Basic Electric Circuits
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Module 11 - State Variable Analysis
Lecture 53 - State Diagram

Namaskar, so in last class we were discussing about the state variable and we also saw that if we have an nth order differential equation it can be converted into n first order equations. And, then we also discussed that these differential equations which are first order in nature can be considered as a state equation and we can define the various state variables. Accordingly, we can sum up the state variable into an equation called the state equation.

We also established that the equation you can write in the matrix form as $\dot{x}(t) = Ax(t) + Bu(t)$. We also discussed that the output equation we can write in the similar fashion. And, we can write $y(t) = Cx(t) + Du(t)$.

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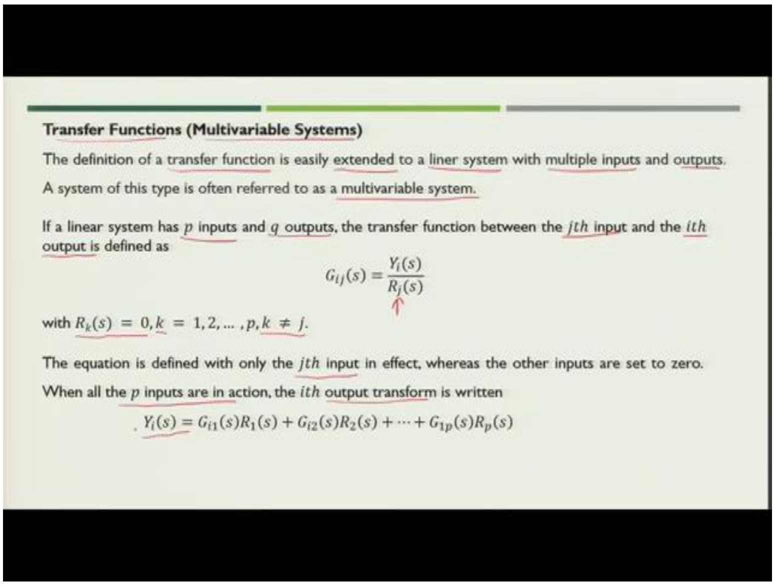

Today we will discuss about the state diagram and before going to the state diagram. Let us first understand the transfer function in case of multi variable system. That means if you have multiple variable in the system, how you will find out the transfer function? Now, the definition of transfer function is easily extended to linear system with multiple inputs and outputs. And this type of system where you have multiple inputs and outputs we call them as multivariable systems.

Now, if a linear system has p inputs and q outputs, the transfer function between jth input and the ith output can be defined as

$$G_{ij}(s) = \frac{Y_i(s)}{R_j(s)}$$

Now, this is applicable when the other input variables are 0. That means $Rk(s) = 0,\ k = 1,\ 2,\ \ldots\ ,\ p,\ k \neq j.$. So, we are considering one input at a time. Now, this is only for considering the jth input in finding out the transfer function. If you consider all the inputs say there are p inputs if you consider all the inputs. You can write the output, transformed output in s domain as

$$Y_i(s) = G_{i1}(s)R_1(s) + G_{i2}(s)R_2(s) + \cdots + G_{1p}(s)R_p(s)$$

Now, similarly for various outputs that is I ranging from 1 to q you can write these equations.

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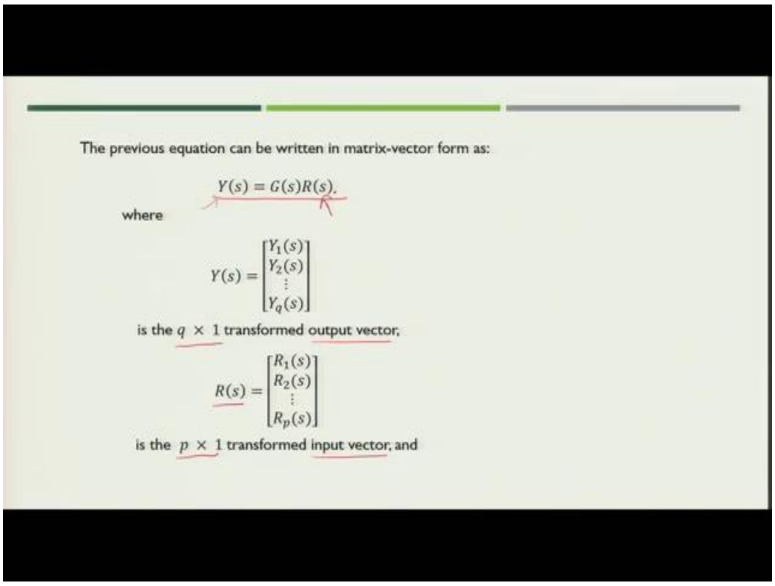

And, when you compile you can write matrix form as

$$Y(s) = G(s)R(s)$$

$$Y(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_q(s) \end{bmatrix}$$

is the $q \times 1$ transformed output vector,

$$R(s) = \begin{bmatrix} R_1(s) \\ R_2(s) \\ \vdots \\ R_p(s) \end{bmatrix}$$

is the $p \times 1$ transformed input vector, and

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{q1}(s) & G_{q2}(s) & \dots & G_{qp}(s) \end{bmatrix}$$

is the $q \times p$ transfer-function matrix.

