Basic Electric Circuits
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Module-9
Sinusoidal Steady-State Analysis 1
Lecture-45
Energy in Coupled Circuit and Ideal Transformer

Namaskar, so in last session we discussed about the magnetically coupled circuit. So, we will continue our decision today and we will talk about energy stored in magnetically coupled circuits. And then also we will discuss about the ideal transformer.

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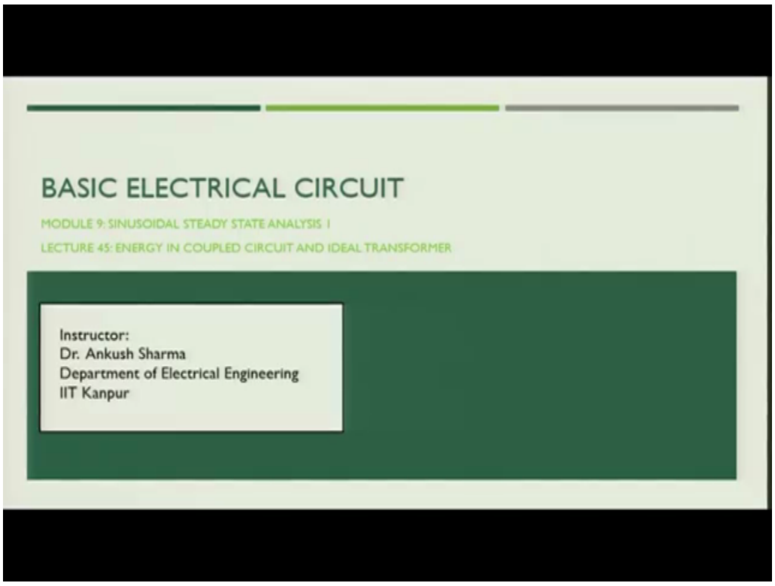

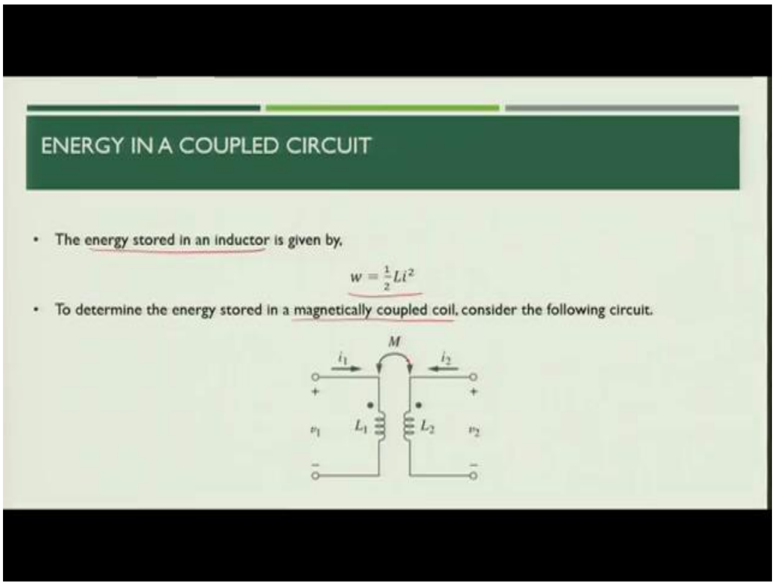

So let us start the discussion of today's session. The energy stored in inductor as we know can be given by,

$$w = \frac{1}{2}Li^2$$

Now, to determine the energy stored in magnetically coupled coil, let us consider the following circuit. Here i_1 and i_2 are the currents on left and right coils, v_1 is the voltage across first coil and v_2 is voltage across second coil. They have inductances as L_1 and L_2 and M is the mutual inductant.

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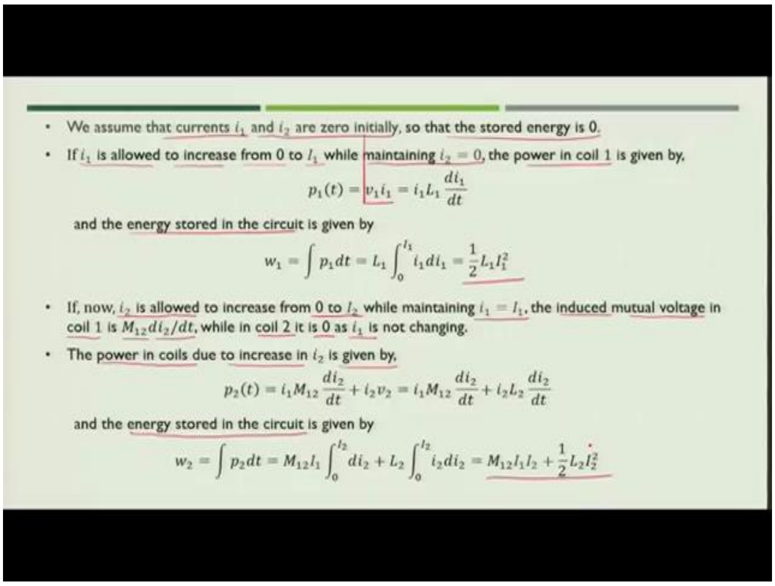

Now let us assume that first the current i_1 and i_2 are initially zero. That means the total energy stored in the circuit is equal to 0. Now i_1 is allowed to increase from 0 to I_1 that is one arbitrary value we are considering. While maintaining current $i_2 = 0$, in that case, the power in coil one can be given by,

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt}$$

and the energy stored in the circuit is given by

$$w_1 = \int p_1 dt = L_1 \int_0^{I_1} i_1 dt = \frac{1}{2} L_1 I_1^2$$

Now if current i_2 is allowed to increase from 0 to I_2 that is again an arbitrary value we are considering. While maintaining the currents same in the first coil, so in that case the induced mutual voltage in first coil will be $M_{12}\frac{di_2}{dt}$.

