Basic Electric Circuits
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Module 11 - State Variable Analysis
Lecture 55 - State Variable Method to circuit Analysis

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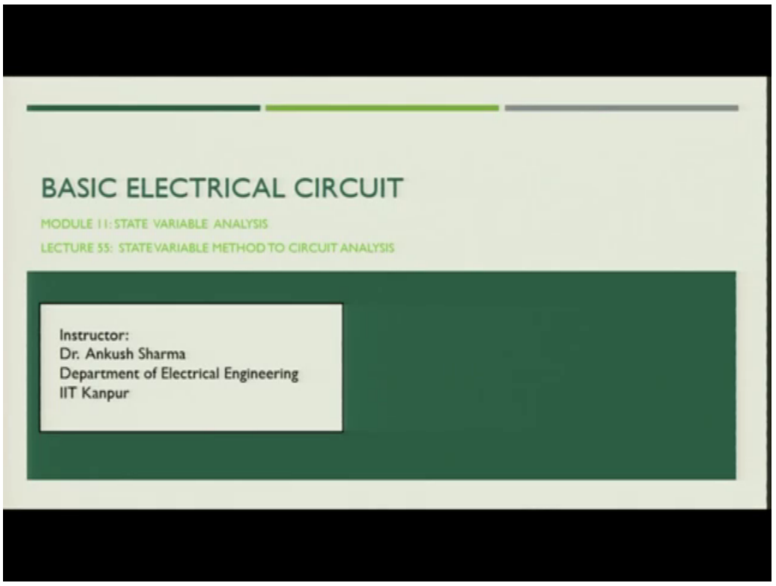

Namaskar, so in last class we discussed about the state transition matrix and the state transition equations. So, today we will discuss about the application of state variable analysis in the circuit analysis.

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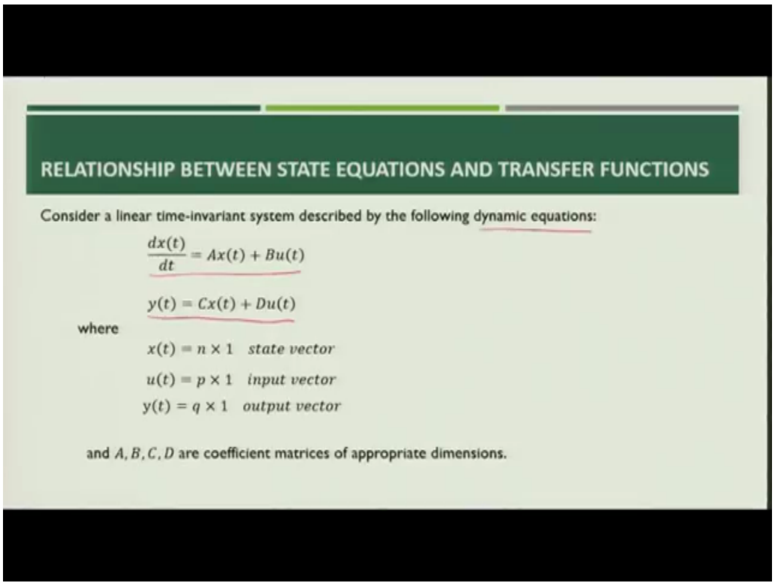

So, let us start our discussion of today's lecture. Before going into the application of state variable in circuit analysis, first let us understand what the relationship between state equations and the transfer functions is. So, we know that the linear time invariant system can be described by following the dynamic equation.

So, first is state equation,

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

the output equation,

$$y(t) = Cx(t) + Du(t)$$

where,

$$x(t) = n \times 1 \quad state\ vector$$

$$u(t) = p \times 1 \quad input\ vector$$

$$\mathrm{y}(t) = q \times 1 \quad output\ vector$$

and $A,\ B,\ C,\ D$ are coefficient matrices of appropriate dimensions.

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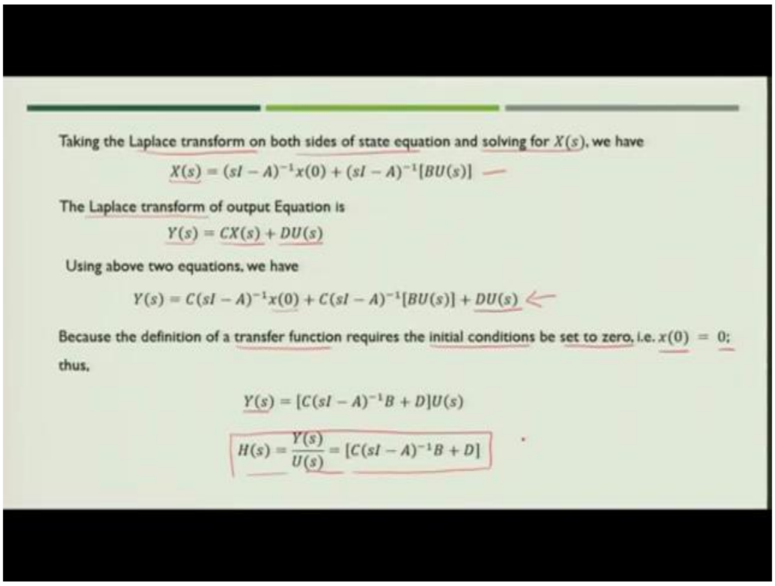

Now, if you take the Laplace transform on both sides of the state equation and rearrange the equation, you get,

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}[BU(s)]$$

Similarly, if you take Laplace transform of the output equation you get,

$$Y(s) = CX(s) + DU(s)$$

Now, in the output equation you can replace $X(s)$ with the value which we have just computed for $X(s)$ and we can write,

$$Y(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}[BU(s)] + DU(s)$$

Now, if you see this equation in this you have the value of $x(0)\ =\ 0$. So, the transfer function requires that initial condition to be set to 0, so this is what we have discussed when we discussed the transfer function in the previous lectures.

