Basic Electric Circuits
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Module 7: Circuit Analysis Using Laplace Transform
Lecture 34: Graphical Approach of convolution Integral

Namashkar, so in the last class we were discussing about the convolution integral, and we started with the graphical approach of calculation of the convolution integral.

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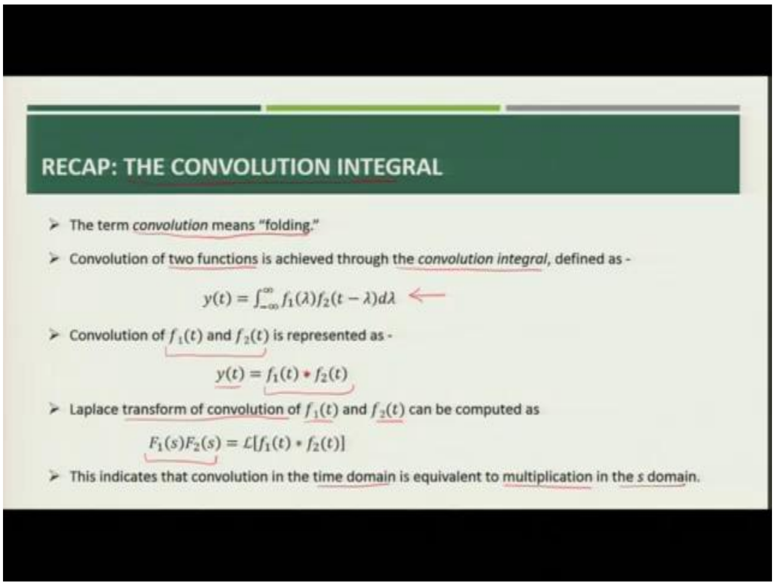

So, we will continue our discussion from yesterday's class. Let us recap what we discussed about the convolution integral. We discussed that convolution is nothings but holding, holding means we take the mirror image of one of the functions which will be part of the convolution integral.

And then convolution of two function is achieved to the convolution integral which is defined as

$$y(t) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda) d\lambda$$

This is what we discussed about the convolution integral.

And then we established that the convolution of f_1 and f_2 is represented as $f(t) = f_1(t) * f_2(t)$. So, this * define that these two functions are being convoluted and the output is $y(t)$. Now, we also established that the Laplace transform of $f_1(t) * f_2(t)$ can be computed by simply multiplying the respective Laplace transforms of the functions as $F_1(s)F_2(s)$. This means that the convolution in time domain is equivalent to multiplication in s domain.

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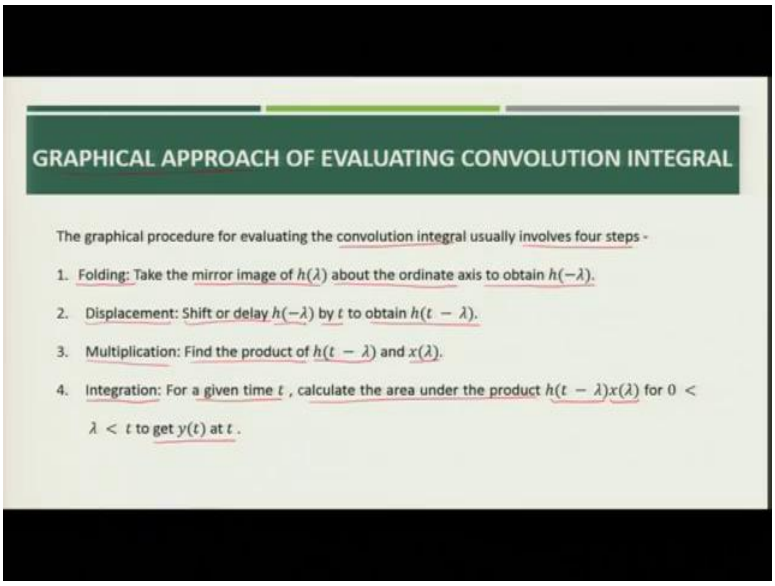

Now we started our discussion on graphical approach of evaluation the convolution integral and we said that the graphical procedure for evaluating the convolution integral, usually involves four steps. Those four steps are, first was folding that means we take that mirror image of one function say $h(\lambda)$ in this case about the ordinate axis, so then we will get function called $h(-\lambda)$. Then second step is displacement where we shift or delay the $h(-\lambda)$ by t so that we get the function $h(t-\lambda)$. Third is multiplication means we find the product of $h(t-\lambda)$ and another function $x(\lambda)$ and then we finally get the integration for the given time t we calculate the area under the product $h(t-\lambda)x(\lambda)$ for $0 < \lambda < t$, and then we get the convolution of function h and x. Now this approach let us try to understand with the help of one example.