While in coil 2, it will be zero because there is no change in I 1. So, $\frac{di_1}{dt}$ will be zero when we are talking about the mutual induced voltage in coil two. For coil two, we will have the mutual voltage and a mutual induced voltage $M_{12}\frac{di_2}{dt}$. Therefore, the power in coils due to increase in current is given by,

$$p_2(t) = i_1 M_{12}\frac{di_2}{dt} + i_2 v_2 = i_1 M_{12}\frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt}$$

and the energy stored in the circuit is given by

$$w_2 = \int p_2 dt = M_{12} I_1 \int_0^{I_1} di_2 + L_2 \int_0^{I_2} i_2 dt = M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

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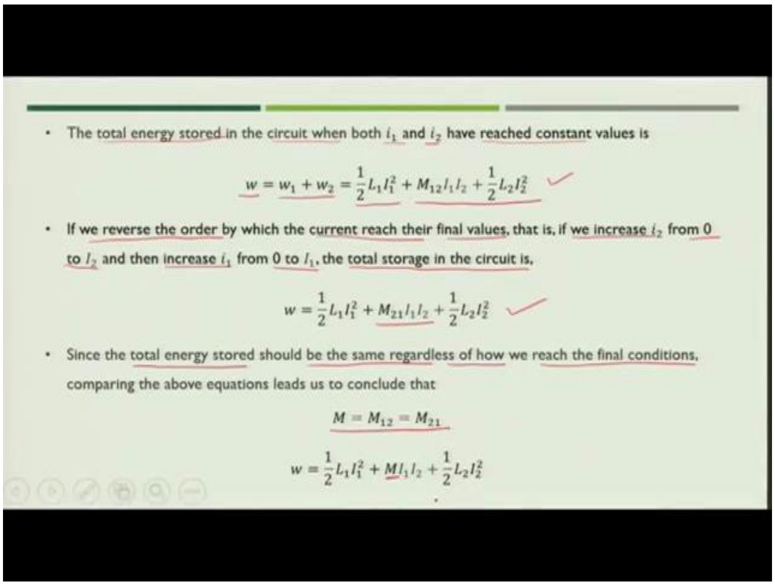

Now in that case, the total energy stored in the circuit will be considered when both i_1 and i_2 have reached their constant value. So total energy stored in the circuit will be,

$$w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

Now if we reverse the order by which the current reach their final values. That means first we will increase i_2 from 0 to I_2 and then increase i_1 from 0 to I_1, In that case, the total energy stored in the circuit can be given as,

$$w = \frac{1}{2}L_1 I_1^2 + M_{21} I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

Now since the total energy stored should be same regardless of how we reach the final conditions. In that case, if you compare these two equations you can simply say that $M = M_{12} = M_{21}$. Total energy stored in the circuit we can write as,

$$w = \frac{1}{2}L_1 I_1^2 + M I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

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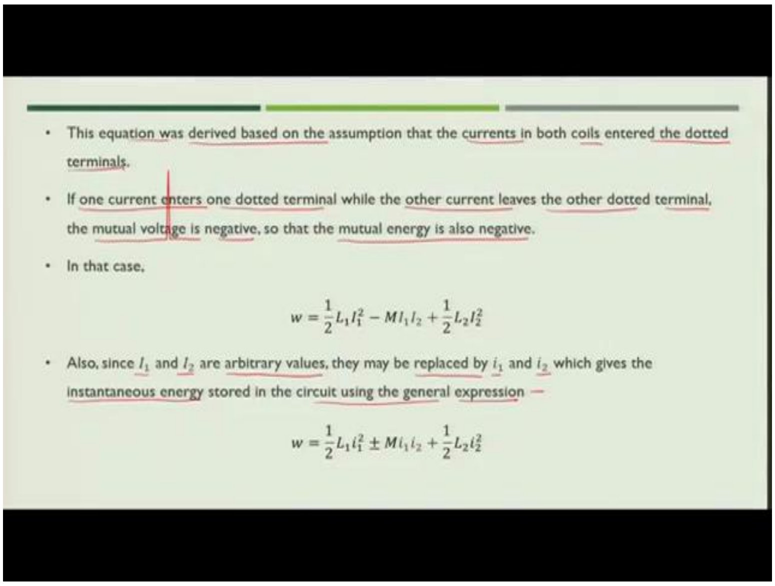

Now this equation we derived by assuming that the currents in both coils are entering the dotted terminal. That means the mutual voltage which induced is in either of the coil will be positive. So in case, if one current enters the dotted terminal while the other leaves the other dotted terminal, the mutual voltage now become negative. In that case, the mutual energy will also be negative, so the total energy stored in the circuit will become,

$$w = \frac{1}{2}L_1 I_1^2 - M I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

Now we have considered I_1 and I_2 are some arbitrary values, so we can replace with any other value like we replace with the generic i_1 *and* i_2 equations the symbols. The instantaneous energy which will be stored in the circuit can be given by some general expression and the general expression is

$$w = \frac{1}{2}L_1 i_1^2 \pm M i_1 i_2 + \frac{1}{2}L_2 i_2^2$$

So, this plus minus will be depending upon how current is entering and leaving the dots.

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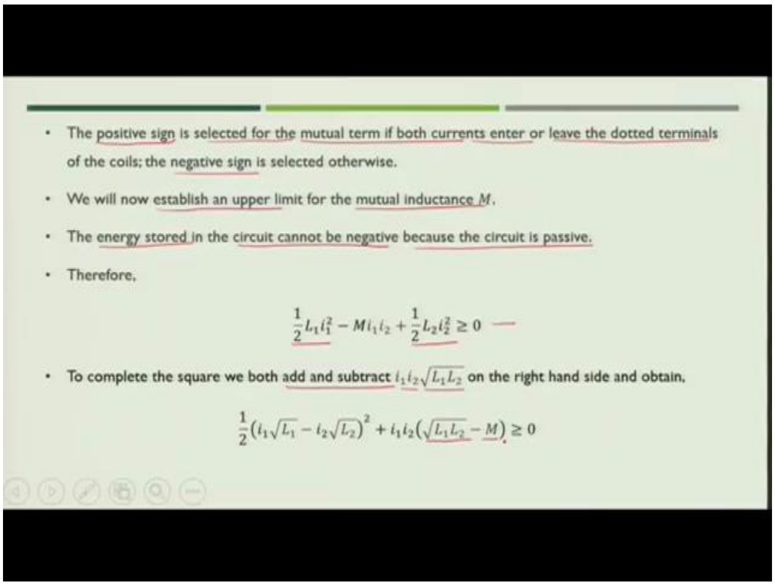