Now, if you set initial condition to 0 and simplify the above equation you get

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = [C(sI - A)^{-1}B + D]$$

So, this is the equation which defines the relationship between state equation and the transfer function. So, if you are given the state equations where you can find out the A, B, C, D coefficients or the coefficient matrices you can simply calculate the transfer function of the system.

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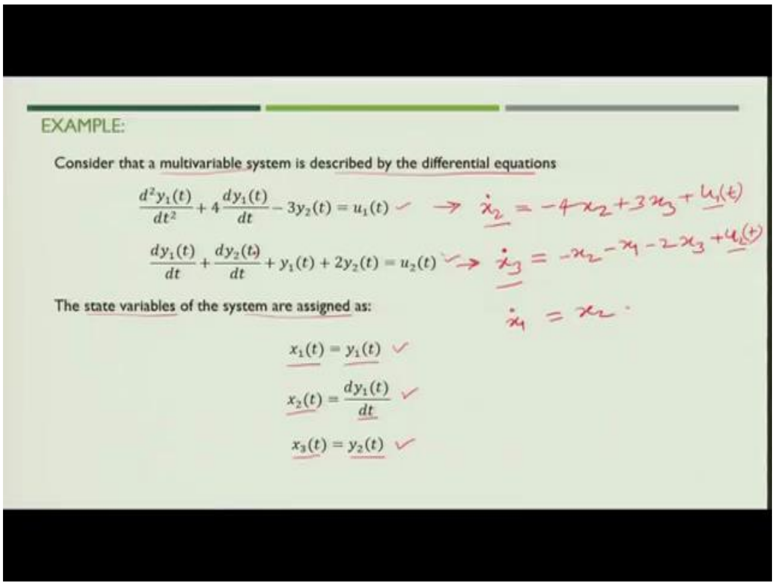

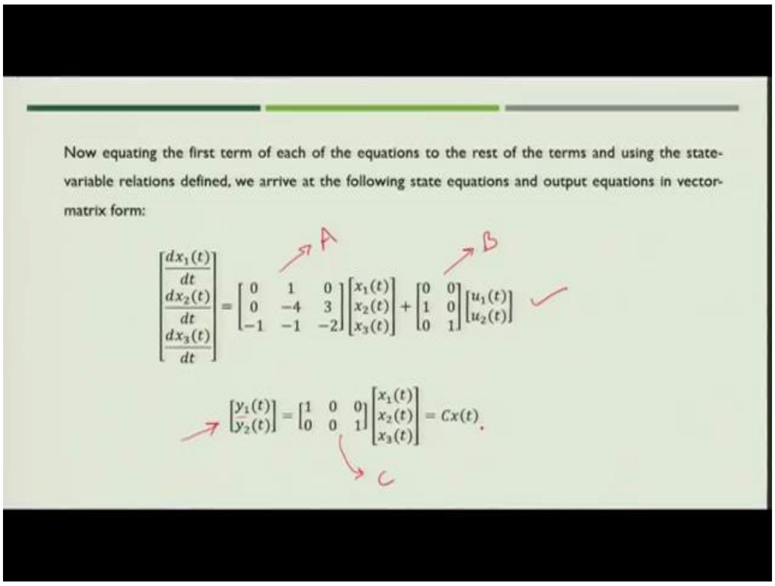

Let us, take one example so that you can understand what we have discussed. Consider a multivariable system which is described with the help of these two differential equations. So, first is $\frac{d^2y_1(t)}{dt^2} + 4\frac{dy_1(t)}{dt} - 3y_2(t) = u_1(t)$, second is $\frac{dy_1(t)}{dt} + \frac{dy_2(t)}{dt} + y_1(t) + 2y_2(t) = u_2(t)$.

Now, let us define the state variables in the system as $x_1(t) = y_1(t)$, so that is one state variable, $x_2(t) = \frac{dy_1(t)}{dt}$ and $x_3(t) = y_2(t)$. So, when we define these 3 state variables for the system, we can write these 2 differential equation in the form of the state variable. So, what you will write for this?

Now equating the first term of each of the equations to the rest of the terms and using the state-variable relations defined, we arrive at the following state equations and output equations in vector-matrix form:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = Cx(t)$$

Now, since we do not have any input variable in the output equation the D coefficient matrix will be 0. So, now this you will get as A, this you will get as B and this is what you have as C matrices, D is of course 0.