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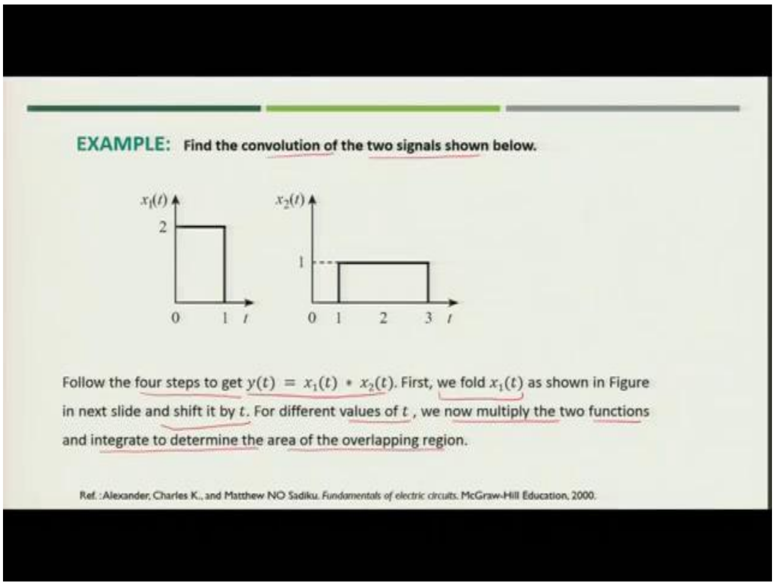

Let us say that we have two functions, one is $x_1(t)$ which is as shown in the figure and $x_2(t)$ is also as shown in figure what we have to do we have to find the convolution of 2 signals that is $x_1(t)$ & $x_2(t)$, so the four steps which we disused we will follow them and try to find out the convolution of $x_1(t)$ & $x_2(t)$,, so as part of our first step to carry out the convolution we first fold $x_1(t)$ so we see when we fold how it look like. And then we will shift by t and finally for different value for t we will now multiply the two functions and then integrate to determine the area of overlapping reasons.

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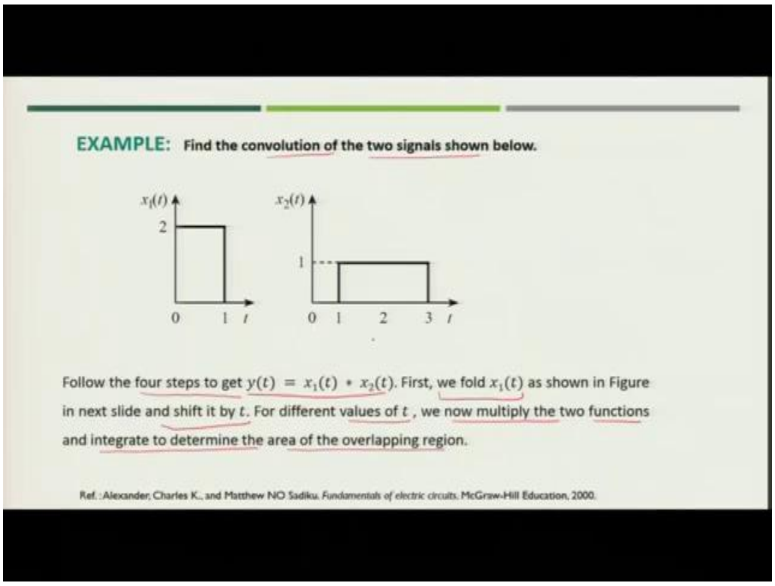