Now the positive sign we will select when mutual term in the mutual the positive sign we select for the mutual term. Both currents enter or leave the dotted terminals otherwise we will consider as negative. Now let us try to establish the upper limit for the mutual inductance. The as we know the energy stored in the circuit cannot be negative because we are talking about the circuit which is passive in nature. So we can conveniently say,

$$\frac{1}{2}L_1 i_1^2 - M i_1 i_2 + \frac{1}{2}L_2 i_2^2 \geq 0$$

Now, we have two square terms in the expression. So what will do? We first create the complete square, so to do that what we will do? We will add and subtract $i_1 i_2 \sqrt{L_1 L_2}$ basically in the expression. So when you add as well as subtract $i_1 i_2 \sqrt{L_1 L_2}$, you can simplify the above equation as

$$\frac{1}{2}\left(i_1\sqrt{L_1} - i_2\sqrt{L_2}\right)^2 + i_1 i_2\left(\sqrt{L_1 L_2} - M\right) \geq 0$$

Now since this is the square term, this can never be negative at most it can be zero. So we can have the updated constant of the criteria that

$$\left(\sqrt{L_1L_2} - M\right) \geq 0 \Rightarrow M \leq \sqrt{L_1L_2}$$

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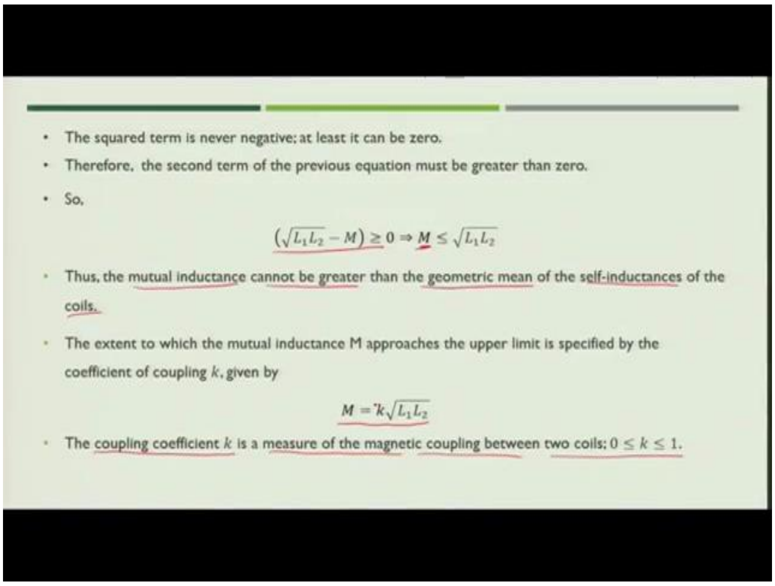

So, we get updated criteria now, so if you simplify you get $M \leq \sqrt{L_1L_2}$. So that means the mutual inductance cannot be greater than the geometric mean of the self inductances of the coil. Now this condition we can converted into equality constant. When you convert this into equality, let us multiply with some constant k. So, we can write $M = k\sqrt{L_1L_2}$, this k is called as coefficient of coupling. So, coefficient of coupling is the measure of magnetic coupling between two coils. The value of k will be $0 \leq k \leq 1$. If k is 1, we will say that the circuit is perfectly coupled. If k is less than 0.5, we say that circuit is loosely coupled and if k is greater than 0.5, we say it is strongly coupled.

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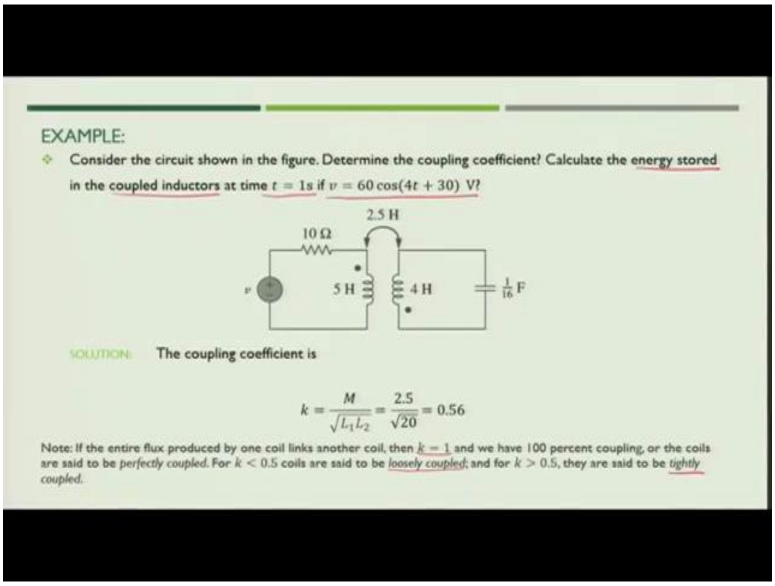

Let us take one example to understand what we discussed related to energy stored in the coupled circuit. We need to find out the energy stored in the couple inductance at $t = 1$s if $v = 60\cos(4t + 30)$ V. Now if you see this circuit the L_1, L_2 are given as 5 and 4 Henry. So k will become,

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

So since k is 0.56, we will say that the circuit is tightly coupled. If k is less than 0.5, we will say that it is loosely coupled and in case k is equal to one will say circuit is perfectly coupled.