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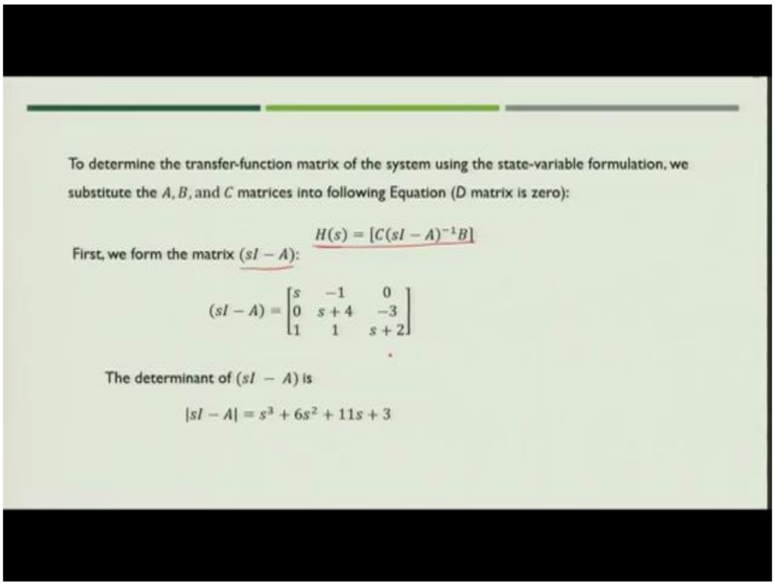

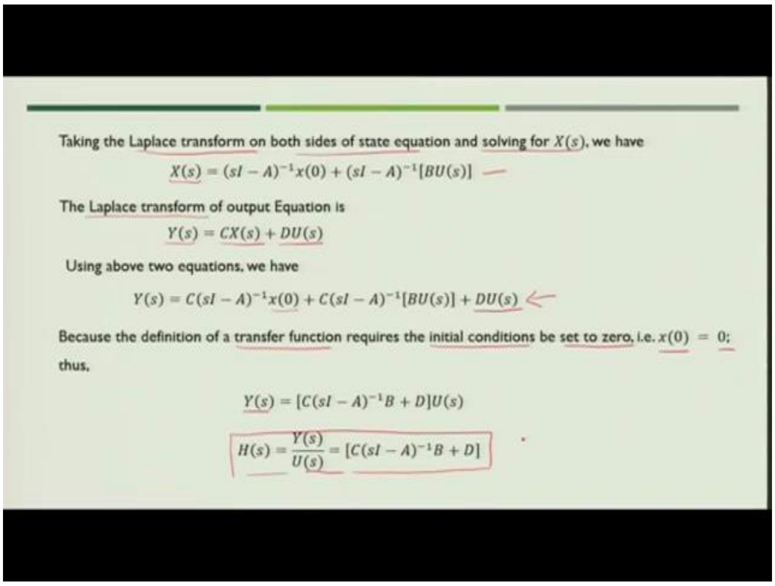

So, now if we have to find out the transfer function of the system, we will use the derivation $H(s) = [C(sI - A)^{-1}B]$. Now, first we need to find out the value of $sI - A$. So, we compile,

$$(sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s+4 & -3 \\ 1 & 1 & s+2 \end{bmatrix}$$

The determinant of $(sI \ - \ A)$ is

$$|sI - A| = s^3 + 6s^2 + 11s + 3$$

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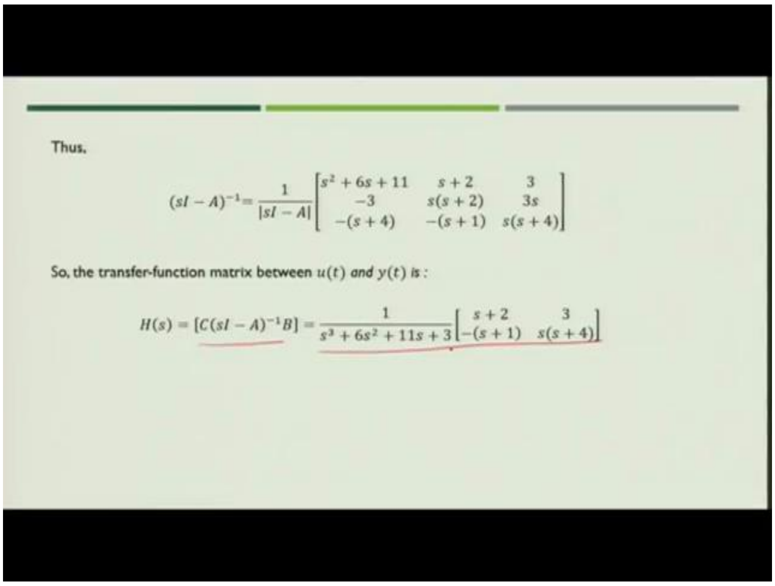

So, the inverse you will get,

$$(sI-A)^{-1}=\frac{1}{|sI-A|}\begin{bmatrix} s^2+6s+11 & s+2 & 3 \\ -3 & s(s+2) & 3s \\ -(s+4) & -(s+1) & s(s+4) \end{bmatrix}$$

So, the transfer-function matrix between $u(t)$ *and* $y(t)$ *is :*

$$H(s)=[C(sI-A)^{-1}B]=\frac{1}{s^3+6s^2+11s+3}\begin{bmatrix} s+2 & 3 \\ -(s+1) & s(s+4) \end{bmatrix}$$

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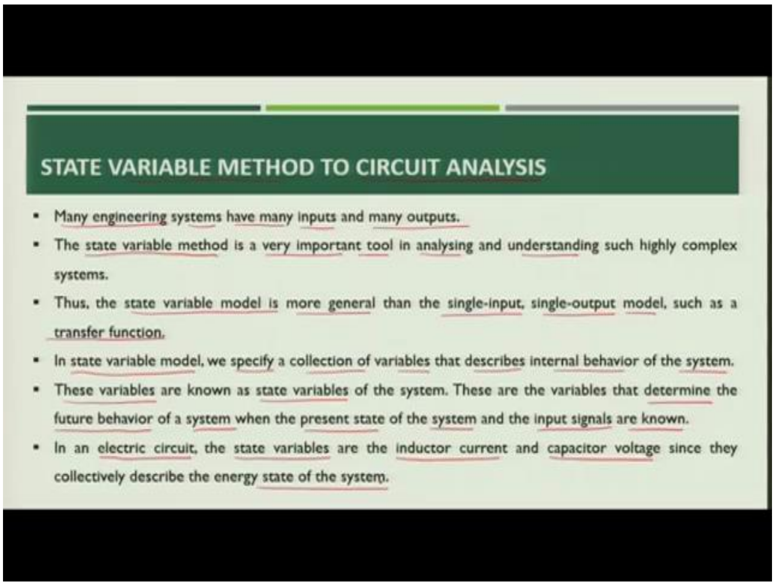