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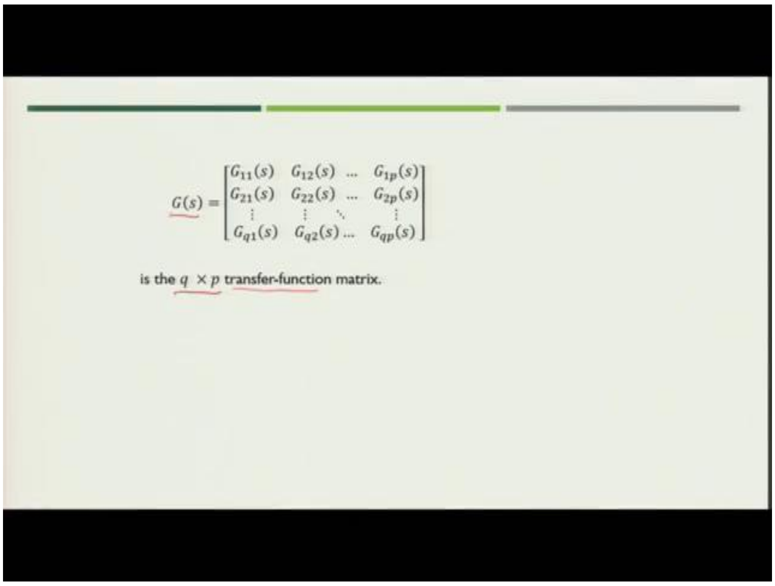

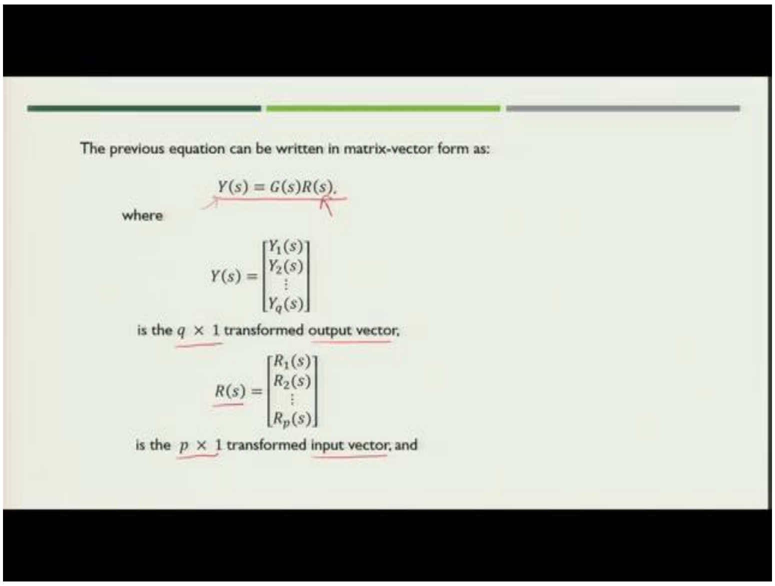

So, this is how you define the matrix which contains the multiple output and multiple input.

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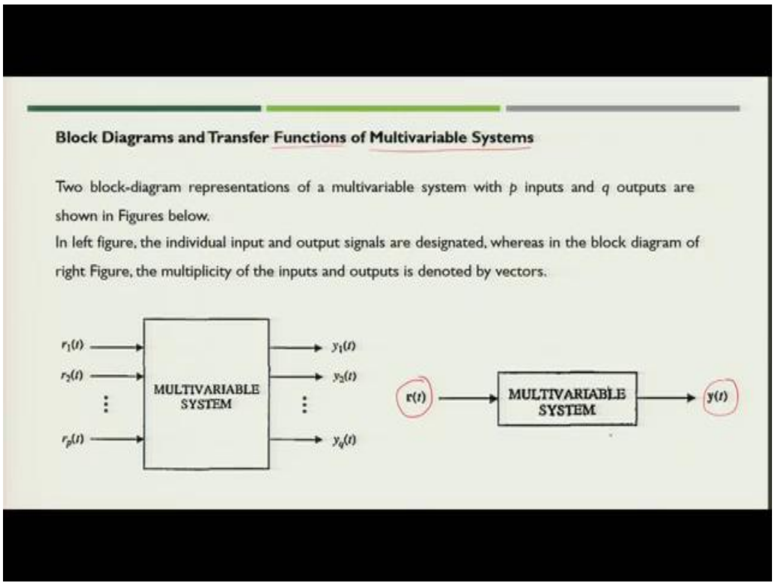

Now, let us see when you have the multiple input, output that means we have a multivariable system, how we will find out the transfer function? So, if you see this block diagram here you have one multivariable system. Where all multiple inputs starting from r_1 to r_p and outputs are from 1 to q. So, in short you can say that it has a vector R as an input and vector Y as an output.

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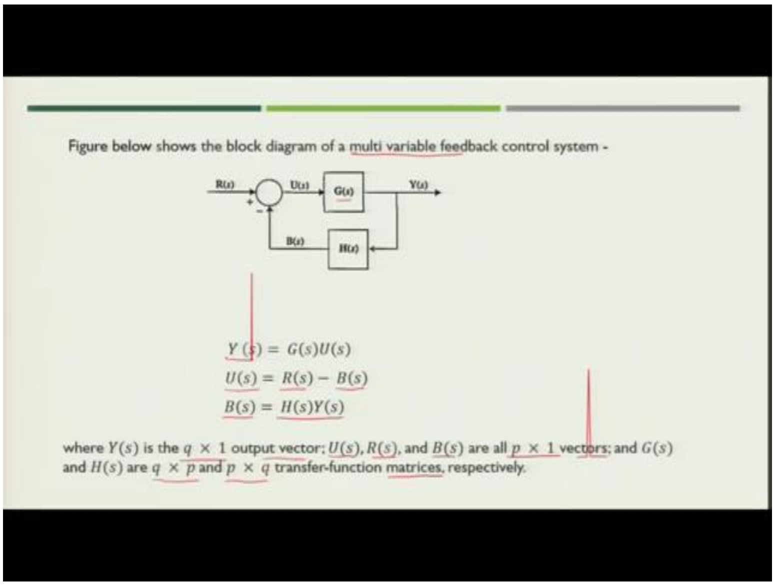