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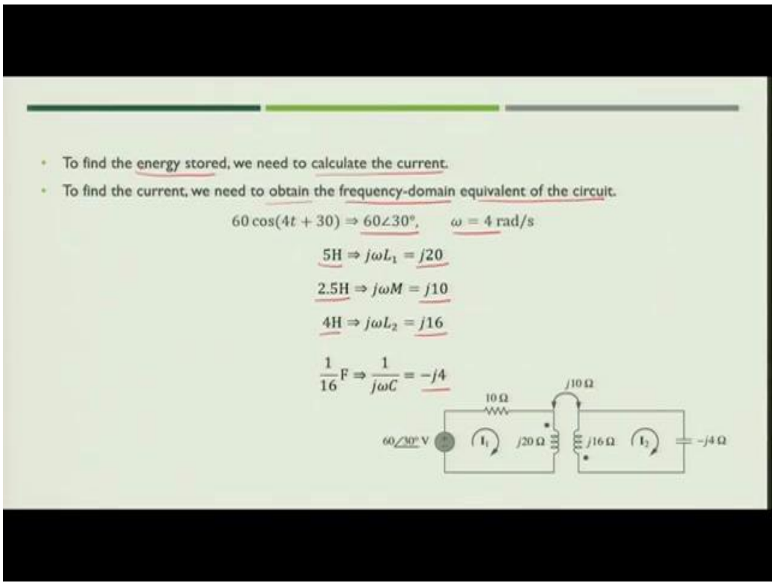

Now the next task is that to find the energy stored, we have to calculate the value of current. So for that, what we need to do? We need to first obtain the frequency domain equivalent of the circuit. So when you converted into frequency domain equivalent,

$$60\cos(4t+30) \Rightarrow 60\angle 30°,\ \ \omega = 4\ \mathrm{rad/s}$$

$$5\mathrm{H} \Rightarrow j\omega L_1 = j20$$

$$2.5\mathrm{H} \Rightarrow j\omega M = j10$$

$$4\mathrm{H} \Rightarrow j\omega L_2 = j16$$

$$\frac{1}{16}\mathrm{F} \Rightarrow \frac{1}{j\omega C} = -j4$$

If you put all those values in the circuit, you will get the circuit in frequency domain. Let us assume that the current $\mathbf{I}_1$ is flowing in the left part of the circuit and $\mathbf{I}_2$ is flowing in the right one of the circuit. If you see these two currents, both are going inside the dot. That means the mutual voltage will be positive.

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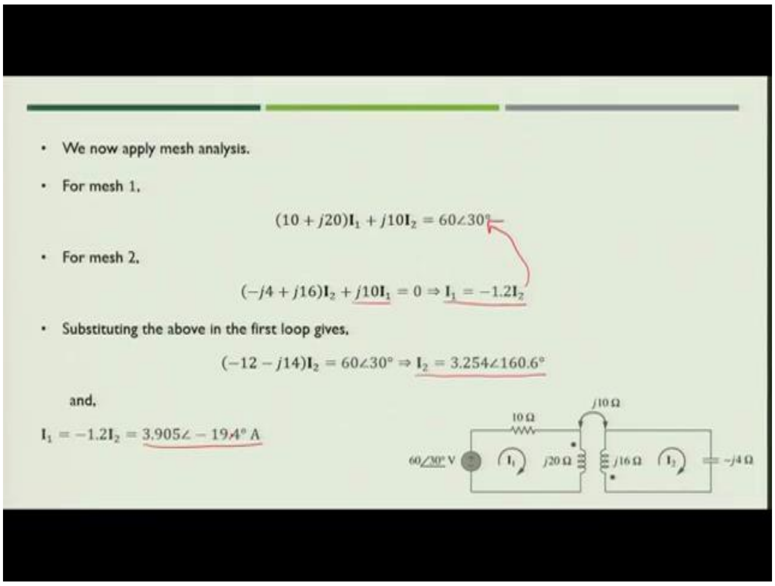

For mesh 1,

$$(10+j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\angle 30°$$

For mesh 2,

$$(-j4+j16)\mathbf{I}_2 + j10\mathbf{I}_1 = 0 \Rightarrow \mathbf{I}_1 = -1.2\mathbf{I}_2$$

Substituting the above in the first loop gives,

$$(-12 - j14)\mathbf{I}_2 = 60\angle 30^\circ \Rightarrow \mathbf{I}_2 = 3.254\angle 160.6^\circ$$

and,

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 = 3.905\angle -19.4^\circ \text{ A}$$

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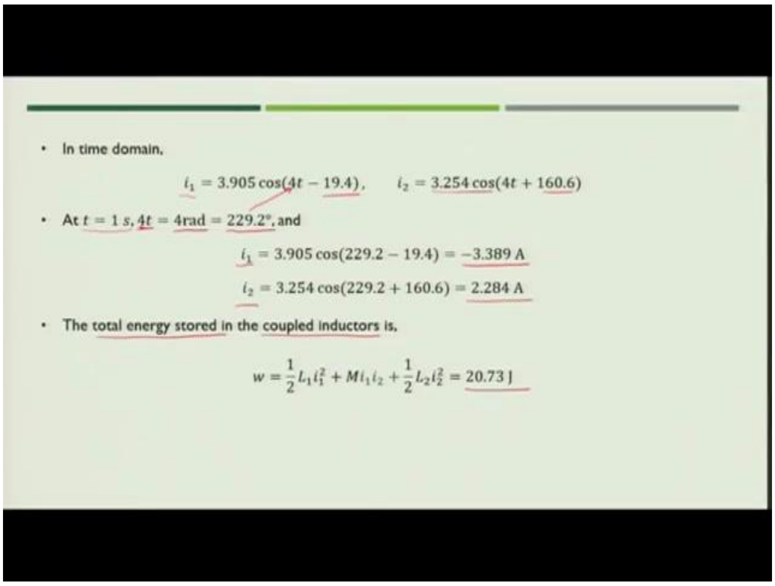

