Basic Electric Circuits
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Module 11 - State Variable Analysis
Lecture 54 - State Transition Matrix

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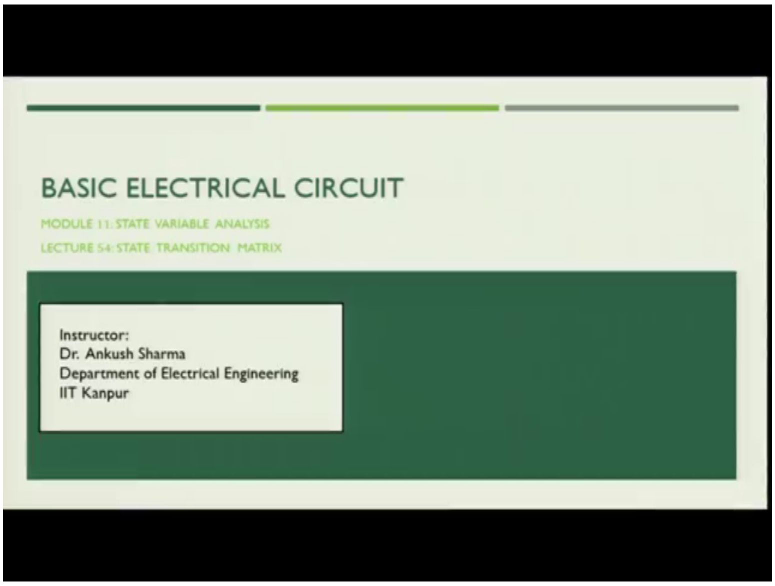

Namaskar, so in last class we were discussing about the state diagram, today we will discuss about the State Transition Matrix.

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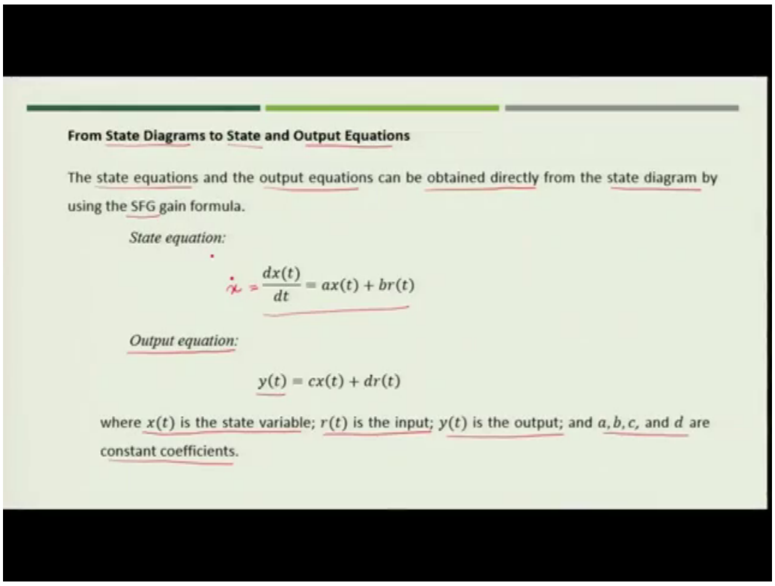

So, let us start the discussing of today's session. So, before going into the state transition matrix, let us first understand how you will create the state and output equations from the state diagram. Now, you may be knowing that the state equation and output equations can be obtain directly.

So, with the help of state diagram, state equation and output equation we know we have discussed in the previous lectures. So, we will see today that how state equation or output equations can be obtained with the help of state diagram for that we use signal flow graph gain formula and find out the state and output equations.

So, what is the state equation? Let us recollect what we discussed in previous lectures. So, state equation is $\frac{dx(t)}{dt} = ax(t) + br(t)$. Then, output equation, output equation can be written as $y(t) = cx(t) + dr(t)$. Now, $x(t)$ is the state variable; $r(t)$ is the input; $y(t)$ is the output; and $a,\ b,\ c,$ and d are constant coefficients. So, in this way you can define the state and output equation.

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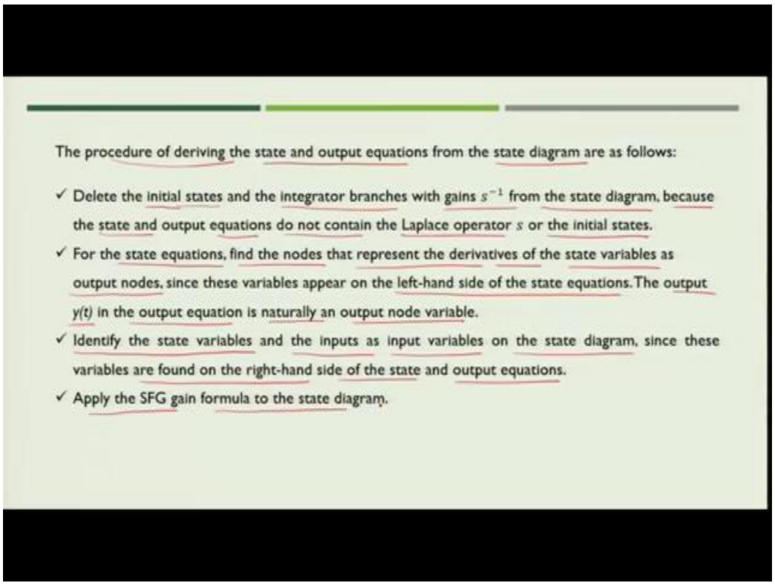

Now, let us try to find out the procedure of deriving the state and output equations from the state diagram. So, in the state diagram you have seen that there are initial states and integrator branches in addition to the gains, which we connect from one node to other node, so when we derive the state and the output equation from the state diagram, we first delete the initial states that is we say like time t is equal to 0 or may be any other time, say t_0 what are the initial states?