So, now if you are asked to find out the transfer function which is a having feedback loop. How will we define in case of multivariable input? So, this $R(s)$ is multivariable input so you will have R as a vector as an input and Y as an output containing the vector of q terms. Now, $G(s)$ is the transfer function so you ca see this can be considered as a transfer matrix which we just saw in this case. So, what we can write so in this case this is your the forward path gain matrix, this is your feedback path gain matrix.

Now, if you find out,

$$Y(s) = G(s)U(s)$$

$$U(s) = R(s) - B(s)$$

$$B(s) = H(s)Y(s)$$

So, as we discussed, $Y(s)$ is the $q \times 1$ output vector; $U(s)$, $R(s)$, and $B(s)$ are all $p \times 1$ vectors; and $G(s)$ and $H(s)$ are $q \times p$ and $p \times q$ transfer-function matrices, respectively.

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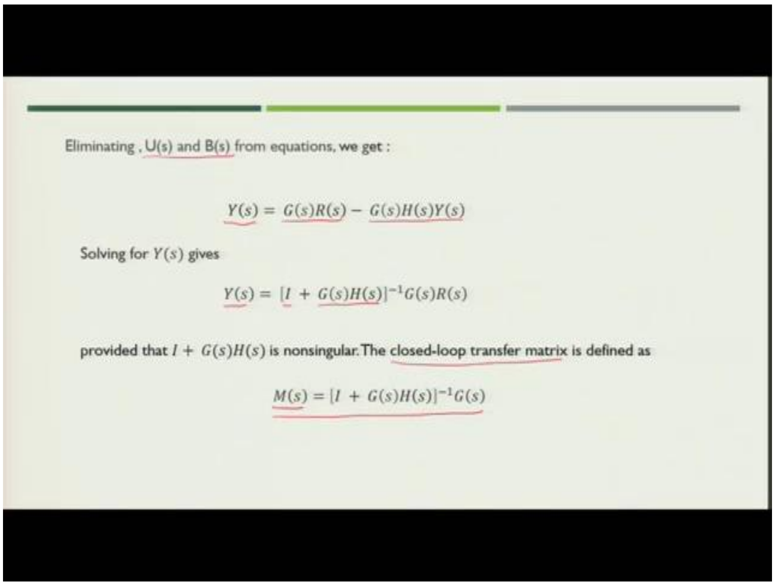

Now, the objective is that we need to eliminate the intermediate variables that is U(s) and B(s). So, when we eliminate U(s) and B(s) from these three set of equations we can write,

$$Y(s) = G(s)R(s) - G(s)H(s)Y(s)$$

Now, when you recompile,

$$Y(s) = [I + G(s)H(s)]^{-1}G(s)R(s)$$

Now, the transfer function is nothing but

So, if you say $\frac{Y(s)}{R(s)} = M(s)$,

$$M(s) = [I + G(s)H(s)]^{-1}G(s)$$

So, this how you will get the transfer function matrix in case of multivariable system.

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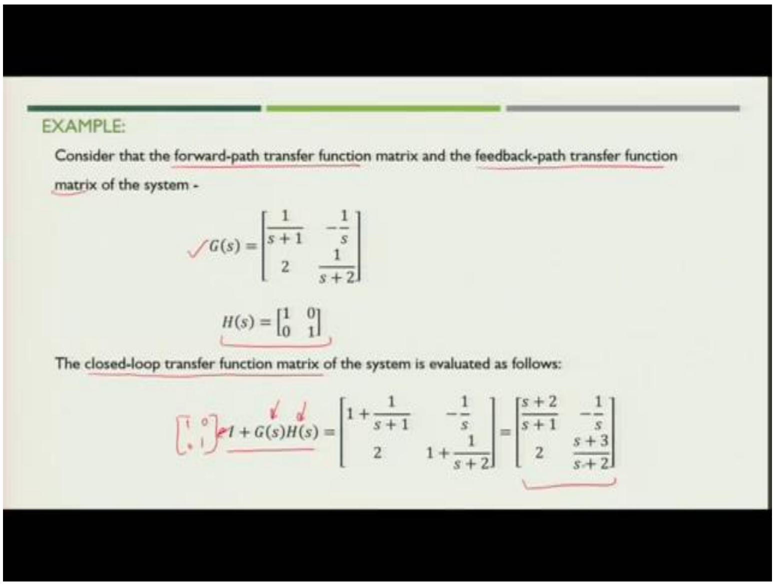

Now, let us take one example suppose forward path transfer function matrix is $G(s)$ given and the feedback path transfer function matrix is $H(s)$ it is given. Now, we need to find the closed loop transfer function matrix so first we will try to find out the value of $I + G(s)H(s)$. So

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix}$$

$$H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The closed-loop transfer function matrix of the system is evaluated as follows:

$$I + G(s)H(s) = \begin{bmatrix} 1 + \frac{1}{s+1} & -\frac{1}{s} \\ 2 & 1 + \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s+1} & -\frac{1}{s} \\ 2 & \frac{s+3}{s+2} \end{bmatrix}$$