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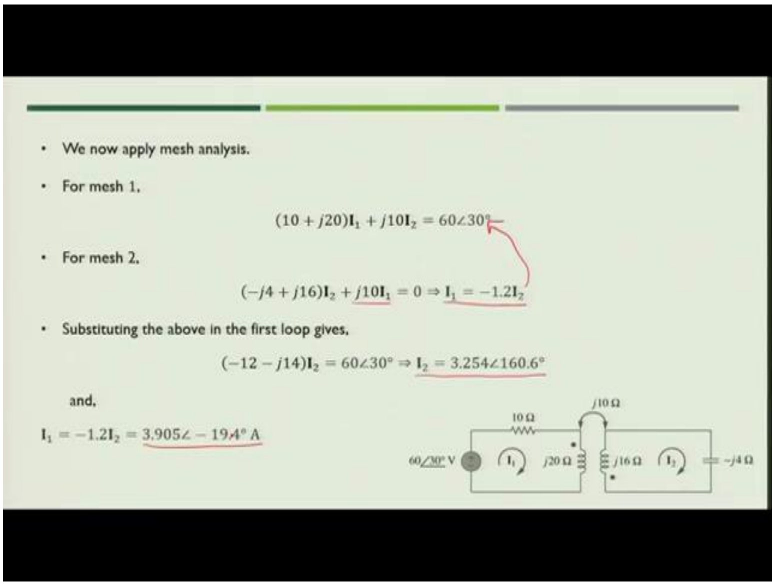

Next task is that we will convert now the values of current we got into time domain. So

$$i_1 = 3.905\cos(4t - 19.4)\,,\;\; i_2 = 3.254\cos(4t + 160.6)$$

Now we need to find the value of energy at time $t = 1\ s$. The value $4t = 4\text{rad} = 229.2$.

So

$$i_1 = 3.905\cos(229.2 - 19.4) = -3.389\text{ A}$$

$$i_2 = 3.254\cos(229.2 + 160.6) = 2.284\text{ A}$$

The total energy stored in the coupled inductors is,

$$w = \frac{1}{2}L_1 i_1^2 + M i_1 i_2 + \frac{1}{2}L_2 i_2^2 = 20.73\text{ J}$$

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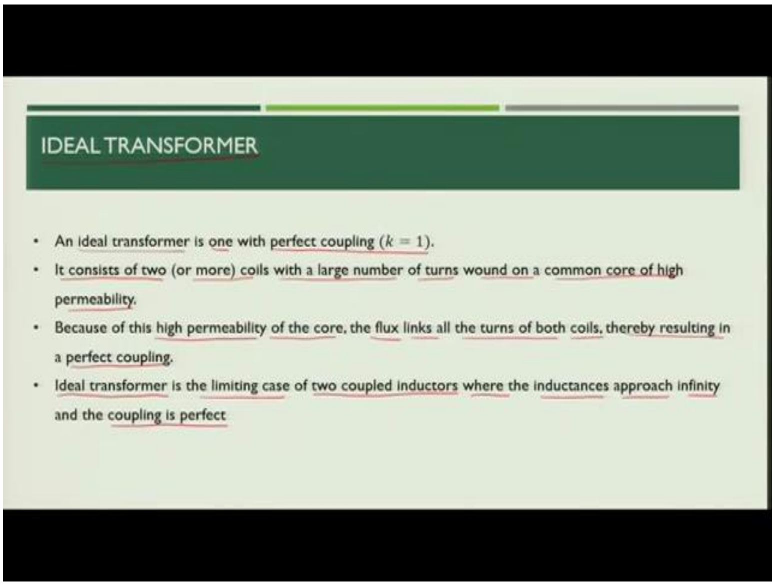

Now let us start the discussion about another topic called ideal transformer. Now what is ideal transformer? The ideal transformer is the one which has perfect coupling. It means that if you have the magnetically coupled circuit and you have the coupling coefficient equal to one that means the circuit which has perfect coupling will be the ideal transformer. It consists of two or more coils with a large number of turns wound on a common core of high permeability.

Since we have high permeability in the transformer core, the flux which will be generated links to all turns of both coils. There by resulting in perfect coupling. The ideal transformer is the limiting case of two coupled inductors where the inductance is approach infinity and the coupling is perfect.

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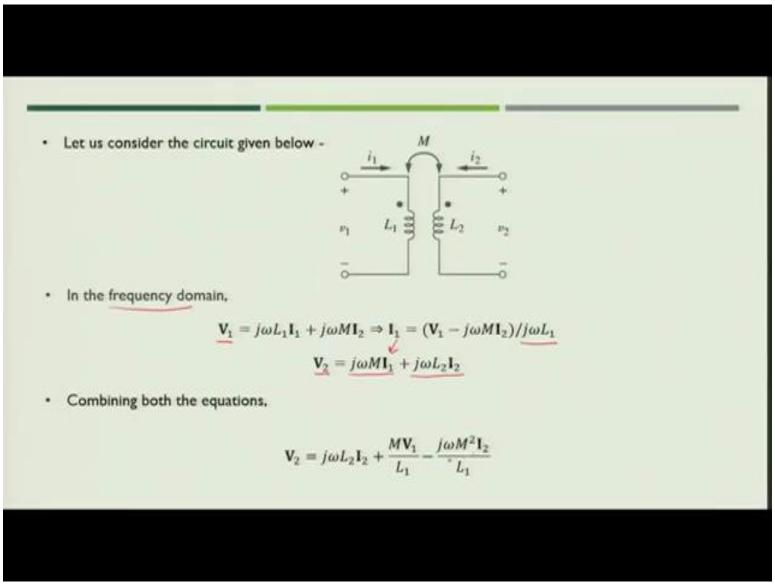

To understand this more clearly will we consider one circuit of the ideal transformer. In this case, if you see v_1 is applied on the left side of the coil, v_2 is applied at the right side of the coil, M is the mutual inductance, L_1 and L_2 are the self inductances of both coils. Now i_1 and i_2 are both entering the dot means the mutual voltage would be positive. We will write the equations for v 1 and v 2 in frequency domain as

$$\mathbf{V}_1 = j\omega L_1\mathbf{I}_1 + j\omega M\mathbf{I}_2 \Rightarrow \mathbf{I}_1 = (\mathbf{V}_1 - j\omega M\mathbf{I}_2)/j\omega L_1$$

$$\mathbf{V}_2 = j\omega M\mathbf{I}_1 + j\omega L_2\mathbf{I}_2$$

Combining both the equations,

$$\mathbf{V}_2 = j\omega L_2\mathbf{I}_2 + \frac{M\mathbf{V}_1}{L_1} - \frac{j\omega M^2\mathbf{I}_2}{L_1}$$