Now, let us start our discussion related to state variable method to our circuit analysis. How we will carry out the circuit analysis with the help of state variable methods? So, we will discuss. Now, since many engineering system we see have multiple input and multiple outputs, so that is why the state variable method is very important tool in analysing and understanding such complex systems which have multiple input and multiple outputs.

So therefore, the state variable model which we have just discussed is more general than the single-input, single-output model which we generally see in case of the transfer function. So, what we do in state variable model? We specify a collection of variables that describe the internal behaviour of the system and these variables are known as state variables of the system. These are the variables that determine the future behaviour of the system when present state of the system and the input signals are known to us.

So, that means the state variable which you find will give you the future behaviour of the system. With respect to our electrical circuit the state variables we generally consider as inductor current and capacitor voltages because they collectively describe the energy state of the system.

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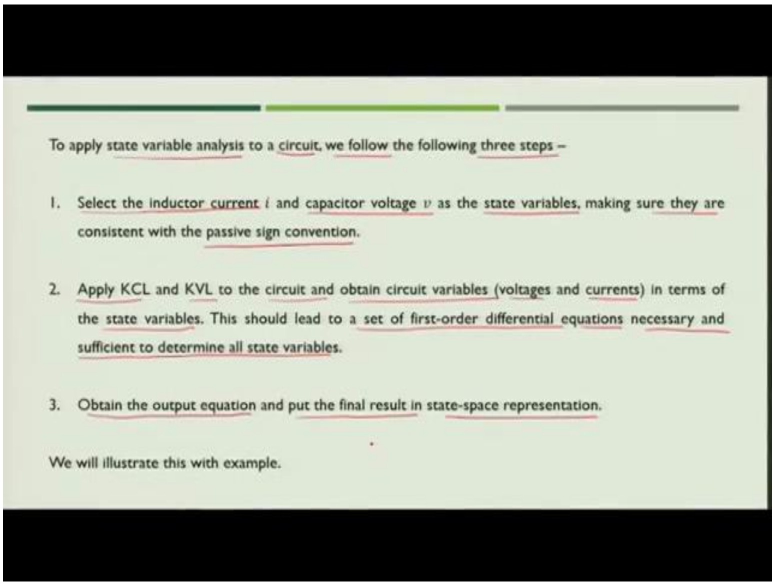

Now, let us see how we will implement it? So, to apply the state variable analysis to our circuit analysis method we follow the following three steps. First step is that what we will select the inductor current i and capacitor voltage v as state variable. We must make sure that they are consistent with the passive sign convention which we discussed in our initial lectures.

Now, let us apply KCL and KVL to the circuit and obtain the circuit variables. So, the circuit variables are voltages and currents in terms of these state variables. This should lead to a set of first order differential equation because when you talk about the voltage and current voltage across capacitor and current at through inductor you will get the differential equations which is necessary and sufficient to determine the state variables.

Now, obtain the output equation and the put the result in state space representation format. So, to see how we can implement this we will take couple of examples so that you can understand this concept very clearly.

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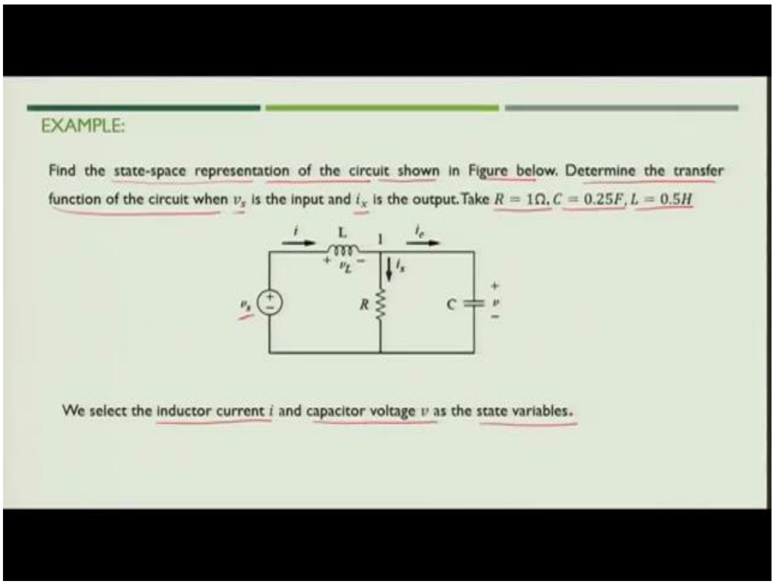

Let us, take one example. Here we have state space, we need to find the state space representations of the circuit which is shown in the figure below. We need to determine the transfer function of the circuit also here vs is the input and ix is the output. So, we need to find the value of ix and vs is given as an input.

The value of resistance $R = 1\Omega$, $C = 0.25F,\ L = 0.5H$. Now, in this circuit what we will do? We will first select inductor current i and capacitor voltage v as the state variables.