We will remove those initially states from the state diagram, we also remove the integrator branches which have gain as s^{-1} from the state diagram. Why we remove it? Because the state and output equations do not contain the Laplace operator that is s or the initial states. So, that is why we remove these two components from our state diagram.

Now, for the state equation we find the nodes that represent the derivatives of the state variables as output nodes, so this will become the left side of the state equation. And then the output $y(t)$ will also be identified in the output equation we will represent $y(t)$ that is naturally an output node variable in the state diagram.

Now, next we will identify the state variables and the inputs as input variable on the state diagram and these variables are found on the right side on the state as well as output equations. And then we finally apply the signal flow graph gain formula to the state diagram.

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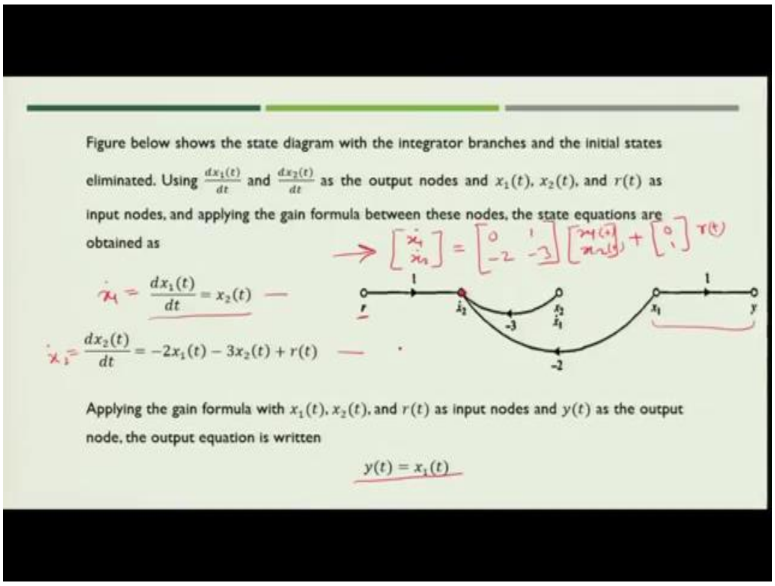

Let us, take one example so that we can understand how we can find out the state and output equation with the help of state diagram, let us see this is the state diagram where we have removed the initial states as well as the $1/s$ that is the integrator section. So, we are left with the only gains and the states. So, if you see here $r(t)$ is the input it is going into the state let say this is the $\dot{x}_2$ given in the state diagram x_1 is connected to y, so from here you can straight away say that $y(t) = x_1(t)$, so this will give you the output equation.

Now, let us try to find out the state equation. So, when you see this node $x_2 = \dot{x}_1 = \frac{dx_1(t)}{dt}$. We get one equation as $\frac{dx_1(t)}{dt} = x_2(t)$. Next node we have

$$\frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) + r(t)$$

So, this is how you will get the state equation from the state diagram. Output equation we have already seen that is $y(t) = x_1(t)$. So, this is how you compile the state and the output equations.

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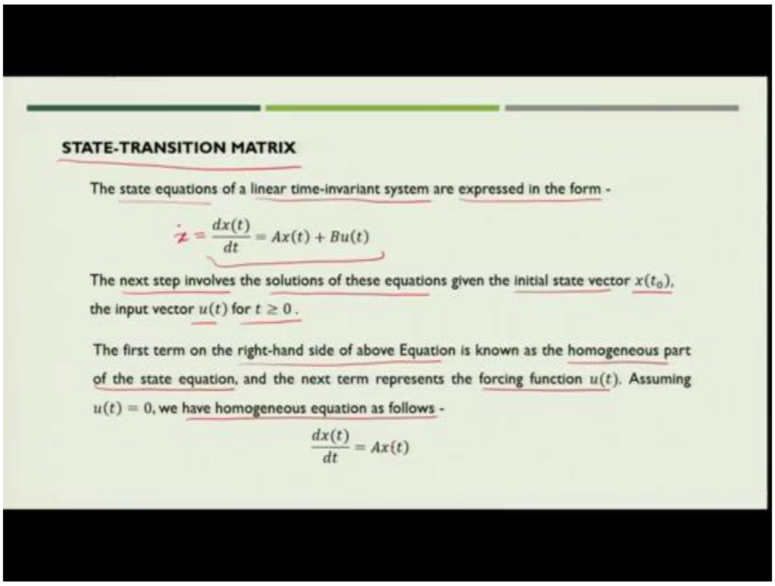

Now, let us talk about the State Transition Matrix. What is a State Transition Matrix? So, here in the previous slide we have seen the state equations of a linear time invariant system are expressed in this form. So, $\frac{dx(t)}{dt} = Ax(t) + Bu(t)$.