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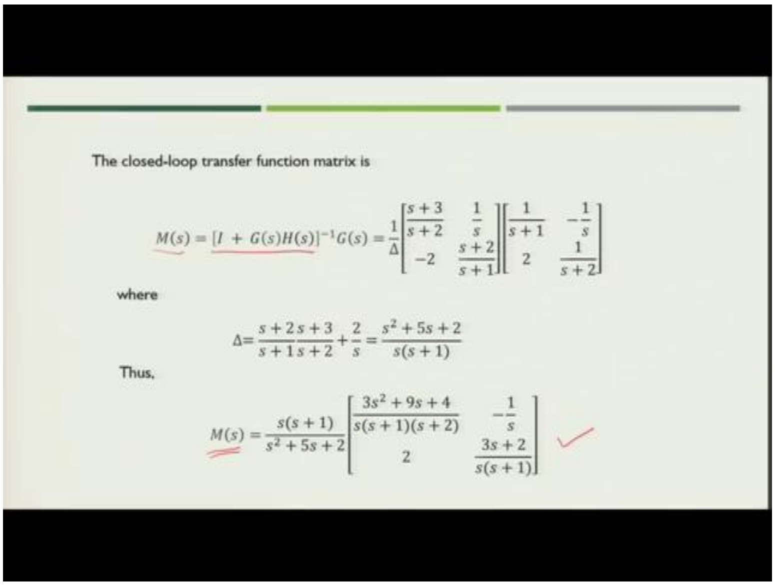

Now, next is we need to find out transfer function matrix, so in that case when we are finding the closed loop transfer function matrix.

$$M(s) = [I + G(s)H(s)]^{-1}G(s) = \frac{1}{\Delta}\begin{bmatrix} \frac{s+3}{s+2} & \frac{1}{s} \\ -2 & \frac{s+2}{s+1} \end{bmatrix}\begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix}$$

where,

$$\Delta = \frac{s+2}{s+1}\frac{s+3}{s+2} + \frac{2}{s} = \frac{s^2+5s+2}{s(s+1)}$$

Thus,

$$M(s) = \frac{s(s+1)}{s^2+5s+2}\begin{bmatrix} \frac{3s^2+9s+4}{s(s+1)(s+2)} & -\frac{1}{s} \\ 2 & \frac{3s+2}{s(s+1)} \end{bmatrix}$$

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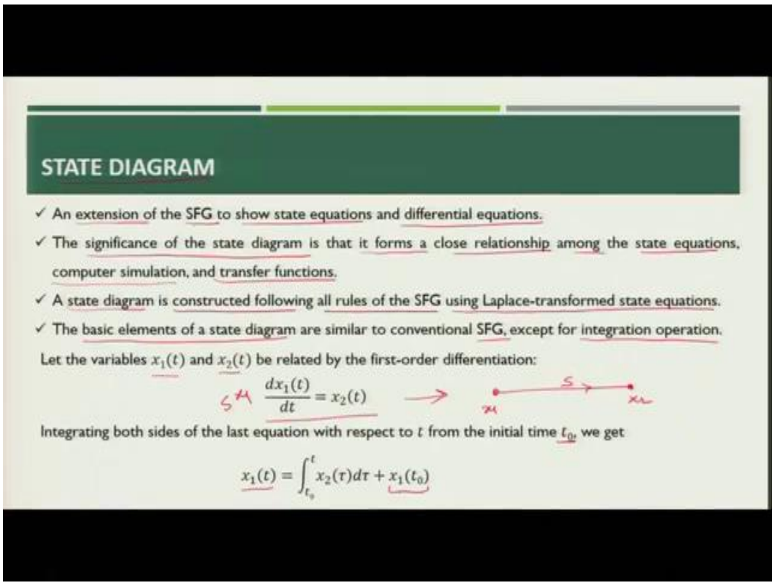

Now, let us talk about the state diagram so what is the state diagram? State diagram is nothing but the extension of the signal flow graph which we discussed in previous lectures. And this shows the state equations so which we discussed and the differential equation. Because, when you talk about the first order differential equation that is nothing but the state equation we are talking about. So, state equation and differential equation can be shown in the extension of signal flow graph that is nothing but the state diagram

Now the significance of the state diagram is that it forms a close relationship among the state equations. The computer simulation and transfer function, how when we see the discussed the state diagram in detail in the in this session. Basically, the state diagram is constructed following all rules of signal flow graph which we have seen till now using Laplace transformed state equation. Now, the difference between signal flow graph and state diagram is the integration operation.

Which we have little bit difference as compared to signal flow graph and the state diagram. So, in this case when we talk about the state diagram let us first see how the integration of operation you will define in the state diagram. Let us see there are two variables $x1(t)$ and $x2(t)$ and this are related with each other as $\frac{dx_1(t)}{dt} = x_2(t)$. So, if you represent them in the state diagram you will say that this is basically x2, this is x1.

