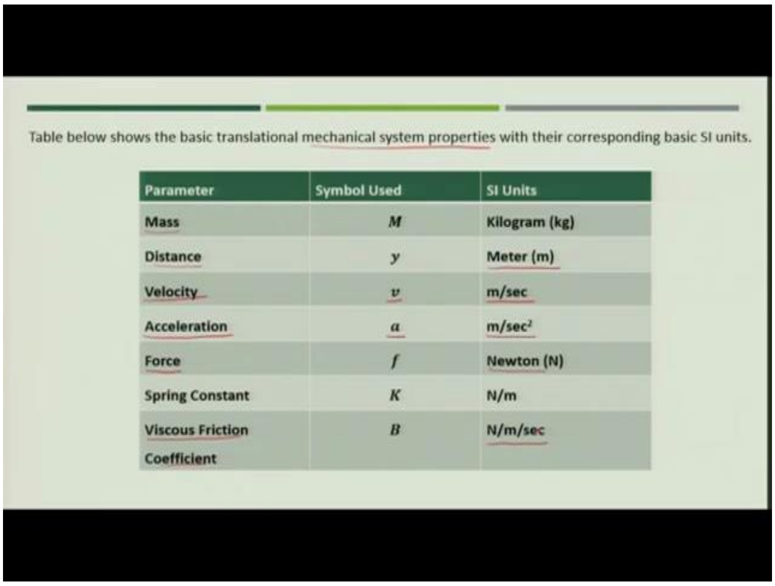

Parameter	Symbol Used	SI Units
Mass	M	Kilogram (kg)
Distance	y	Meter (m)
Velocity	v	m/sec
Acceleration	a	m/sec^2
Force	f	Newton (N)
Spring Constant	K	N/m
Viscous Friction Coefficient	B	N/m/sec

Now, you see this is the table where you can see the generally utilized mechanical system properties and their corresponding the SI unit which we take like mass is generally represented by capital M and the SI unit is Kilogram for the mass. For distance we generally say small y which is represented in meter, velocity is small v and the SI unit is meter per second.

Similarly, acceleration is represented with small a and the SI unit is meter per second square. Force we represent with small f sometimes you will also see capital F is being used and the SI unit is Newton. Spring constant just we saw is represented by capital K and the SI unit is Newton per meter. Viscous friction coefficient with just saw it is represented with capital B and the SI unit for viscous friction coefficient is n by Newton per meter per second.

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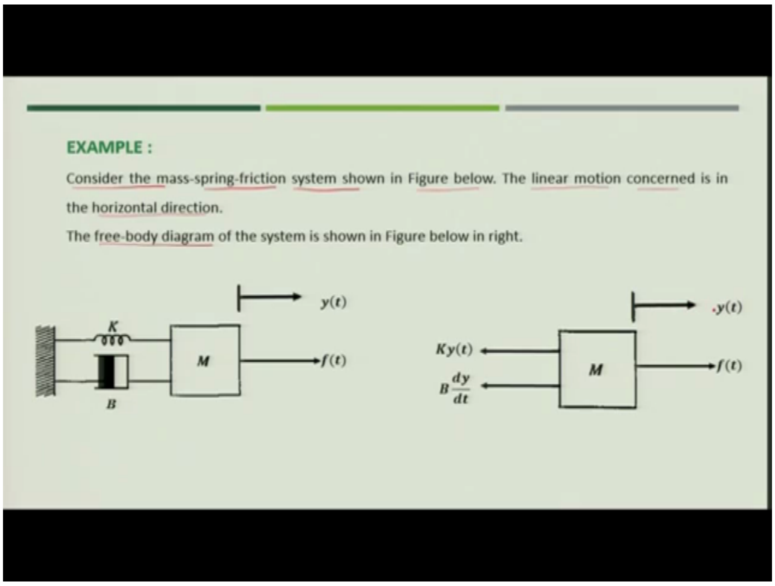

Now, let us understand the model, mechanical model which we just discussed with the help of one example. Now, let us consider the mass spring friction system which is shown in the figure below. The linear motion concerned in this is the horizontal direction, so the mass is moving in the horizontal direction. So, it is represented with $y(t)$, so displacement will be in the horizontal direction. So, it is having spring with spring constant K and you represent the friction with dashpot and the viscous friction coefficient which we considered is B and the force f which is applied to the mass.

Now, if you create the free body diagram of the system, how you will represent? There will be mass M, the force is applied on this mass will be one that is the spring friction, spring tension rather we should say. So, we represent it by $Ky(t)$. Viscous friction we represent with $B\,\frac{dy(t)}{dt}$ and then force which is responsible for mass to move in this direction.