Now, the next step is the finding out the solution of these equation. Let us, assume that there is an initial state vector $x(t_0)$, the input vector $u(t)$ for $t \geq 0$. Now, if you see this equation the right-hand side of the above equation is known as homogeneous part of the state equation and next term is forcing function that is $u(t)$. Now, if you assume for the time being that $u(t) = 0$ then we have homogeneous equation. So, homogeneous equation is nothing but $\frac{dx(t)}{dt} = Ax(t)$.

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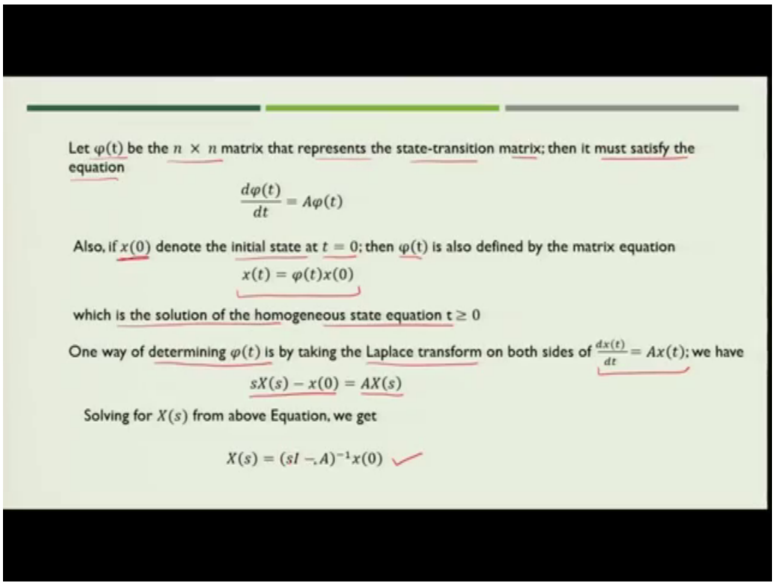

Now, let us assume that there is a matrix $\varphi(t)$ which is $n \times n$ matrix and it represents the state transition matrix. So, in that case if it is the state transition matrix it should satisfy the following equation, that is $\frac{d\varphi(t)}{dt} = A\varphi(t)$. Now, if $x(0)$ denote the initial state at $t = 0$; then $\varphi(t)$ is also defined by the matrix equation and you can write,

$$x(t) = \varphi(t)x(0)$$

So, state transition matrix is nothing but the matrix which correlates any the states at time $t = 0$ to any state of time t. So, this equation which you have got $x(t) = \varphi(t)x(0)$ is the solution of homogeneous state equation. Now, we need to find out the value of $\varphi(t)$ in term of the other parameters which we have like we have A, B and x, so what we will do? The value of $\varphi(t)$ can be determined by taking the Laplace transform on both sides of this equation that is the homogeneous equation we have got that is $\frac{dx(t)}{dt} = Ax(t)$.

So, when you take the Laplace transform on both sides, what you can write? For this side you can write

$$sX(s) - x(0) = AX(s)$$

Now, if you rearrange the terms you can simply write

$$X(s) = (sI - A)^{-1}x(0)$$

So, this is what you get the value of $X(s)$. Now, if you see we are taking the inverse of this matrix that is $sI - A$.

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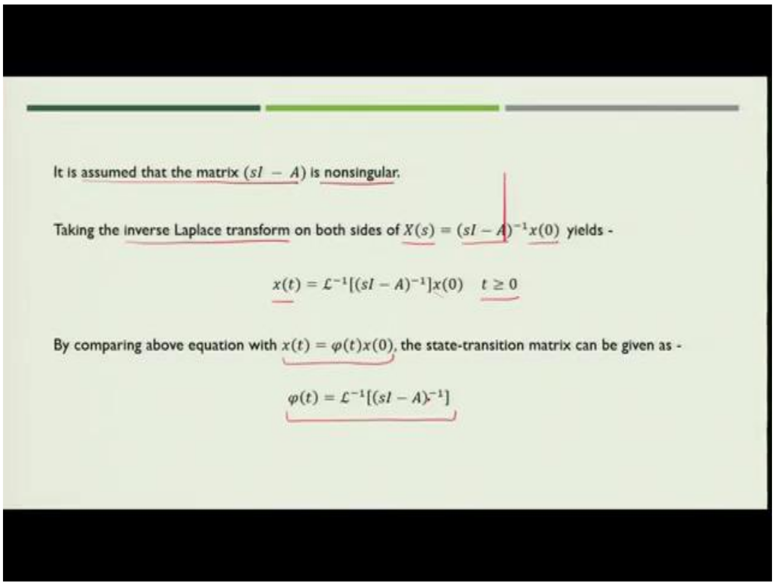