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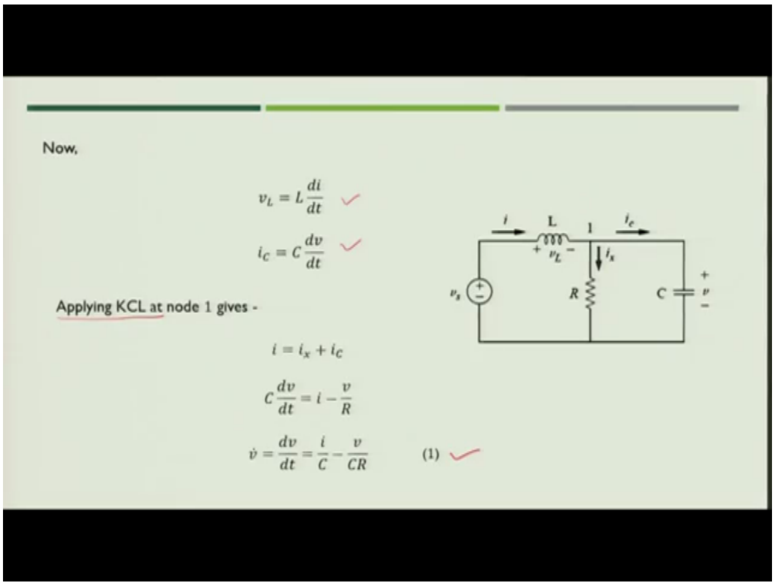

So, when we see them as a state variable, what we can first find? We can find

$$v_L = L\frac{di}{dt}$$

$$i_C = C\frac{dv}{dt}$$

These are the equations which we have already discussed in our previous lectures.

Now, let us apply the KCL at node 1. So, when you apply KCL at node 1 what you can write?

$$i = i_x + i_C$$

$$C\frac{dv}{dt} = i - \frac{v}{R}$$

$$\dot{v} = \frac{dv}{dt} = \frac{i}{C} - \frac{v}{CR}$$

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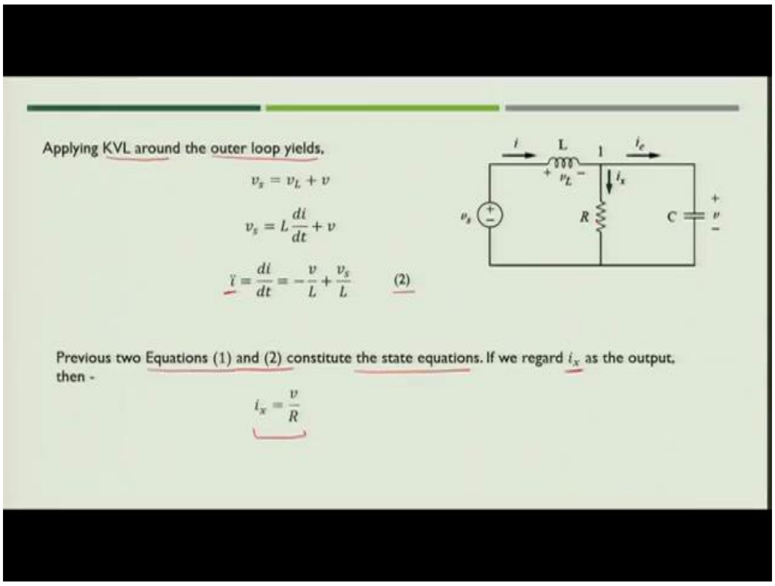

Now, next we apply the KVL around the outer loop we get

$$v_s = v_L + v$$

$$v_s = L\frac{di}{dt} + v$$

$$\ddot{i} = \frac{di}{dt} = -\frac{v}{L} + \frac{v_s}{L}$$

Now, from the above equations we get the state equation. Now, if we take i_x as an output then $i_x = \frac{v}{R}$.

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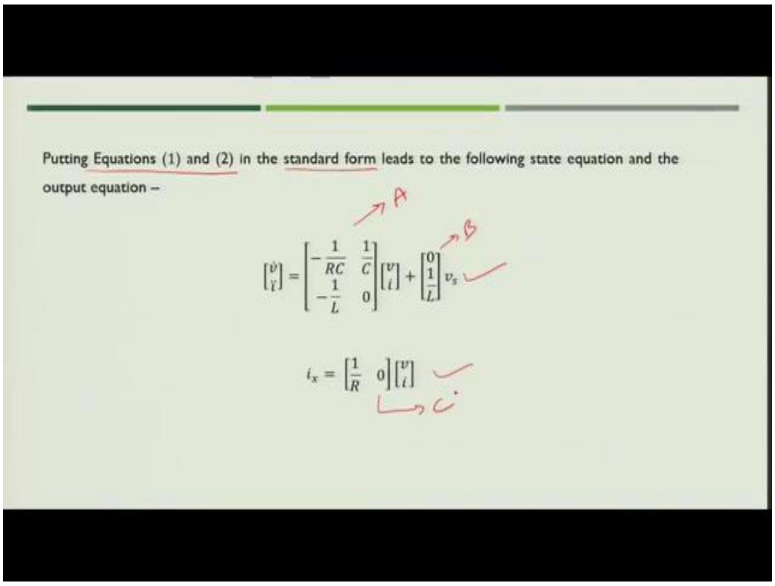