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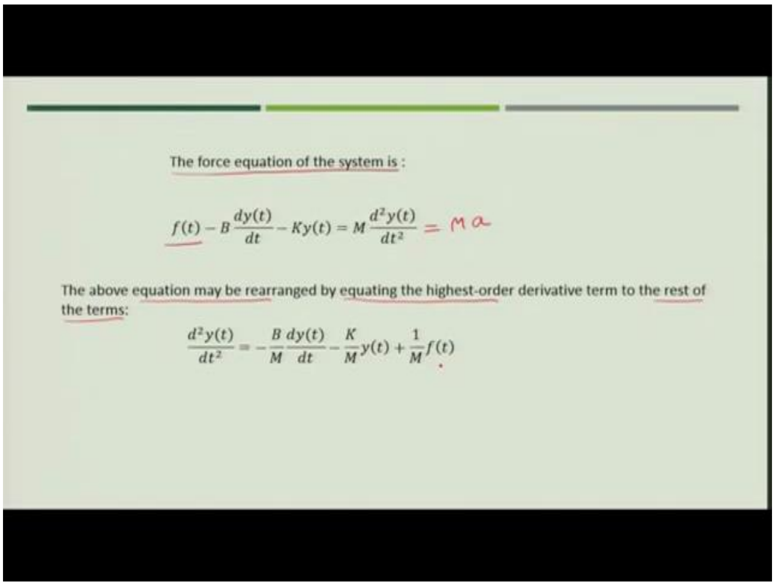

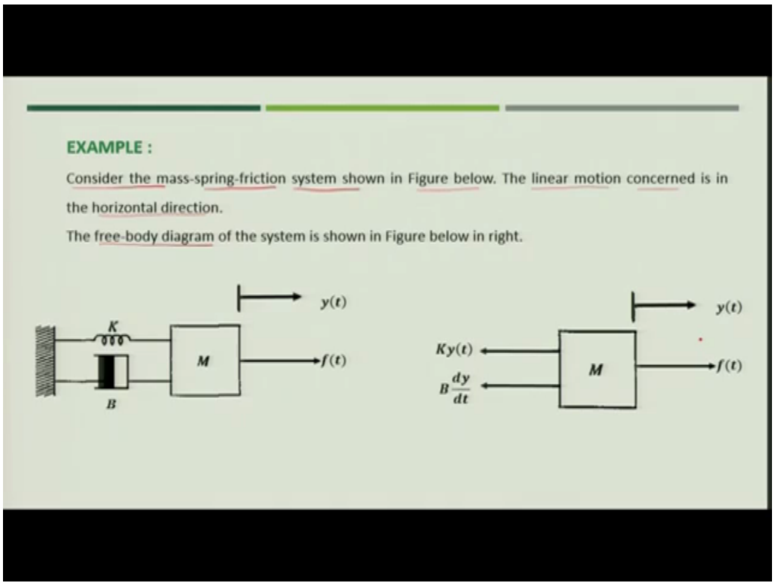

Now, if we compile the equation which is force equation of the system, what we can write? We can write f t, so that is the force which is applying in this direction since these two are in the opposite direction. What we can write?

$$f(t) - B\frac{dy(t)}{dt} - Ky(t) = M\frac{d^2y(t)}{dt^2}$$

So, f that is the net force applied on the mass minus the viscous friction minus spring tension you get the net force, which is responsible for acceleration of the object. Now, let us arrange the above equation by equating the highest order derivative term to the rest of the terms. What can we write? We can write,

$$\frac{d^2y(t)}{dt^2} = -\frac{B}{M}\frac{dy(t)}{dt} - \frac{K}{M}y(t) + \frac{1}{M}f(t)$$

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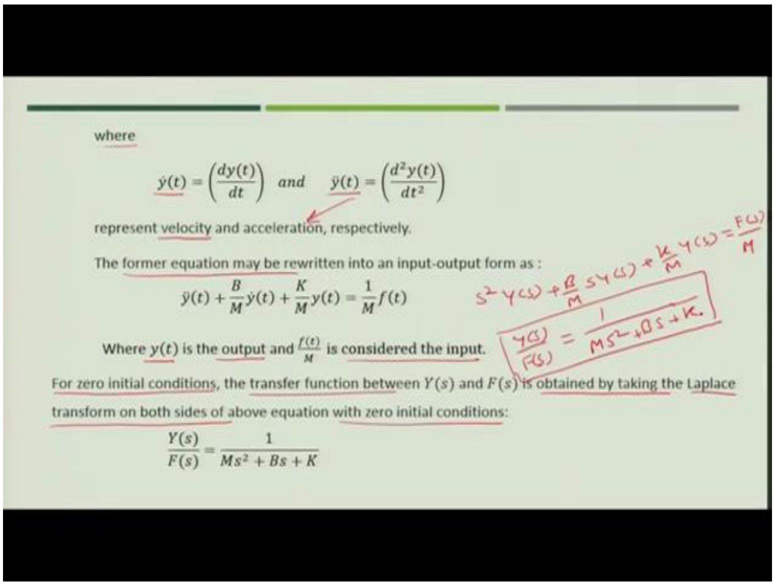