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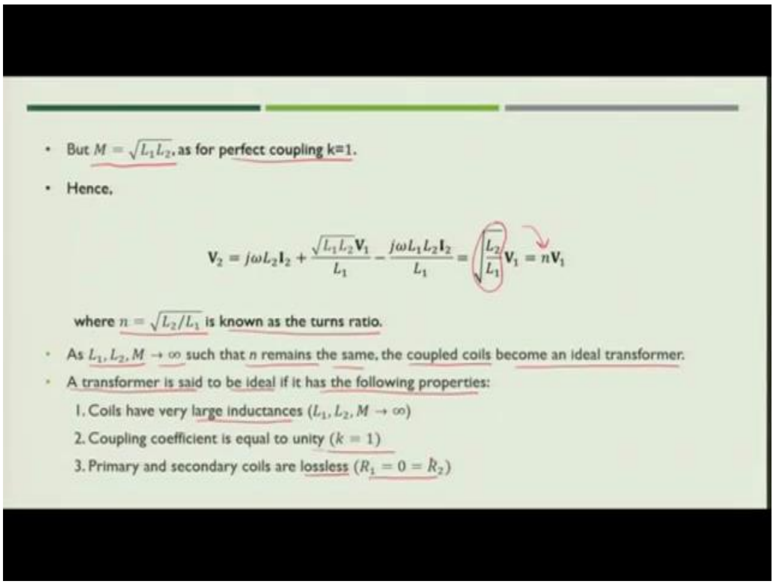

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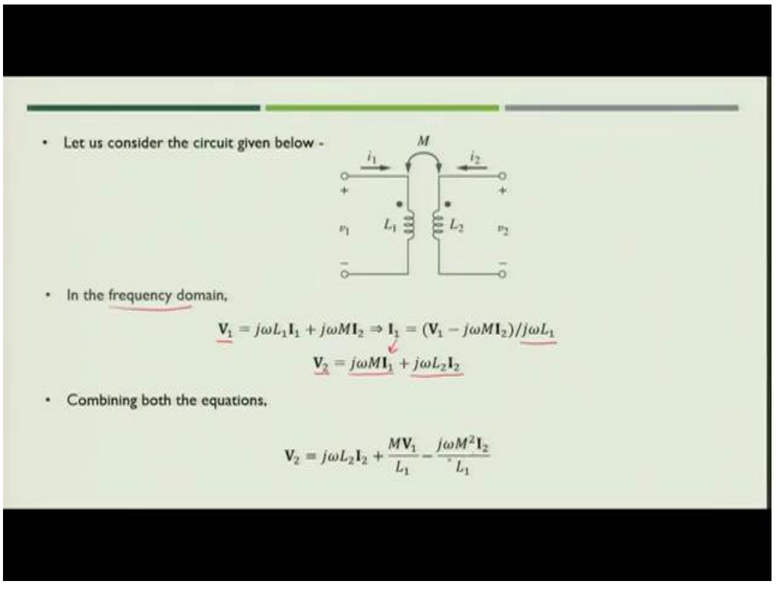

Now you know that $M = \sqrt{L_1 L_2}$ in case of perfect coupling. So, when you consider k is equal to one means perfectly coupled coils, $M = \sqrt{L_1 L_2}$. Now this value we will put in the previous equation and when we simplify, we get

$$\mathbf{V}_2 = j\omega L_2 \mathbf{I}_2 + \frac{\sqrt{L_1 L_2}\mathbf{V}_1}{L_1} - \frac{j\omega L_1 L_2 \mathbf{I}_2}{L_1} = \sqrt{\frac{L_2}{L_1}}\mathbf{V}_1 = n\mathbf{V}_1$$

Now what is n? n is popularly known as the turns ratio of the transformer. So, if you see the expression, if $L_1,\ L_2,\ M \to \infty$ in such a manner that n remains the same, the coupled coils will become the ideal transformer. To summarise we can say the transformer is said to be ideal if it has the following properties.

1. Coils have very large reactances (L_1, L_2, $M \to \infty$)

2. Coupling coefficient is equal to unity ($k = 1$)

3. Primary and secondary coils are lossless ($R_1 = 0 = R_2$)

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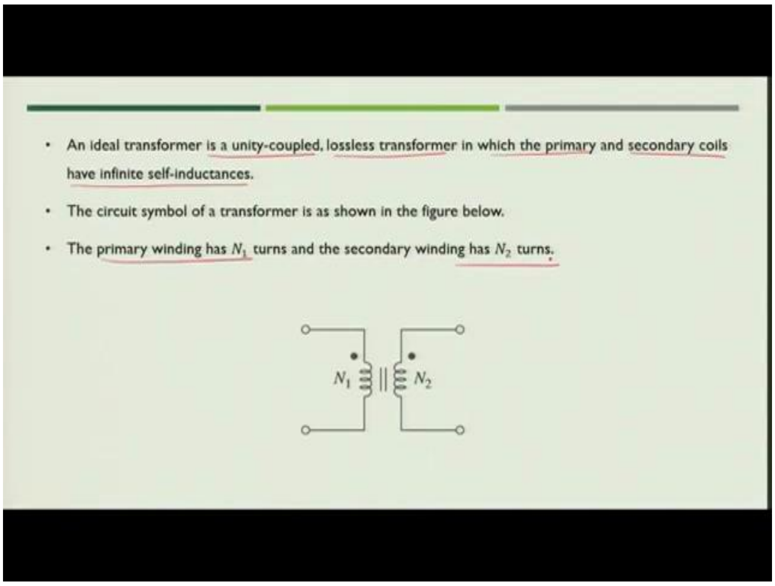

Now an ideal transformer is unity coupled lossless transformer in which primary and secondary coils have infinite self inductances. The circuit symbol of the transformer we generally show like this where we have the coil one and two. And have the turns ratio N_1, N turns number of turns in primary coil as N_1. Number of turns in secondary coil as N_2 and the double lines in between two coils shows that the core is present in the transformer. Now primary winding is having N_1 turns and secondary is having N_2 turns.