We have to first take the integration at on both sides we can write

$$x_1(t) = \int_{t0}^{t} x_2(\tau)d\tau + x_1(t0)$$

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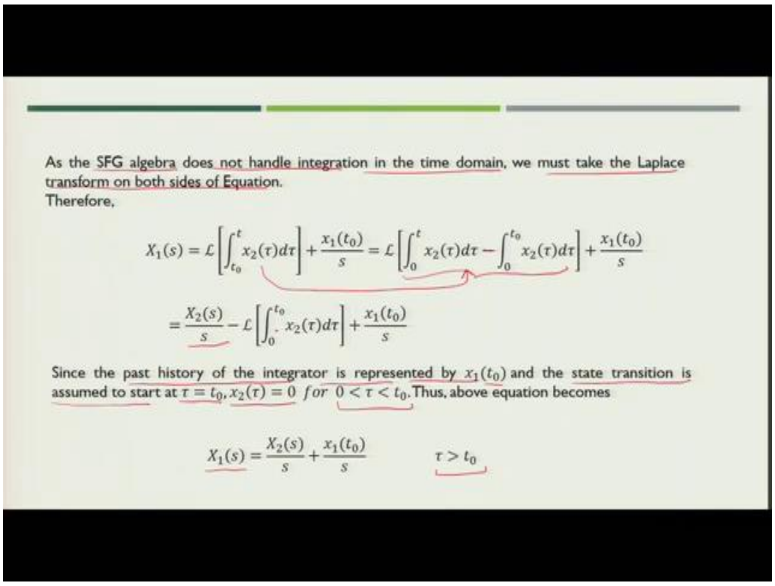

Now, if you take the Laplace transform, so as we know that signal flow graph algebra does handle integration in time domain. So, what we have to do? We have to take the Laplace transform on both sides. So, we can write,

$$X_1(s) = \mathcal{L}\left[\int_{t_0}^{t} x_2(\tau)d\tau\right] + \frac{x_1(t_0)}{s} = \mathcal{L}\left[\int_{0}^{t} x_2(\tau)d\tau - \int_{0}^{t_0} x_2(\tau)d\tau\right] + \frac{x_1(t_0)}{s}$$

So, this can be equivalently represented with the help of the integration which we have broken into two parts. Now, if you see the Laplace transform of this you can simply write it as,

$$X_1(s) = \frac{X_2(s)}{s} - \mathcal{L}\left[\int_{0}^{t_0} x_2(\tau)d\tau\right] + \frac{x_1(t_0)}{s}$$

And the state transition is assumed to start at tau is equal to t naught means we are transiting at $\tau = t_0$ that means $x_2(\tau) = 0 \;\; for \;\; 0 < \tau < t_0$. It means that this section you can replace by 0. So, finally what you get,

$$X_1(s) = \frac{X_2(s)}{s} + \frac{x_1(t_0)}{s} \qquad\qquad \tau > t_0$$

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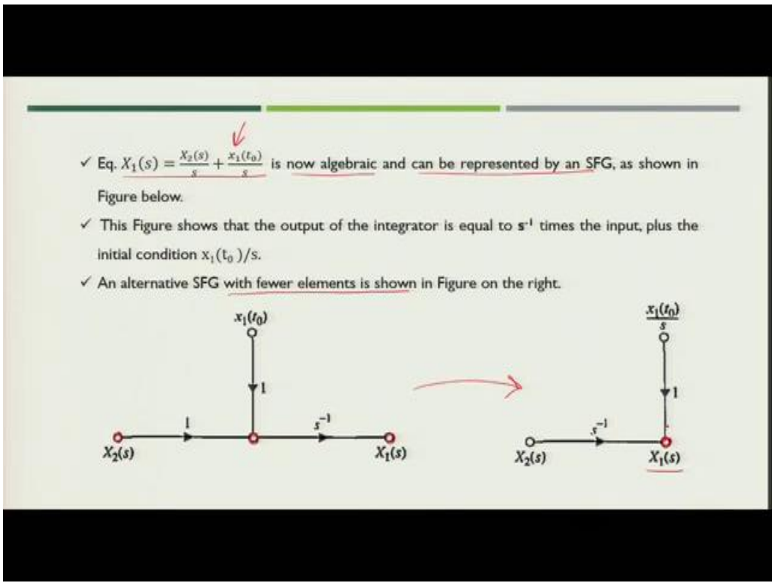

Now, the equation that is we have just found that is $X_1(s) = \frac{X_2(s)}{s} + \frac{x_1(t_0)}{s}$ is now algebraic and can be represented by the signal flow graph. How we will represent? If we see this figure you have two variables one is $X_2(s)$ and another is $X_1(s)$. And we are summing up here that is summing block. So, at this point this will be $X_2(s) + x_1(t_0)$. And then it is multiplied by $1/s$. So, finally you get $\frac{X_2(s)}{s} + \frac{x_1(t_0)}{s}$ which is nothing but the equation which we just derived in case of the integration which we just performed.

Alternatively, you can also represent it in less number of connections. So, say if this block is representing $X_1(s)$ so $1/s$ you can put in both cases. So, here when you sum up you will get $\frac{X_2(s)}{s} + \frac{x_1(t_0)}{s}$. So, both figures are representing the same equation. So, the advantage of this figure is that you need fewer element to show the equation which we just derived.