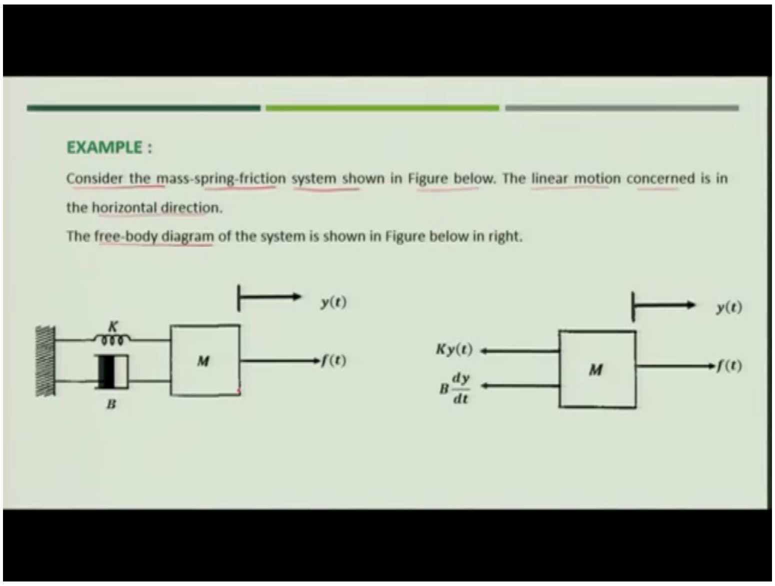

Now, $\dot{y}(t) = \left(\frac{dy(t)}{dt}\right)$ *and* $\ddot{y}(t) = \left(\frac{d^2y(t)}{dt^2}\right)$, represent velocity and acceleration, respectively.

So, the previous equation you can write as,

$$\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{1}{M}f(t)$$

Now, $y(t)$ *is the output and* $\frac{f(t)}{M}$ *is considered the input* to the system. Now, if you consider the zero initial conditions the transfer function that is between $Y(s)$ and $F(s)$ can be obtained by taking the Laplace transform on both sides of the equation with zero initial conditions. So, if you take the Laplace transform of this, what you can write? This will be

$$\frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

This is what you get the transfer function of the mechanical system which you just considered. So, if you see it is like what we saw in case of electrical systems. So, you can say that mechanical system which we just discussed is analogous to some electrical system which have the same transfer function as we saw in case of this mechanical system.

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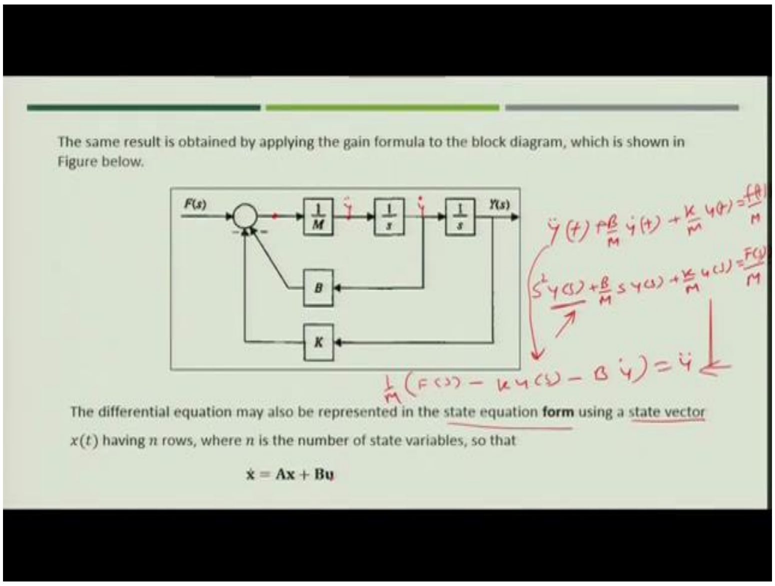