So, the important property which we must assume here is that the matrix $sI - A$ is nonsingular means the determent of this particular matrix is not equal to 0. Now, let us take the inverse Laplace transform on both sides,

$$x(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x(0) \quad t \geq 0$$

Now, this you can compare with our initial equation which we found in terms of state transition matrix that is xt that is the state at any time t is related to the initial state with the help of state transition matrix. So, when you compare both of them you can simply get the value of state transition matrix

$$\varphi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}] .$$

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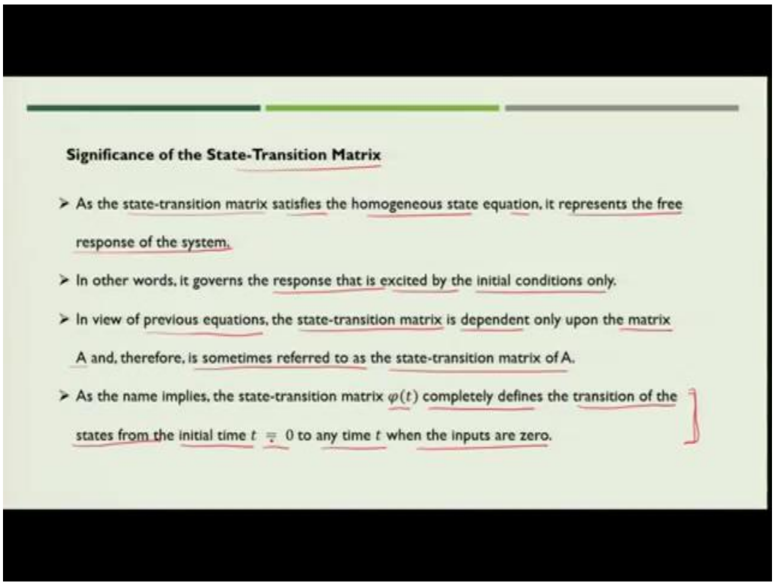

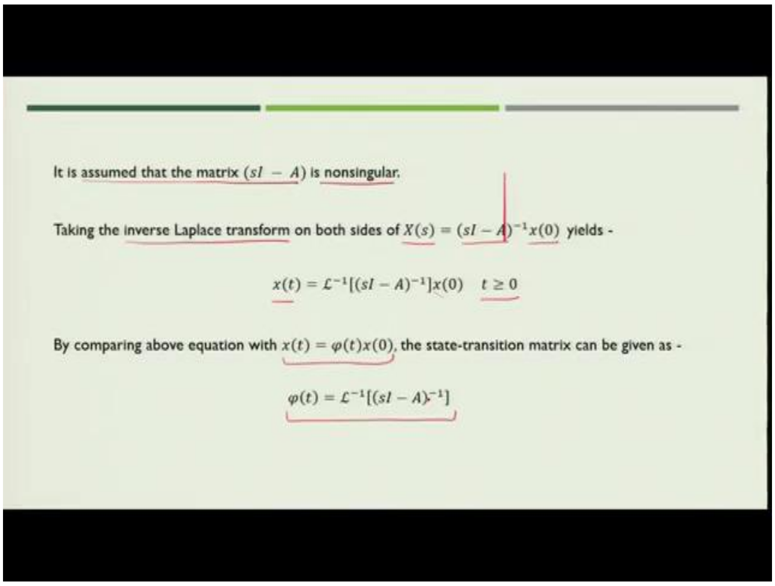

Now, what is the significance of state transition matrix? As the state transition matrix satisfies the homogeneous state equation it represent the free response of the system. Free response means the response that is excited by the initial conditions only keeping the forcing function that is $u(t) = 0$.

Now, view of previous equation which we have just see in the previous slide the state transition matrix is dependent only upon the matrix A. So, if you see this the state transition matrix is depending upon the matrix A that is why sometimes it is referred to as the state transition matrix of A.

Now, as the name applies, you see the state transition matrix, so the state transition matrix completely defines the transition of the state from the initial time t is equal to 0 to any time t when the inputs are zero. So, this is one of the most important property of the state transition matrix because with the help of this we can completely find the value of x at any time t given the initial state that is state at time t is equal to 0.

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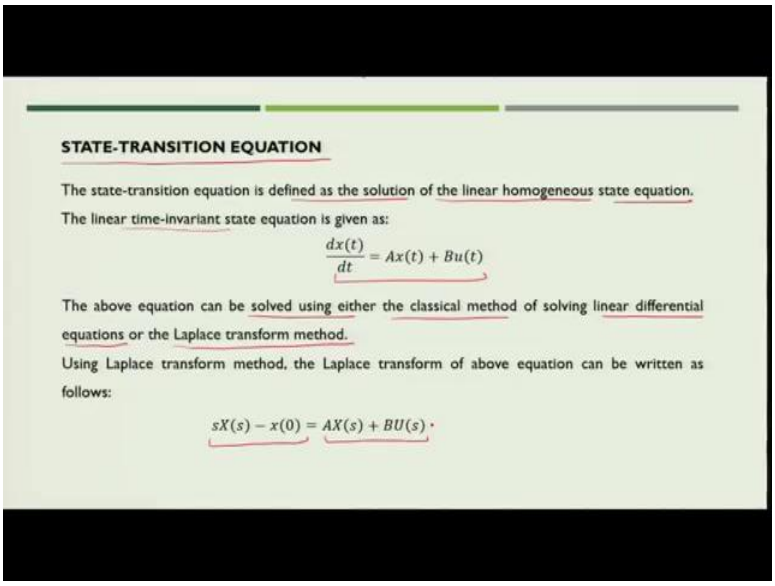