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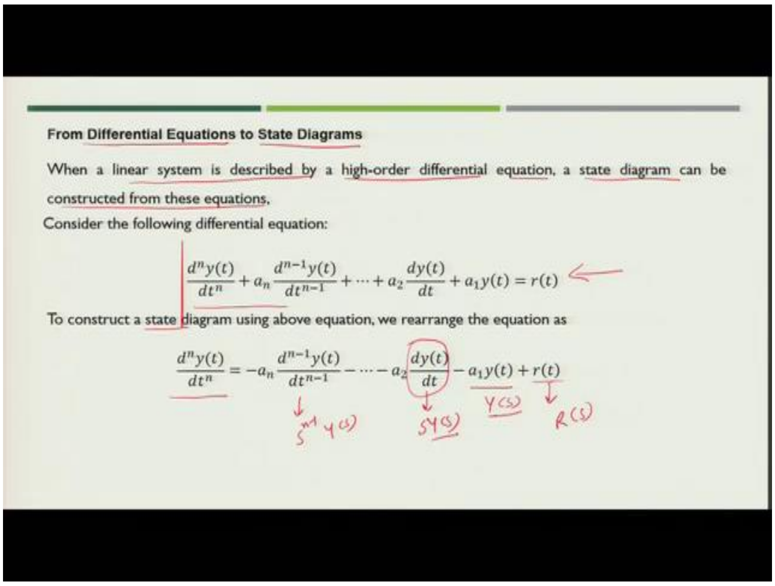

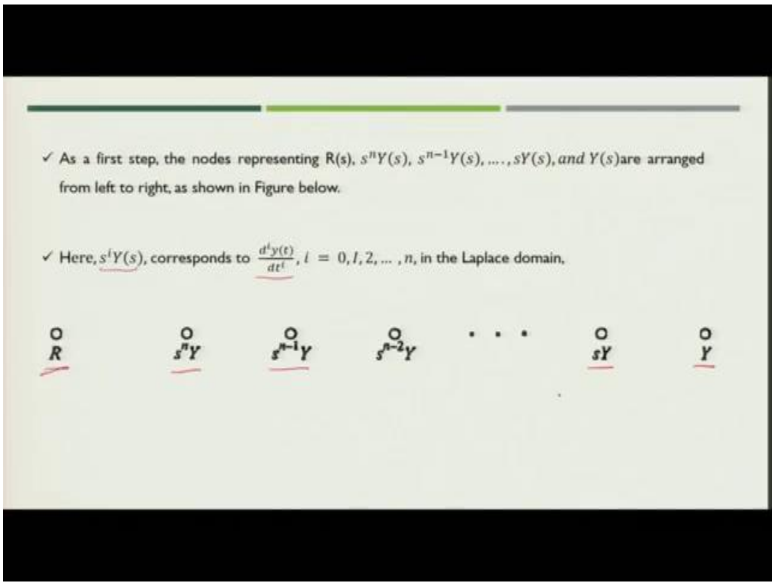

Now, let us see how you will convert the differential equations into state diagrams. So, when the linear system is described by higher order differential equations the state diagram can be constructed from these equations. How we will construct? Let us consider one differential equation. Which is of nth order differential equation so that is

$$\frac{d^n y(t)}{dt^n} + a_n \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_2 \frac{dy(t)}{dt} + a_1 y(t) = r(t)$$

So, this is suppose you have got this differential equation and you have to convert this differential equation into state diagram. We have to construct, we have to rearrange this equation as shown in this equation. That is we will write,

$$\frac{d^n y(t)}{dt^n} = -a_n \frac{d^{n-1} y(t)}{dt^{n-1}} - \cdots - a_2 \frac{dy(t)}{dt} - a_1 y(t) + r(t)$$

So, what we have done? We have kept this as left component and rest we have shifted to right side. So, we got this equation from the previous equation. So, you know that individual components you can represent as their Laplace transform. So, what we will do we will this particular component you can replace with the Laplace transform of Rt. So, we have now around total n element for one as the out as the forcing function on the right side. So, these are generally considered as a node. So, when you represent them as a node how you will show them in the figure.

So, we will see y so we have put 1 node as Y then next is s into Ys so we have put sY and so on. So, we have got up to sn minus 1Y and then snY for the last component. And, then we add R so, we have put all the elements which we have seen in the equation as nodes. So, what is siYs so the ith order of the differential component that is diY by dti is nothing but siY. So, for nth order component we written snY. Now, we have defined the nodes next we use this equation to define the links.

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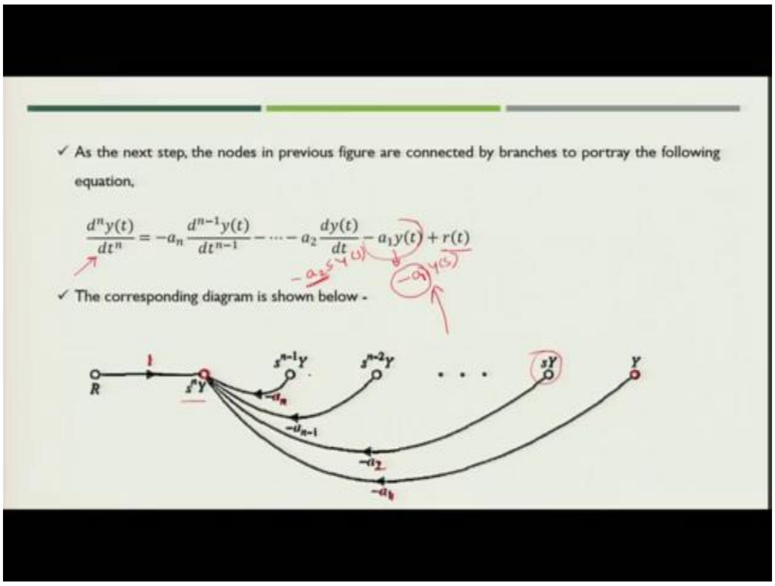