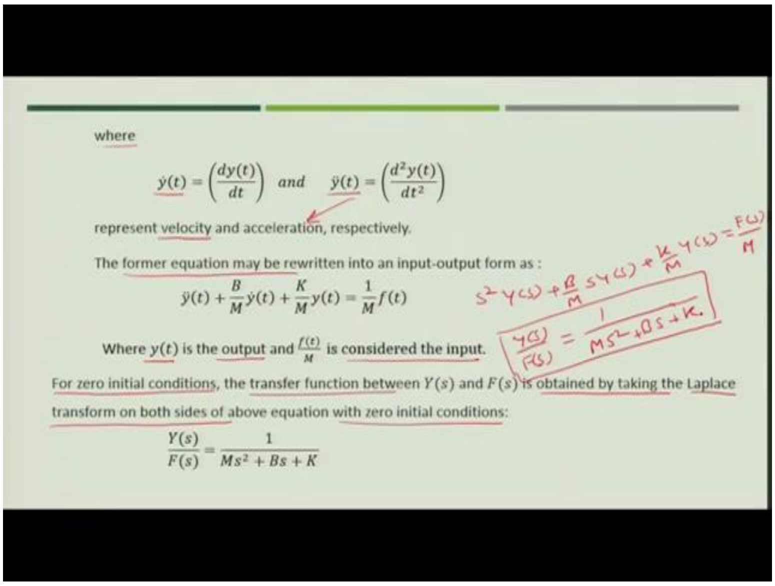

Now, how you will represent with the help of block diagram? So, if you see the equation, equation is $\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{1}{M}f(t)$. Now, if you take the Laplace transform it will become $s^2Y(s) + \frac{B}{M}sY(s) + \frac{K}{M}Y(s) = \frac{F(s)}{M}$. Now, $Y(s)$ is the output and $F(s)$ is the input.

So, how you will represent with a help of signal flow graph? So, if you see the particular equation if $Y(s)$ is the output, the $\dot{y}(t)$ is $sY(s)$, $\ddot{y}(t)$ is $s^2Y(s)$. So, what you can do now? You can see what is the output at this note? So, at this note will be y dot by M. Now, you compile this, how you will compile? You compile f s minus K into y s minus B into y dot then you multiply with 1 by M, you get y double dot.

So, if you see this particular equation if you take f by M into K by M into y minus B by M into y dot is equal to y double dot. So, with the help of assigning the particular values of y dot and y double dot you can simply find out that the equation which we just saw can be represented with the help of signal flow graph. So, you can cross verify with the help of the equation which we just written in case of the mechanical system. Now, next we can also use the State equation using the state vector for representing the same system. So, let us say that the State equation is $\dot{x} = Ax + Bu$.

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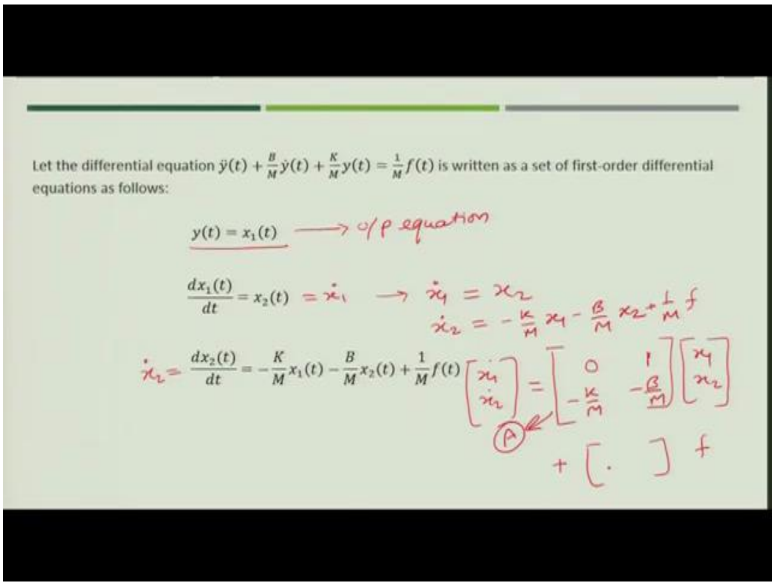

And your equation is $\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{1}{M}f(t)$. Now, what you have to do? You have to define the State variables. We will define $y(t) = x_1(t), \dot{y}(t) = x_2(t)$.

So, what you can write?

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{1}{M}f(t)$$

$$y(t) = x_1(t)$$

So, when you compile this, what you can write?