Now, let us talk about the state transition equation. So, the state transition equation is defined as the solution of linear homogeneous state equation. So, the equation which we have linear time invariant that is $\frac{dx(t)}{dt} = Ax(t) + Bu(t)$ this is the state equation we have. Now, the above equation can be solved using either the classical method of solving the linear differential equation or we can utilize the Laplace transform. So, if we use the Laplace transform, the above equation you can write in Laplace domain as

$$sX(s) - x(0) = AX(s) + BU(s)$$

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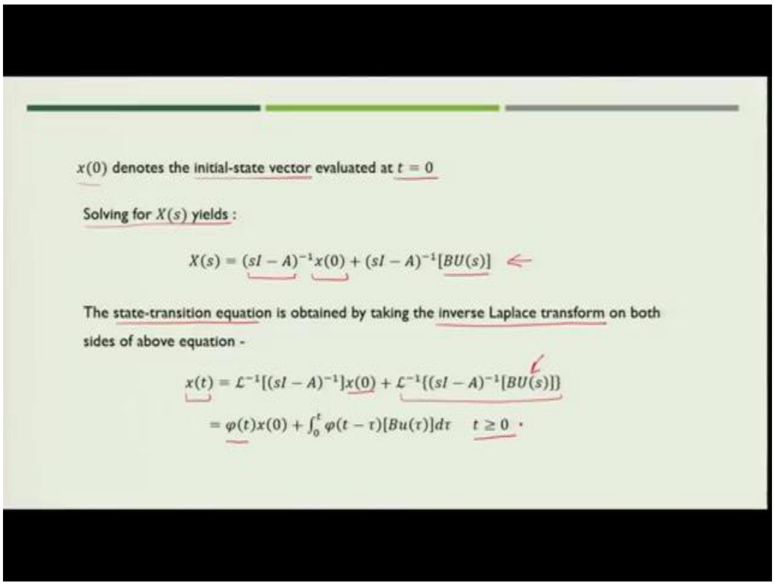

So, $x(0)$ denotes the initial-state vector evaluated at $t = 0$. Now, if you solve for Xs this equation what you get? You get,

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}[BU(s)]$$

Now, the state transition equation you can obtain from this equation with the help of taking the inverse Laplace transform on both sides. So, here now you get the value

$$x(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x(0) + \mathcal{L}^{-1}\{(sI - A)^{-1}[BU(s)]\}$$

$$= \varphi(t)x(0) + \int_0^t \varphi(t - \tau)[Bu(\tau)]d\tau \quad t \geq 0$$

So, this term you have got in addition to the term which we saw in case of homogeneous output, homogeneous response of the system because now you have added the forcing function in the system. So, what you can write now? You get this term nothing but the state transition matrix and then you get x at time t is equal to 0, so you get $\varphi(t)x(0)$ plus the inverse Laplace transform of this component. So, we will use the fundamental theorem of inverse Laplace transform, you can write this as $\int_0^t \varphi(t - \tau)[Bu(\tau)]d\tau \quad t \geq 0$.

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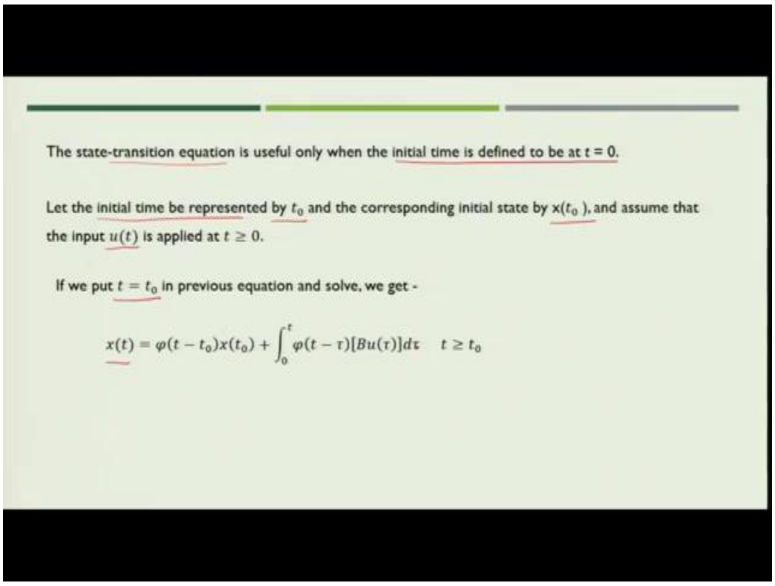