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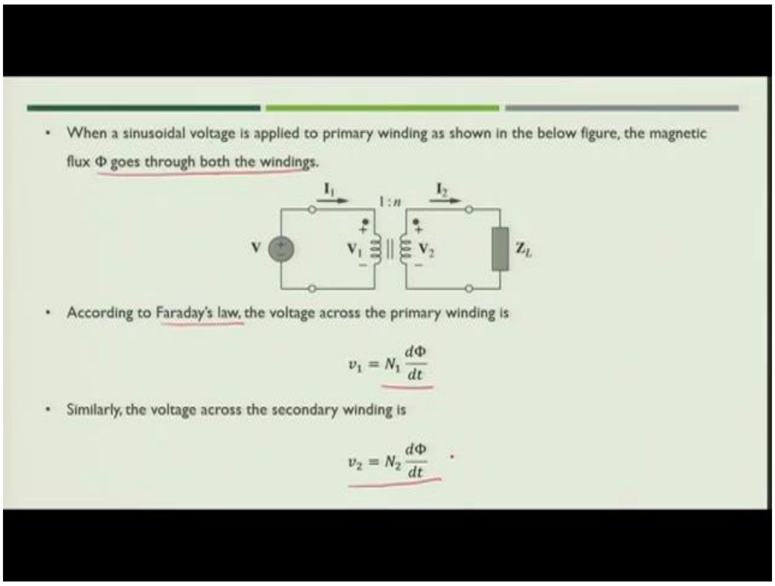

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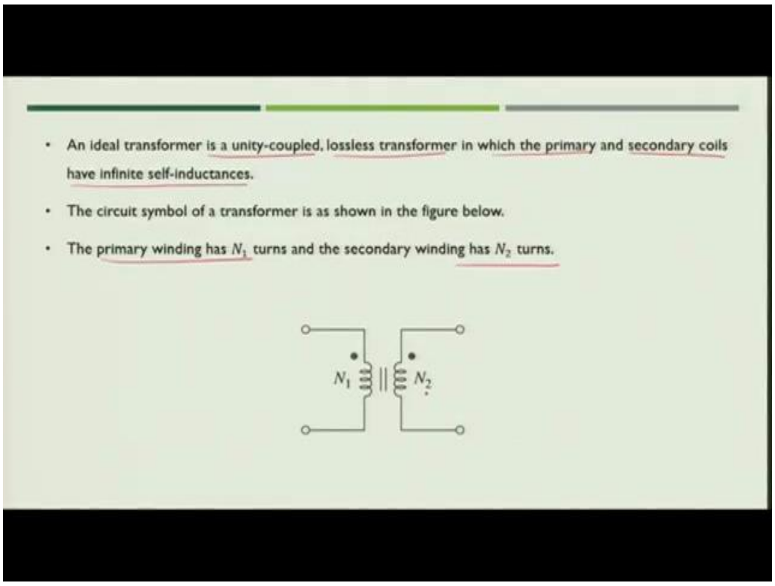

So what we will do now? We will apply this sinusoidal voltage in the transformer circuit which we just saw. And the magnetic flux in that case, generated will goes through both of the windings. You can simply apply the Faradays law to find out the voltages v_1 and v_2. When you apply voltage in the primary coil, so what will happen? You can say,

$$v_1 = N_1 \frac{d\Phi}{dt}$$

Similarly, the voltage across the secondary winding is

$$v_2 = N_2 \frac{d\Phi}{dt}$$

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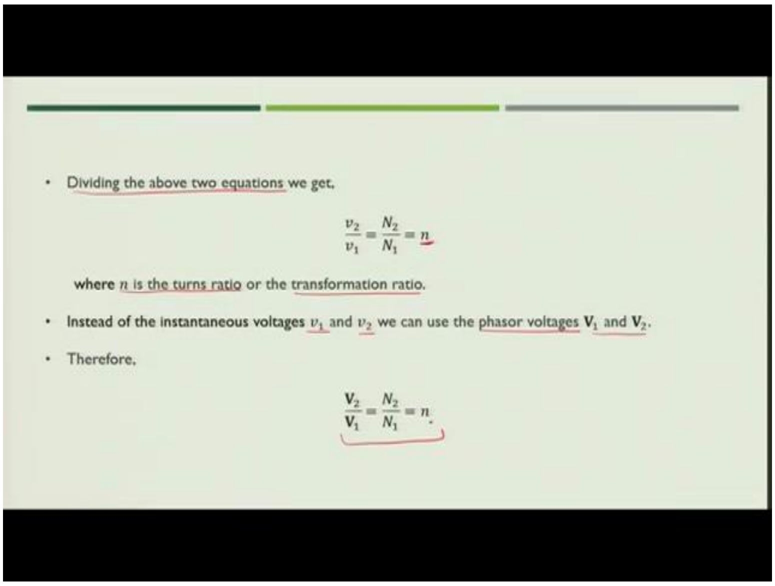

Now if you divide both of the equations, we get

$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n$$

which is nothing but the turns ratio of the transformer.

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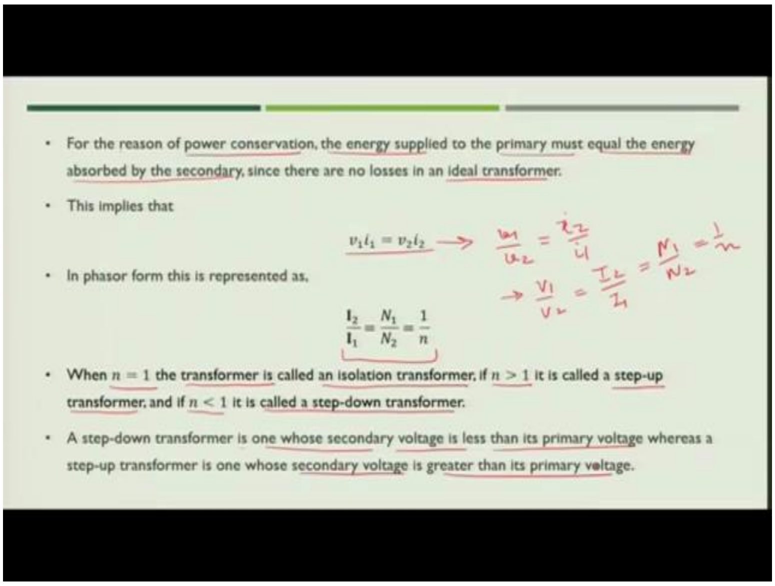

Now the for the reason of power conservation the energy supplied to the primary should be equal to energy absorbed by the secondary. So, when you follow the law of energy conservation, the value that is $v_1 i_1 = v_2 i_2$ in case of ideal transformer. If you write like,

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2} = \frac{1}{n}$$

When $n = 1$, we say transformer is called as a isolation transformer. When $n > 1$ it is called a step-up transformer, and if $n < 1$ it is called a step-down transformer. Step-down transformer is the one whose secondary voltages is less than the primary voltage. Whereas in case of step up transformer the secondary voltage is greater than its primary voltage.