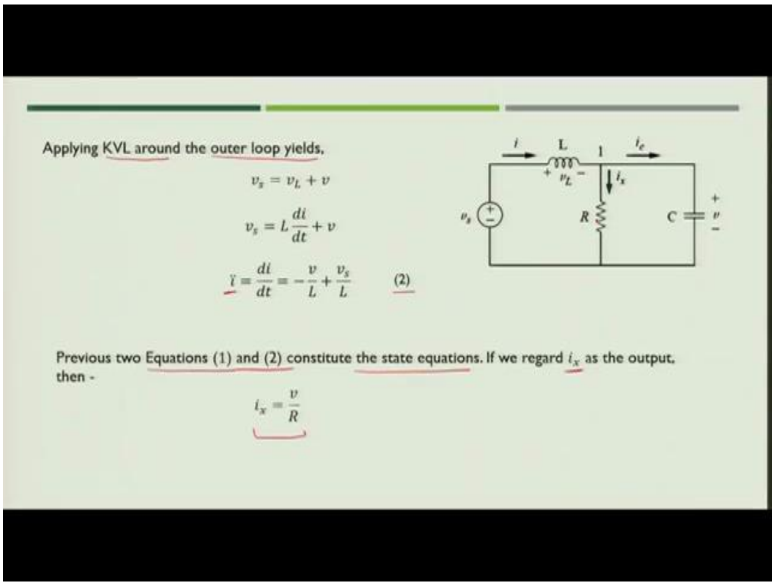

So, if you compile these equations, we get the following state equation.

$$\begin{bmatrix}\dot{v}\\ \ddot{i}\end{bmatrix} = \begin{bmatrix}-\frac{1}{RC} & \frac{1}{C}\\ -\frac{1}{L} & 0\end{bmatrix}\begin{bmatrix}v\\ i\end{bmatrix} + \begin{bmatrix}0\\ 1\\ \overline{L}\end{bmatrix} v_s$$

$$i_x = \begin{bmatrix}\frac{1}{R} & 0\end{bmatrix}\begin{bmatrix}v\\ i\end{bmatrix}$$

Now, you have got A, you have got B and you have got C of course D is 0.

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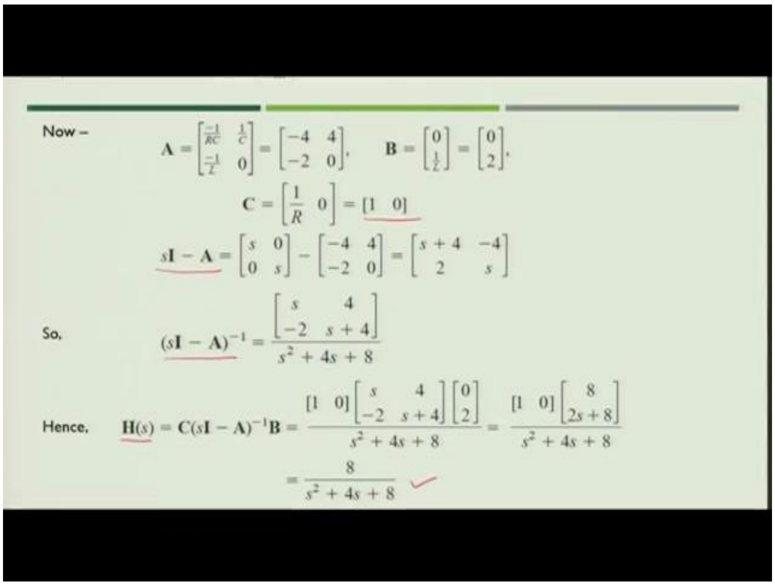

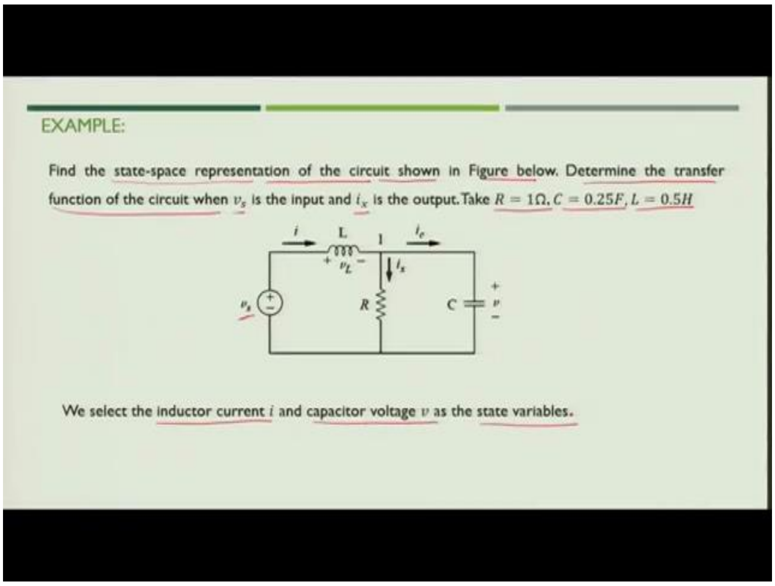

So, what we will do? We have

$$\mathbf{A} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} = [1 \quad 0]$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & 4 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s+4 & -4 \\ 2 & s \end{bmatrix}$$

So, now you have got A, B and C.