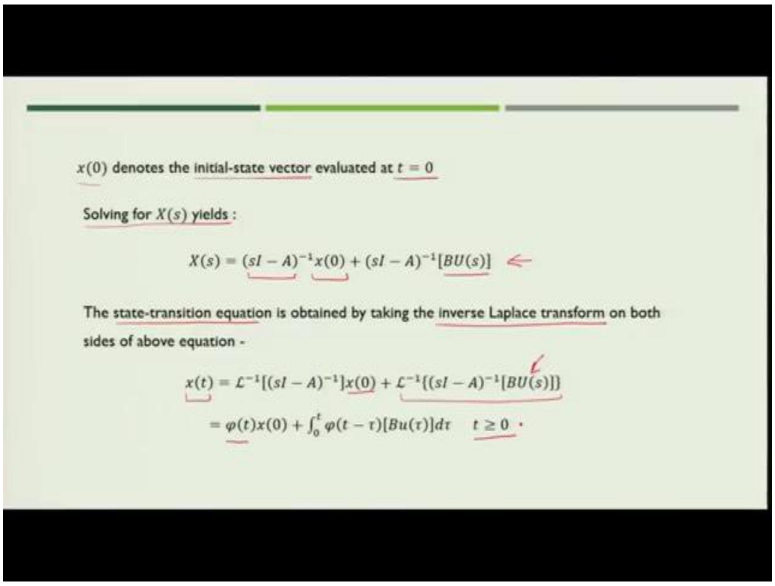

So, this is what you got the state transition equation that is the output $x(t)$ when you have forcing function present in the system. Now, this what we have defined when initial time is defined time t is equal to 0. Now, if initial time is say any other time t_0 the corresponding initial state will now become x(t_0) and we assume that there is an input that is $u(t)$ which is applied at any time t greater than 0. So, if you put $t = t_0$ in the previous equation and simplify you will get now,

$$x(t) = \varphi(t - t_0)x(t_0) + \int_0^t \varphi(t - \tau)[Bu(\tau)]d\tau \quad t \geq t_0$$

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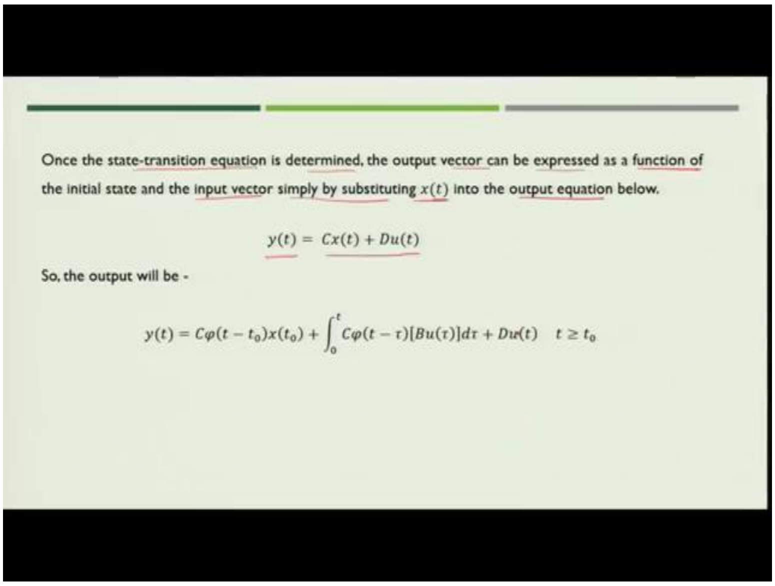

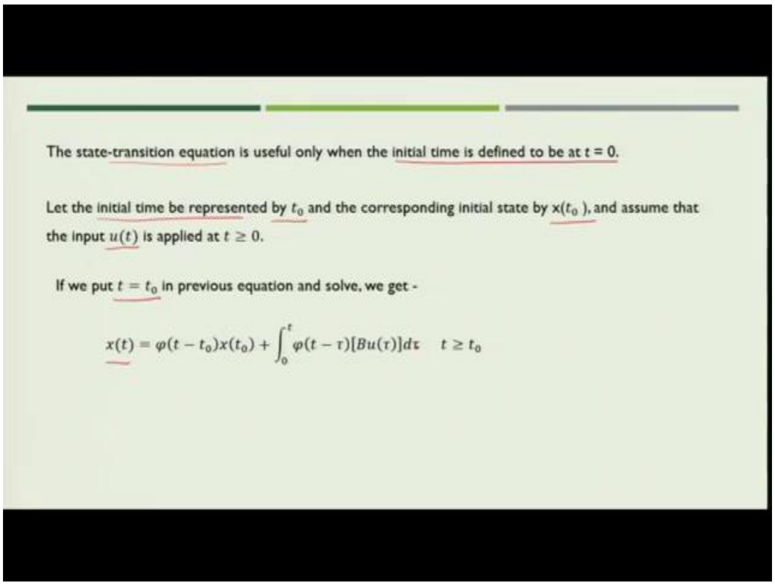

Now, once the state transition equation is determined, the output vector you can express as the function of initial state and the input vector simply by substituting $x(t)$ into the output equation. So, what is the output equation? We have output equation as $y(t) = Cx(t) + Du(t)$. The output which you will get is

$$y(t) = C\varphi(t-t_0)x(t_0) + \int_0^t C\varphi(t-\tau)[Bu(\tau)]d\tau + Du(t) \quad t \geq t_0$$

So, this is what we get the output equation for the system where the input is available.