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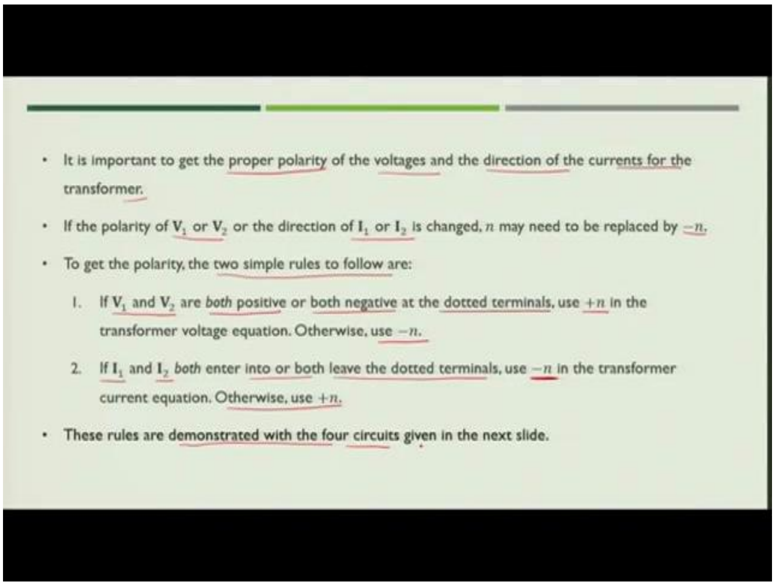

So now it is important to get the proper polarity of the voltages and directions of the currents for the transformer. So if the polarity of $\mathbf{V}_1$ or $\mathbf{V}_2$ or the direction of $\mathbf{I}_1$ or $\mathbf{I}_2$ is changed, the value of n which we have just saw, we need to be replaced by $-n$. So how to find out whether we should use n or $-n$. We will simply use the following rules.

1. If $\mathbf{V}_1$ and $\mathbf{V}_2$ aare *both* positive or both negative at the dotted terminals, use $+n$ in the transformer voltage equation. Otherwise, use $-n$.

2. If $\mathbf{I}_1$ and $\mathbf{I}_2$ *both* enter into or both leave the dotted terminals, use $-n$ in the transformer current equation. Otherwise, use $+n$.

If we follow these two rules, the possible combinations for voltage and current are four. So, we can have four possible combinations of circuits for the ideal transformer. Let us see what are those four circuits?

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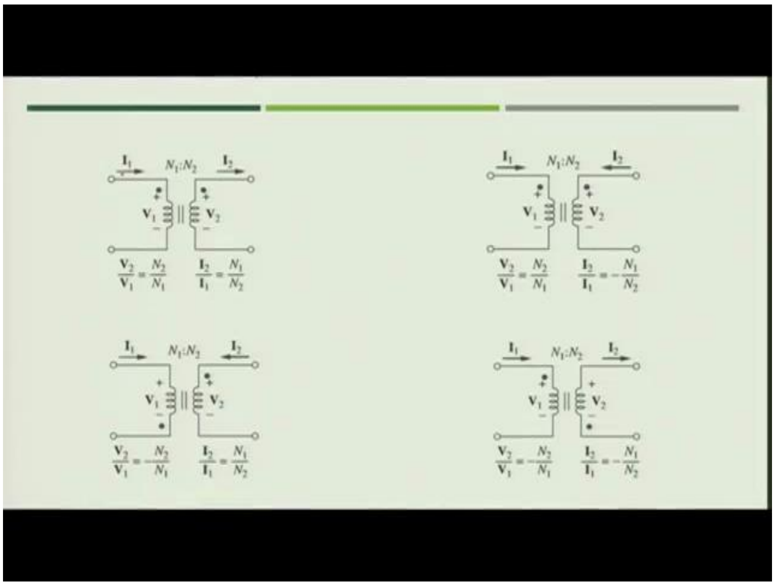

In first, if you see $\mathbf{V}_1$ and $\mathbf{V}_2$ both are positive at the dotted terminal. So, we can say,

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1}$$

Now since current $\mathbf{I}_1$ is going inside and current $\mathbf{I}_2$ is going outside the dotted terminal,

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2}$$

Now in the second case, the voltage terminals are positive at the dotted side of the terminal. So we write,

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1}$$

But now the direction of the current is opposite. So,

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = -\frac{N_1}{N_2}$$

And the third case, the value of the voltages in left side the dotted terminal has negative polarity of the voltage. And at the right side you have positive polarity of the voltage. So, we will write

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = -\frac{N_2}{N_1}$$

Now in case of current, in both coils if you see in this case,

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2}$$

Now in fourth case, your voltages are having the same polarity as we saw in this case, where V 1 is positive at the dotted terminal while V 2 is negative at the dotted terminal. So,

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = -\frac{N_2}{N_1}$$

While in case of current, if you see current is going I 1 is going inside the dotted terminal. At the same time, I 2 is also going inside the dotted terminal because if you trace the path of the current. I 2 will go inside the dotted terminal from here and come out from the upper side. So,

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = -\frac{N_1}{N_2}$$

So, with this we can close our today's discussion in which we discussed about the energy stored in magnetically coupled circuits. And also discussed about the ideal transformer and their various connections and the polarity. We will continue our discussion related to the ideal transformer. Thank you.