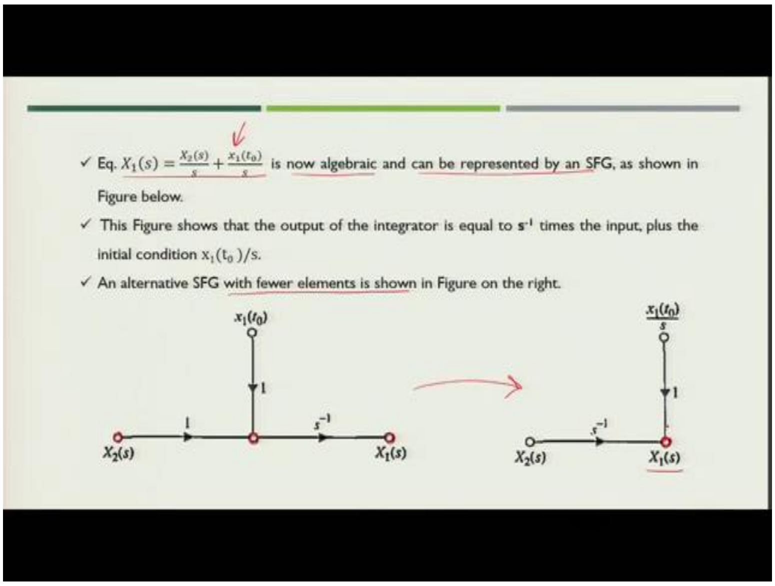

So, if you see the equation which we have just rearranged so what we can now see we can see the connections. So, if you see this particular node which is nothing but representing this particular component that is $\frac{d^n y(t)}{dt^n}$. So, that means this will be summation of all other components. So, rt does not have any component coefficient there so you will connect r with s to the power ny with only unit gain. Now, let us see this particular component so in s domain you can write as minus a1ys.

So, what will happen? You coefficient for this particular component is minus a1, so, what you will do? You will connect the y with s to the power ny with minus a1. Similarly, for this you can write minus a2 into sys so sys so you see here and this is connected with s to the power nys with the coefficient minus a2. So, you got minus a2 as the connection and similarly you connect all of them. So, what you get finally that is s to the power ny is nothing but r minus an s minus s to the power n minus y minus an minus 1 s to the power n minus 2y and so on up to minus a2sy minus a1y, so this particular figure represents the differential equation which we just written.

Now, the important thing which is still is left is that how we will co relate the nodes which are connect, which are adjacent to each other. How we will connect them? So just previous case when we discussed, we discuss that how x1 and x2 related. So, this knowledge we just discussed, we will utilized in connecting the components.

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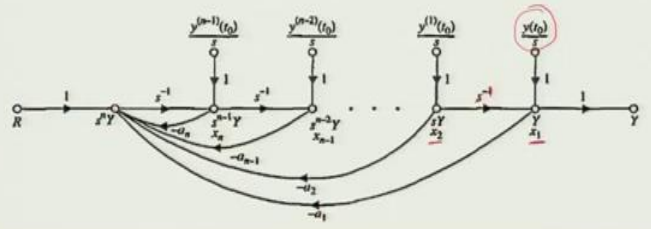
- Finally, the integrator branches with gains of s^{-1} are inserted, and the initial conditions are added to the outputs of the integrators.
- The complete state diagram is drawn as shown in Figure below.
- The outputs of the integrators are defined as the state variables, $x_1, x_2, \ldots, x_n$.
- This is usually the natural choice of state variables once the state diagram is drawn.



As the SFG algebra does not handle integration in the time domain, we must take the Laplace transform on both sides of Equation.
Therefore,

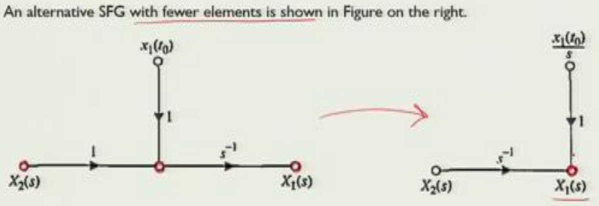
$$X_1(s) = \mathcal{L}\left[\int_{t_0}^{t} x_2(\tau)d\tau\right] + \frac{x_1(t_0)}{s} = \mathcal{L}\left[\int_{0}^{t} x_2(\tau)d\tau - \int_{0}^{t_0} x_2(\tau)d\tau\right] + \frac{x_1(t_0)}{s}$$

$$= \frac{X_2(s)}{s} - \mathcal{L}\left[\int_{0}^{t_0} x_2(\tau)d\tau\right] + \frac{x_1(t_0)}{s}$$

Since the past history of the integrator is represented by $x_1(t_0)$ and the state transition is assumed to start at $\tau = t_0$, $x_2(\tau) = 0$ for $0 < \tau < t_0$. Thus, above equation becomes

$$X_1(s) = \frac{X_2(s)}{s} + \frac{x_1(t_0)}{s} \qquad \tau > t_0$$

- Eq. $X_1(s) = \frac{X_2(s)}{s} + \frac{x_1(t_0)}{s}$ is now algebraic and can be represented by an SFG, as shown in Figure below.
- This Figure shows that the output of the integrator is equal to s^{-1} times the input, plus the initial condition $x_1(t_0)/s$.
- An alternative SFG with fewer elements is shown in Figure on the right.