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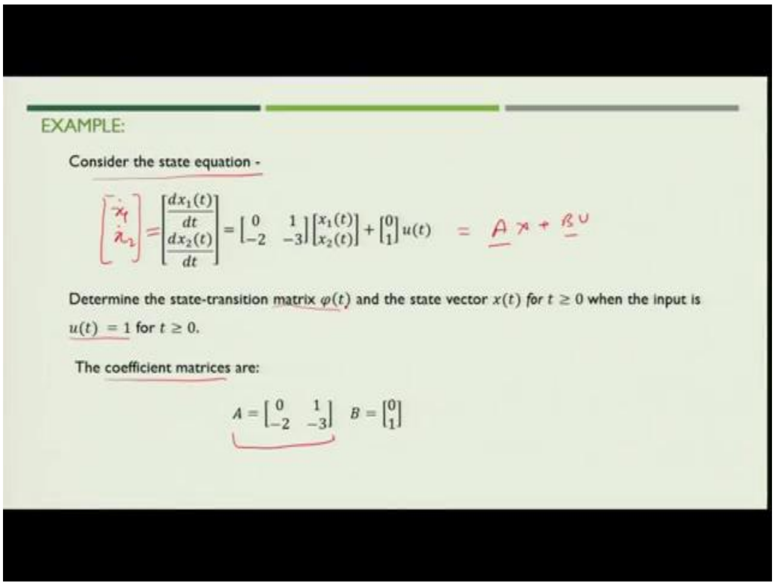

Now, let us take an example and try to find out the state transition matrix first. So, let us consider the state equation that is say

$$\begin{bmatrix}\frac{dx_1(t)}{dt}\\ \frac{dx_2(t)}{dt}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

So, when you compare this you get $A = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$ $B = \begin{bmatrix}0\\ 1\end{bmatrix}$.

So, now we have got coefficient matrices in the question it is given that $u(t) = 1$ for $t \geq 0$ that is why this is 1. So, now we need to find out first the state transition matrix.

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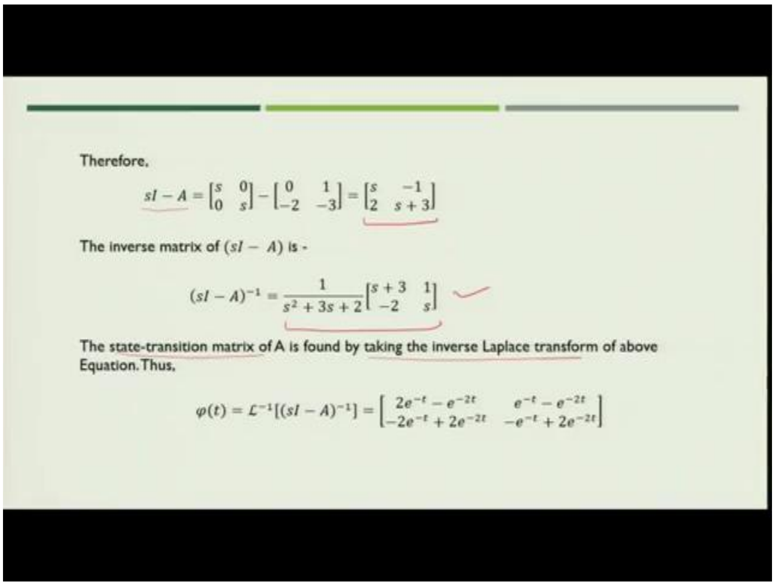

So, what will be the state transition matrix? First let us try to find out the value of $sI - A$, so

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

The inverse matrix of $(sI - A)$ is -

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Now, the state transition matrix you can find by taking the inverse Laplace transform of this. So, when you take the inverse Laplace transform you get,

$$\varphi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

So, this is simply you can get by taking the inverse Laplace transform of the matrix.

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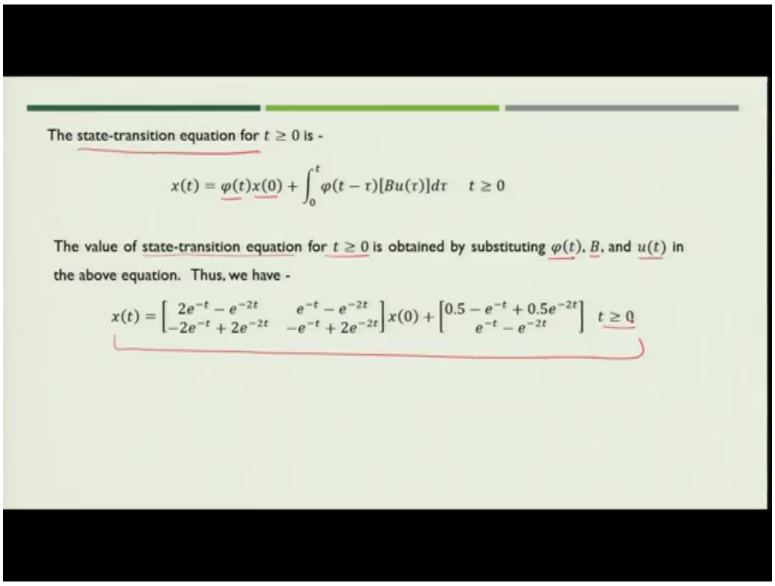