The next task is that we find first the value of

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} s & 4 \\ -2 & s+4 \end{bmatrix}}{s^2 + 4s + 8}$$

So, now you can simply say that the transfer function related to the electrical circuit which we have in the example is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{[1 \quad 0]\begin{bmatrix} s & 4 \\ -2 & s+4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}}{s^2 + 4s + 8} = \frac{[1 \quad 0]\begin{bmatrix} 8 \\ 2s+8 \end{bmatrix}}{s^2 + 4s + 8}$$

$$= \frac{8}{s^2 + 4s + 8}$$

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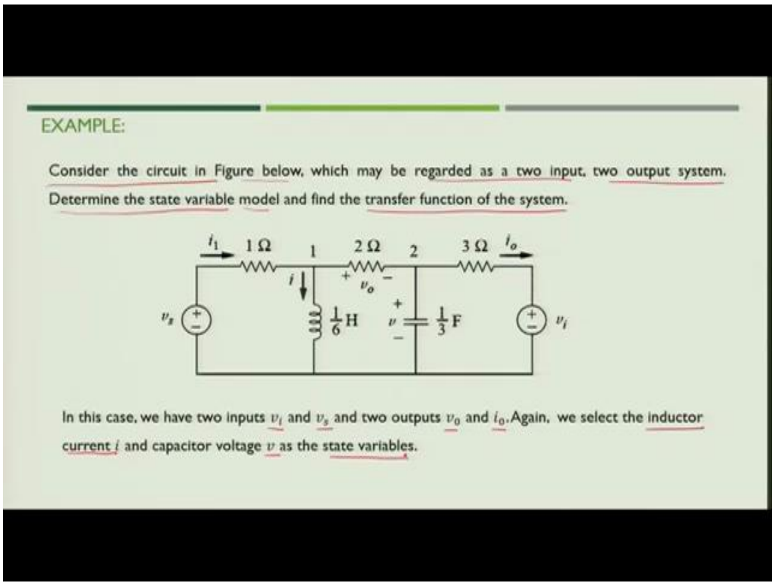

Now, let us take another example. So, consider the circuit which is shown in the figure below, which may be regarded as a two input and two output system. In the previous case we saw the simple single output system. Here we have 2 input and 2 output system. In this case we need to determine the state variable model and the transfer function of the system.

Now, what we will do in this case we have two inputs. What are those inputs? v_i and v_s are the two inputs in the circuit and we have two outputs so we need to find the value of v_0 and i_0, v_0 is the voltage across 2 ohm resistance and i_0 is the current following through the 3 ohm resistance. So, v_i and v_s are considered as an input in this case and v_0 and i_0. are the outputs. So, as we did in the previous case. In this case also we select inductor current and capacitor voltage as the state variables.

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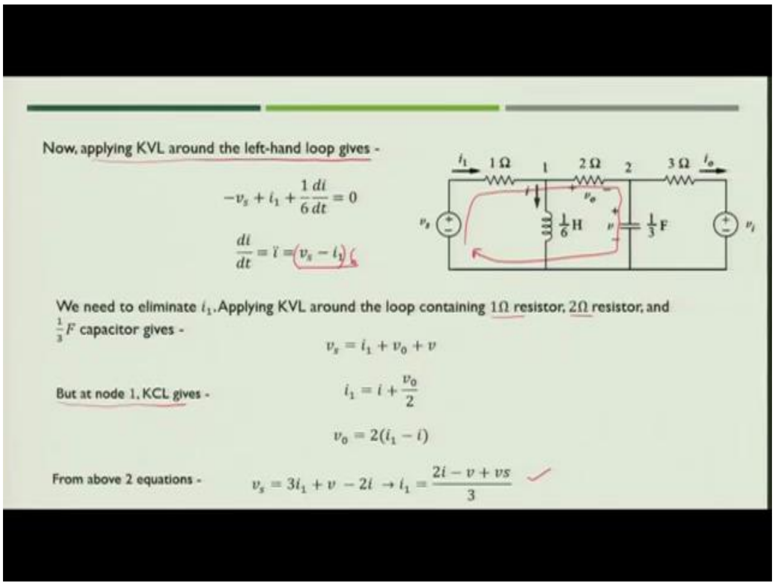

So, first what we will do? We will apply KVL around the left hand loop so this is the left hand loop. If you apply KVL what you can write?

$$-v_s + i_1 + \frac{1}{6}\frac{di}{dt} = 0$$

Now if you rearrange you can write,

$$\frac{di}{dt} = \ddot{\imath} = v_s - i_1$$

We need to eliminate i_1. Applying KVL around the loop containing 1Ω resistor, 2Ω resistor, and $\frac{1}{3}F$ capacitor. So, if you apply KVL around this, what you get?

$$v_s = i_1 + v_0 + v$$

But at node 1, KCL gives

$$i_1 = i + \frac{v_0}{2}$$

$$v_0 = 2(i_1 - i)$$

From above 2 equations,

$$v_s = 3i_1 + v - 2i \rightarrow i_1 = \frac{2i - v + v_s}{3}$$

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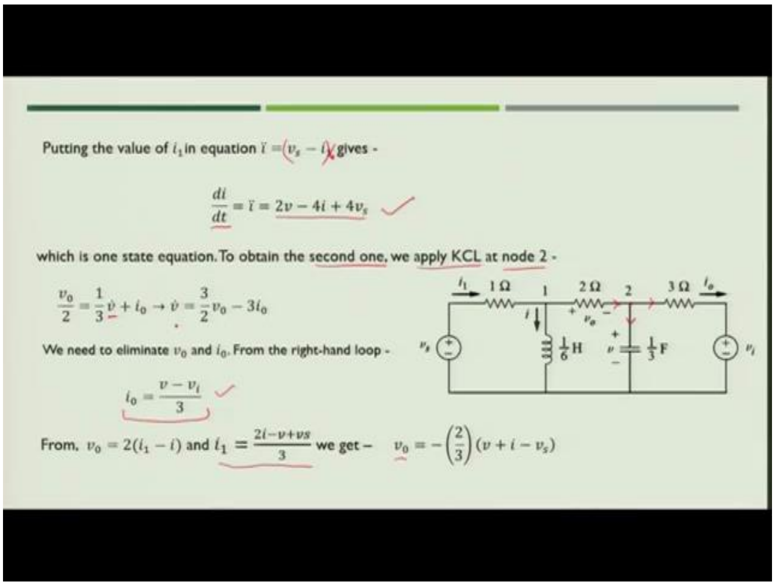