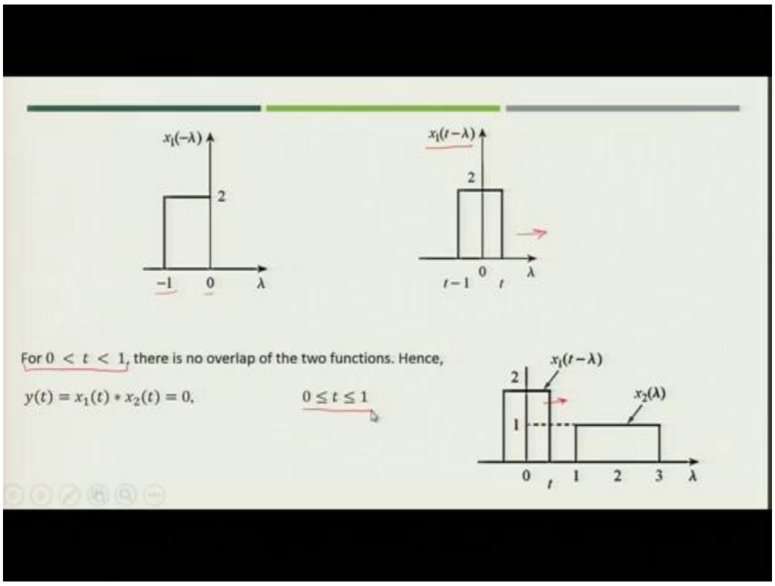

So, what we will do we will first take the mirror image so the mirror image of $x_1(t)$ will be like as shown in the figure, so now it will become $x_1(-\lambda)$ you will take the λ as an axis so now instead of having shape like this it has become the mirror image now the shape is like this.

And then we will shift the function by t so now if you shift it by t it will shift in the λ axis and it will range now between $t-1$ to t instead of -1 to 0 which was there in the mirror image, now it is having the range between $t-1$ to t. Now, if you start moving this particular function that is $x_1(t-\lambda)$ and you shift over $x_2(\lambda)$ because here it will become now $x_2(\lambda)$, so we will put both of these signal on the same axis so what will happen you will start now shifting this integral with respect to the various values of t and then we will find out what would be the integral in the cases of different values of t.

So first we will see when $0 < t < 1$, the maximum value of t will be one in this case so when you have 1 coming at this point it means that there will be no overlap between these two functions so what we can say, we say

$$x_1(t) * x_2(t) = 0,\ \ 0 < t < 1$$

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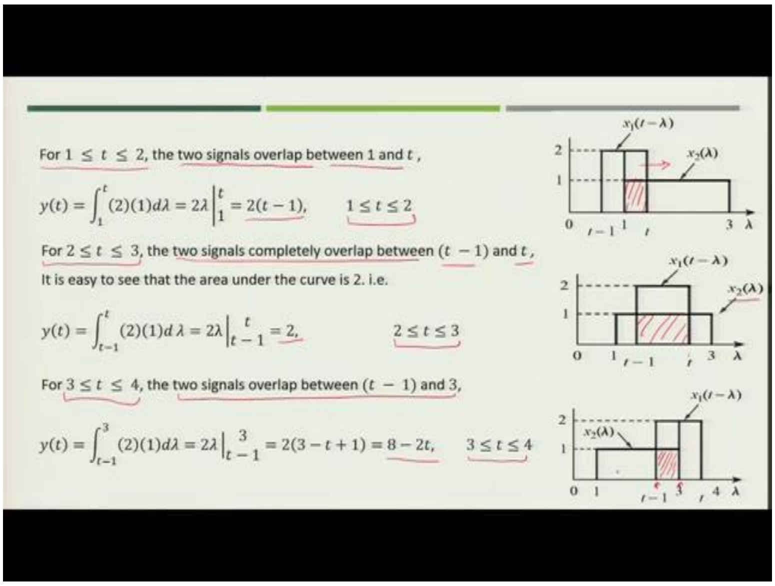

Next let us see for $1 < t < 2$. So, what will happen now the two signals will overlap between 1 and t so if you are shifting this function which was earlier here further then you will see that for some time that is between 1 and t this function will overlap, right.