Now, the state transition equation for $t \geq 0$ is,

$$x(t) = \varphi(t)x(0) + \int_0^t \varphi(t - \tau)[Bu(\tau)]d\tau \qquad t \geq 0$$

Now, the value of state transition equation for $t \geq 0$ is obtained by you have found the value of $\varphi(t)$, B, and $u(t)$ is also given in the question.

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} x(0) + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \quad t \geq 0$$

So, this is the state transition equation which you will get for any time t greater than 0.

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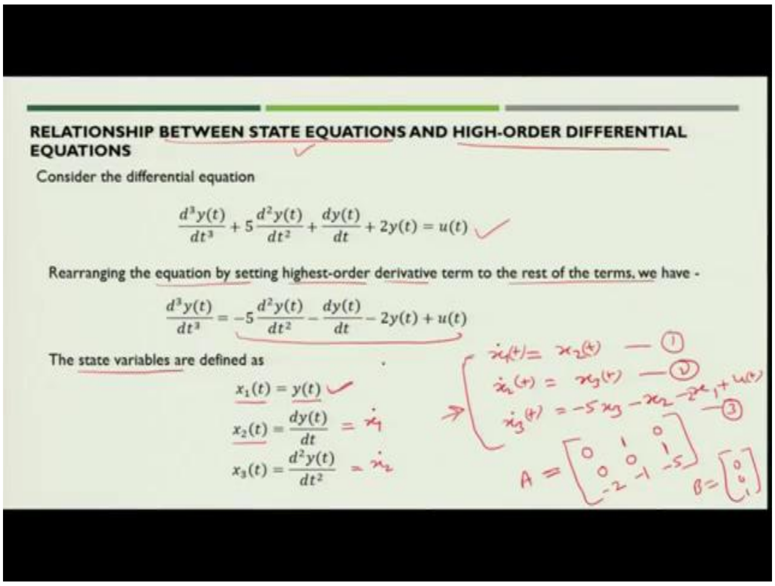

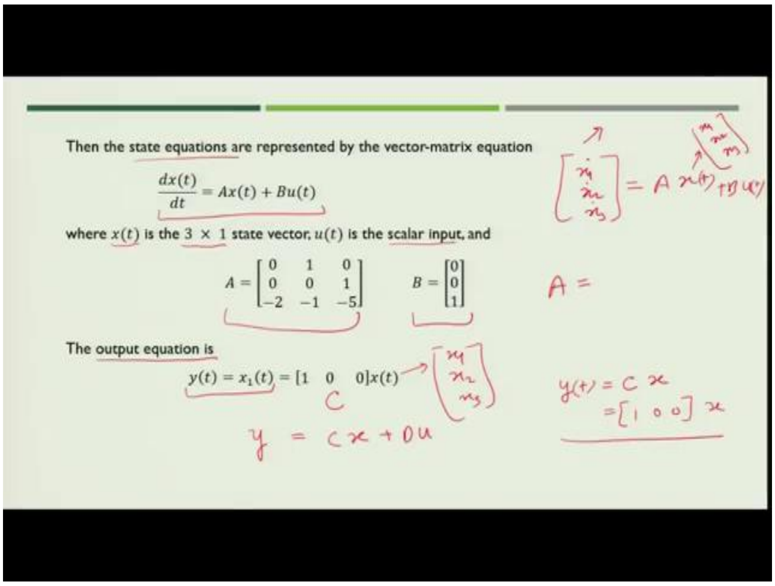

Now, let us see the relationship between state equations and the high order differential equations. So, let us consider one differential equation, so that is the third order equation you have $\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 2y(t) = u(t)$. So, this is the higher order differential equation is given we need to find this, we need to convert it into the state equations.

So, first thing what we must do? We have to first rearrange the equation by setting the highest order derivative term to the rest of the terms, so if you take this on the left so this will become

$$\frac{d^3y(t)}{dt^3} = -5\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) + u(t)$$

Now, let us define the state variables. So,

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt}$$

$$x_3(t) = \frac{d^2y(t)}{dt^2}$$

Then the state equations are represented by the vector-matrix equation as

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

Now, what is the value of A? If you see these three equations which you have derived you can simply find the value of A and B as,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The output equation is

$$y(t) = x_1(t) = [1 \quad 0 \quad 0]x(t)$$

So, with this we can close our today's discussion in which we discussed about the state transition matrix as well as the state transition equation. Thank you.