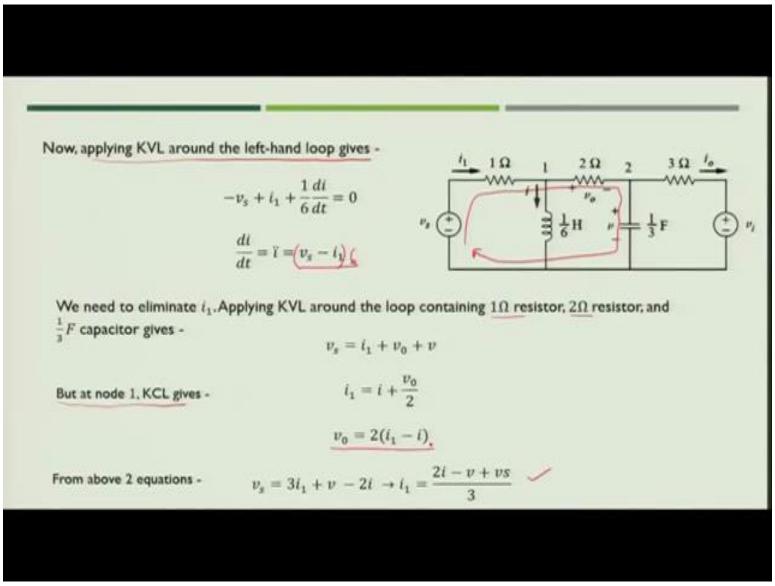

Now, putting the value of i_1 in equation $\ddot{i} = v_s - i_1$. So, this is

$$\frac{di}{dt} = \ddot{i} = 2v - 4i + 4v_s$$

Now, you got one equation. Now, to obtain the second one what we do? We apply KCL at node 2. So, if you apply KCL at node 2 what you get?

$$\frac{v_0}{2} = \frac{1}{3}\dot{v} + i_0 \rightarrow \dot{v} = \frac{3}{2}v_0 - 3i_0$$

Now, to eliminate what we will do? We get from the right hand loop

$$i_0 = \frac{v - v_i}{3}$$

This we get form this loop and from the previous discussion that is $v_0 = 2(i_1 - i)$ and

$i_1 = \frac{2i - v + v_s}{3}$.

So, if you put these values and simplify, we get,

$$v_0 = -\left(\frac{2}{3}\right)(v + i - v_s)$$

Now, we have got the value of i_0 and v_0, we put these values in this above equation and simplify.

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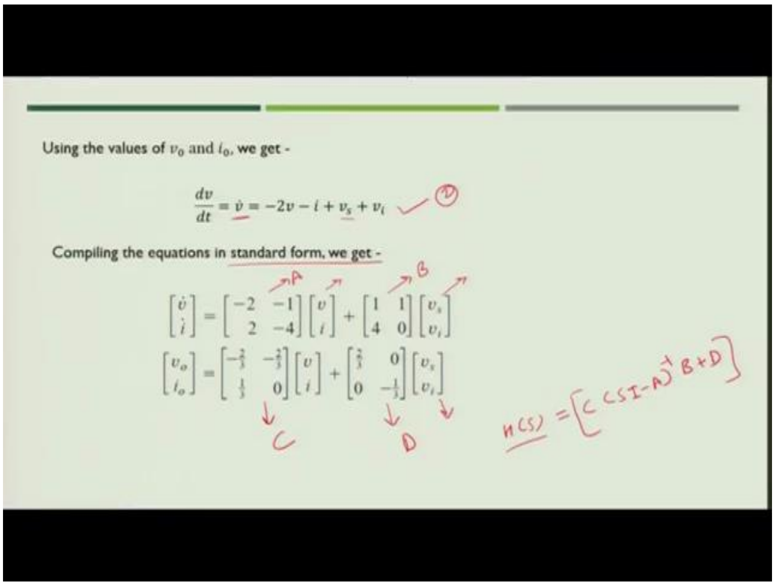

When we simplify, we get

$$\frac{dv}{dt} = \dot{v} = -2v - i + v_s + v_i$$

So, now you have got the second state equation this was the first one, so now if you compile it in the form of the matrix the state equation which you can now write

$$\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ v_i \end{bmatrix}$$

$$\begin{bmatrix} v_o \\ i_o \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} v_s \\ v_i \end{bmatrix}$$

So, with this you can now understand that how you can convert this particular the circuit into a state space representation. Now, you have A, B, C and D so the transfer function you can simply calculate $C(sI - A)^{-1}B + D$ so put the values of A, B, C, D in this matrix and simplify you get the transfer function Hs. So, with this we can close our today's discussion in this session we discussed mainly about the application of (state space) state variable equations in the circuit analysis. We will continue our discussion related to other aspects of state variable in our next lecture. Thank you.