So, if you see the component at this side x2 is connected with x1 how you are connecting? You are connecting with sy is connected with y with s to the power n minus1 plus the initial condition for y that is y t naught by s. So, this is what you got from the equation which we just derived. So, because if you see x1, x1 is nothing but x2 by s into x1 t naught by s in our case x1s is y x2s is sy. So, what we will do? We will use that information and connect the adjacent nodes. So, in this way you can connect all the adjacent nodes with 1 by s and put the initial condition.

Because, when you divide the nth order differential equation into n first order equations. You will have the equation as we discussed in the case. So, you will get these equations in Laplace domain where you have initial conditions for every state variable. So, we will utilize that information and we will put the initial conditions of all the state variable and connect them with 1 by s. So, this is the final state diagram which you will get, and this will represent this particular differential equation.

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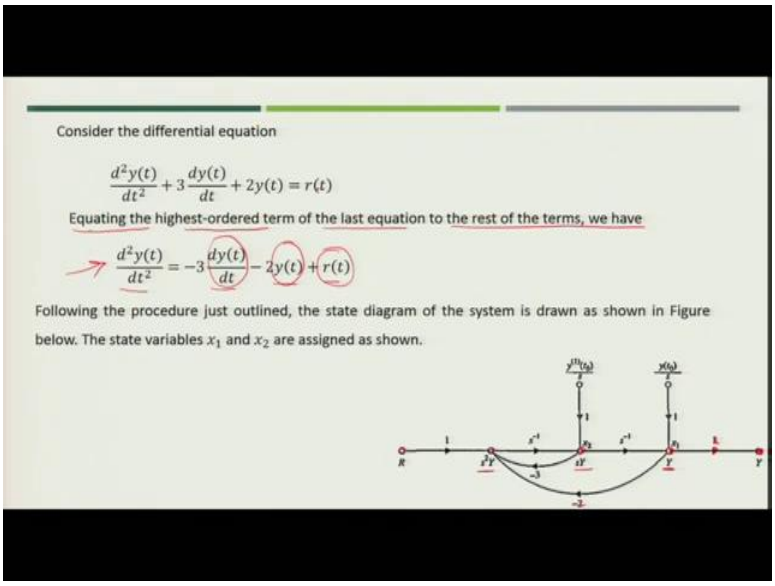

Now, let us take one example so that you can better understand let us see there is one differential equation that is $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = r(t)$. Now, what we will do? We will equate the highest order term of the last equation to the rest of the terms. So, what we can write? We can write $\frac{d^2y(t)}{dt^2} = -3\frac{dy(t)}{dt} - 2y(t) + r(t)$. So, here we will create 1 2 3 4 nodes now this node is basically being extended to give you better clarity.

So, you can add y with unit gain so that you can have the clear understanding from the state diagram. So, we have added these four nodes one is R, second is $y(t)$ so you will replace with

Y, third is $\frac{dy(t)}{dt}$ so will replace with sY and $\frac{d^2y(t)}{dt^2}$ you will replace with s^2Y. Now, what procedure we followed previously we will just use that procedure. So, the y is adding 2s square y with minus 2 as coefficient so we have minus 2 as a coefficient dy by dt you multiply with minus 3 2s square y.

So, we get these two connections and r is having 1 as coefficient so r is connected to s square y with 1 as the gain. So, now what we will do? We will connect these two nodes with each other with the help of the integration function which we discussed. So, we will connect with 1 by s square will connected to sy with 1 by s. And, plus the initial condition you have for y1 t naught by s and then for y we have yt naught by s as a initial condition. So, finally you get the state diagram for the differential equation which we just shown in the slide.

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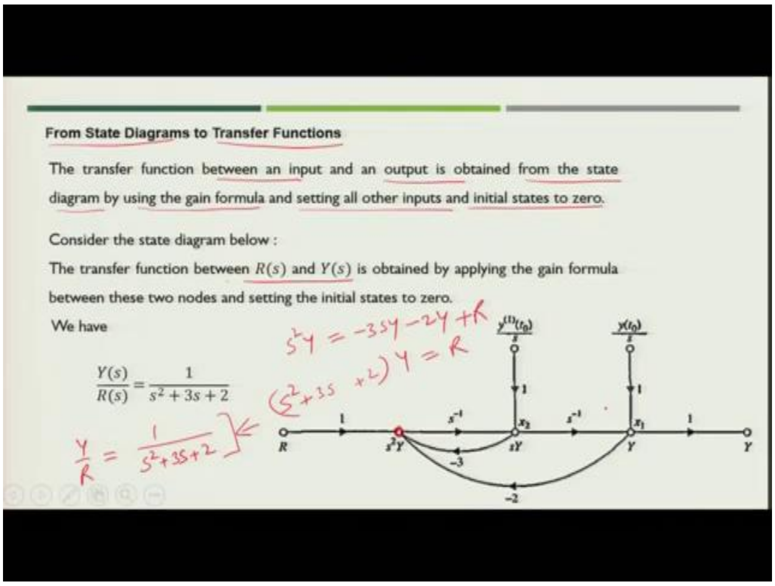

Now, the next question comes when you are asked to find the transfer function from the state diagram. So, what you will do? So transfer function between input and output is obtained from the state diagram by using the gain formula and setting all other inputs and initial state to 0. So, what we will do? We will set the initial states to 0 and then we will equate input to output. So, R and Ys is obtained by using the gain formula so what we will do?

Transfer function is

$$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + 3s + 2}$$

So, with this we can close our today's session in which we discussed mainly the state diagram. Now, we will see next how we will use the state diagram to find out the state equations, thank you.