So, what you can do now you can define the convolution of these two function by using convolution integral as,

$$y(t) = \int_{1}^{t} (2)(1)d\lambda = 2\lambda \Big|_{1}^{t} = 2(t-1),\ 1 < t < 2$$

Now we will shift this signal further and this direction then we will try to find out the value of the convolution of two signals $2 < t < 3$. So, when you will see for $2 < t < 3$ are completely overlapping and the overlap is completely between $(t-1)$ and t because the function $x_1(t-\lambda)$ ranges between $t-1$ to t.

So, what will happen,

$$y(t) = \int_{t-1}^{t} (2)(1)d\,\lambda = 2\lambda \Big|_{t-1}^{t} = 2, \qquad 2 < t < 3$$

Next when the time $3 < t < 4$ means some portion of this function $x_1(t-\lambda)$ would be overlapping between x_2 and x_1 and what would be the overlap period these signals would be overlapping between $(t - 1)$ and 3. So,

$$y(t) = \int_{t-1}^{3} (2)(1) d\lambda = 2\lambda \Big|_{t-1}^{3} = 2(3 - t + 1) = 8 - 2t, \qquad 3 < t < 4$$

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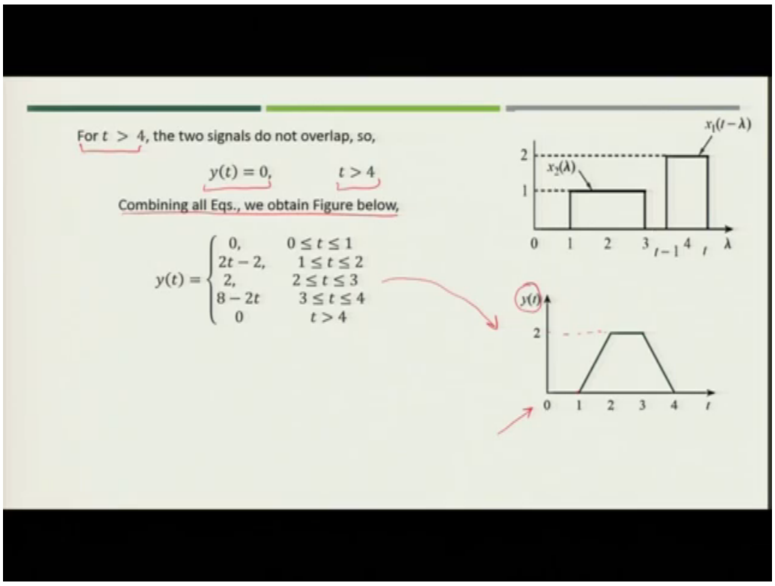

Now for time $t > 4$ you will see that now the signal are not overlapping so what we can say is $y(t) = 0$, for $t > 4$. Now if you combine all the equations, we can say that the convolution

$$y(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 2t - 2, & 1 \le t \le 2 \\ 2, & 2 \le t \le 3 \\ 8 - 2t & 3 \le t \le 4 \\ 0 & t \ge 4 \end{cases}$$

Now if you plot what you will get, you will get that up to 1 it is 0 and then it is incrementing because it is $2t - 2$ then it is a constant with maximum value of 2. And then again decreasing with value $8 - 2t$ and then finally 0. So, this function which you have now got $y(t)$, is the convolution of two functions x_1 and x_2 which were defined in the example. Now the next question would be how you will utilize the convolution integral in circuit analysis.