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{pmatrix}0 & 1\\ -\frac{K}{M} & -\frac{B}{M}\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}^T \frac{f(t)}{M}$$

So, this is nothing but the matrix A we have got. For f you have only one element that is $\frac{1}{M}$ which is coming against $\dot{x}_2$, so you will have only one element in that, so that is 0 and 1.

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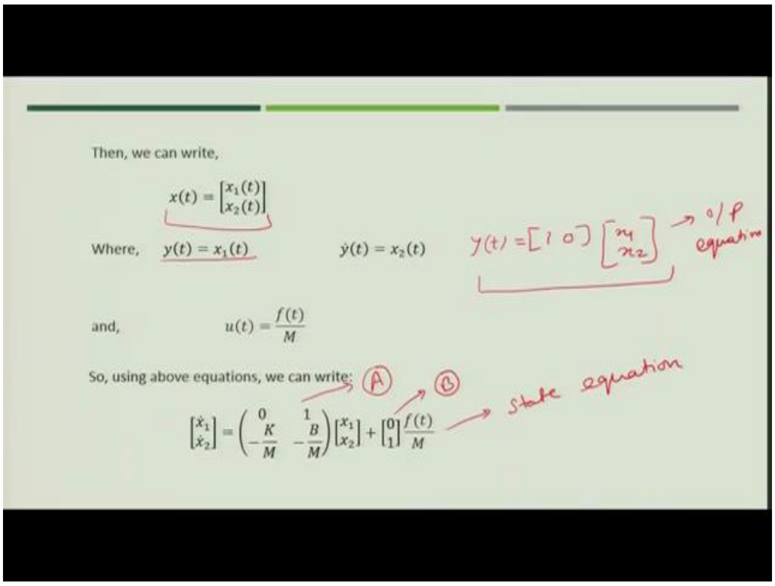


Let the differential equation $\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{1}{M}f(t)$ is written as a set of first-order differential equations as follows:

$$y(t) = x_1(t) \longrightarrow \text{o/p equation}$$

$$\frac{dx_1(t)}{dt} = x_2(t) = \dot{x}_1 \rightarrow \dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{K}{M}x_1 - \frac{B}{M}x_2 + \frac{1}{M}f$$

$$\dot{x}_2 = \frac{dx_2(t)}{dt} = -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{1}{M}f(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} f$$

where

$$\dot{y}(t) = \left(\frac{dy(t)}{dt}\right) \quad and \quad \ddot{y}(t) = \left(\frac{d^2y(t)}{dt^2}\right)$$

represent velocity and acceleration, respectively.

The former equation may be rewritten into an input-output form as :

$$\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{1}{M}f(t)$$

$$s^2 Y(s) + \frac{B}{M} sY(s) + \frac{K}{M} Y(s) = \frac{F(s)}{M}$$

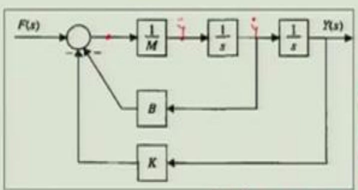
$$\boxed{\frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}}$$

Where $y(t)$ is the output and $\frac{f(t)}{M}$ is considered the input.

For zero initial conditions, the transfer function between $Y(s)$ and $F(s)$ is obtained by taking the Laplace transform on both sides of above equation with zero initial conditions:

$$\frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

The same result is obtained by applying the gain formula to the block diagram, which is shown in Figure below.



$$\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{f(t)}{M}$$

$$s^2 Y(s) + \frac{B}{M} sY(s) + \frac{K}{M} Y(s) = \frac{F(s)}{M}$$

$$\frac{1}{M}\left(F(s) - KY(s) - B\dot{y}\right) = \ddot{y}$$

The differential equation may also be represented in the state equation **form** using a state vector $x(t)$ having n rows, where n is the number of state variables, so that

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

And this is the A which you have got, this is B which you have got basically in this case when we write we can write $1/M$ also or you can take M out also. So, in that case when you take M out it will become f/M and the elements will be 0, 1. So, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $y(t) = x_1(t)$ is your output equation. So, you can also write $y(t) = [1\ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

This you have got as output equation, and this is what you have got State equation. So, you can now understand that the mechanical system can also be represented with the help of either the transfer function method or you can create the equivalent the block diagram or you can convert into signal flow graph or you can represent it in the form of State equation.

So, we close our today’s session with the discussion related to the mechanical input which we just had. We continue our discussion, and we will try to understand some other mechanical properties of this system. And will create the mathematical models of those systems. Then finally we will move on to comparison of electrical with the other mechanical as well as other types of systems. Thank you.