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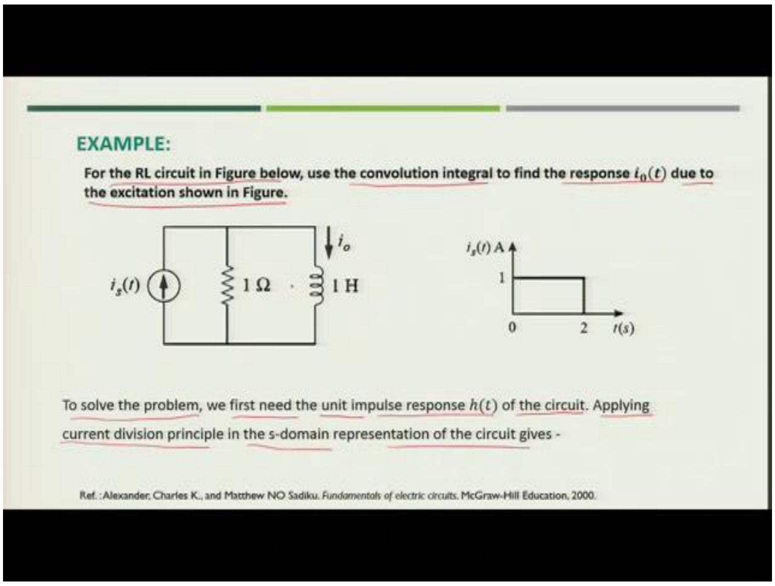

So, let us see with one another example that how you will utilize the convolution integral in solving the circuit. Let us take one RL circuit which is shown in the figure below, we will use the convolution integral to find the response i_0 at any time t due to the excitation shown in the figure. So now this is the excitation which is applied at $t = 0$ with value 1 and it becomes 0 at $t = 2$.

So now to solve this problem we must first use the technique which we discuss that we will first point the unit impulse response that is $h(t)$ of the circuit. So, when we use the unit impulse response what we have to do we apply the correct division principle in s domain and find the value of i_0 so let us first convert this circuit into s domain.

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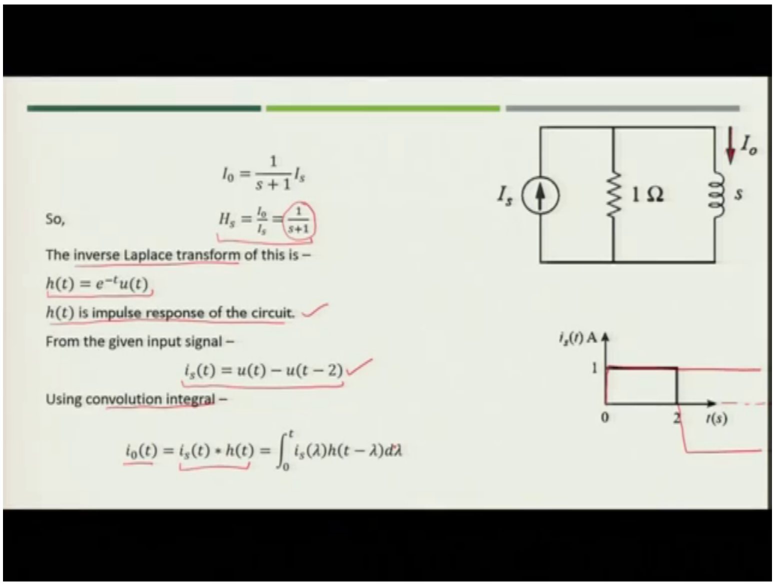

So when you convert into s domain the current source will become I_s resistance will remain same and the value of inductor that is 1H will become s.

In this case and the current is I_0 we have to find out the transfer function of this particular circuit that is I_0/I_s, so let us first let us try to find out the value of I_0. I_0 is nothing but if you use the current division here so when you use current division the value of

$$I_0 = \frac{1}{s+1} I_s$$

So this you will get with the help of current division, then you find out the transfer function

$$H_s = \frac{I_0}{I_s} = \frac{1}{s+1}$$

Now if you take the inverse Laplace transform of H_s what you get that you get

$$h(t) = e^{-t}u(t)$$

And this is basically the impulse response of the circuit. Now let us see the current source which is given as

$$i_s(t) = u(t) - u(t-2)$$

So what what will happen only for this period the Is will be one, otherwise it will be zero. So when you convert Is into two unit step functions you will represent it like $u(t) - u(t-2)$ because it is delayed by two seconds.

Now you have i_s, you have got the impulse response of the circuit what we have to do, we have to find the value of i_0 at any time t with the help of convolution integral so what we will do now we will take the convolution of Is and h functions so what we can write we can $\int_0^t i_s(\lambda)h(t-\lambda)d\lambda$.

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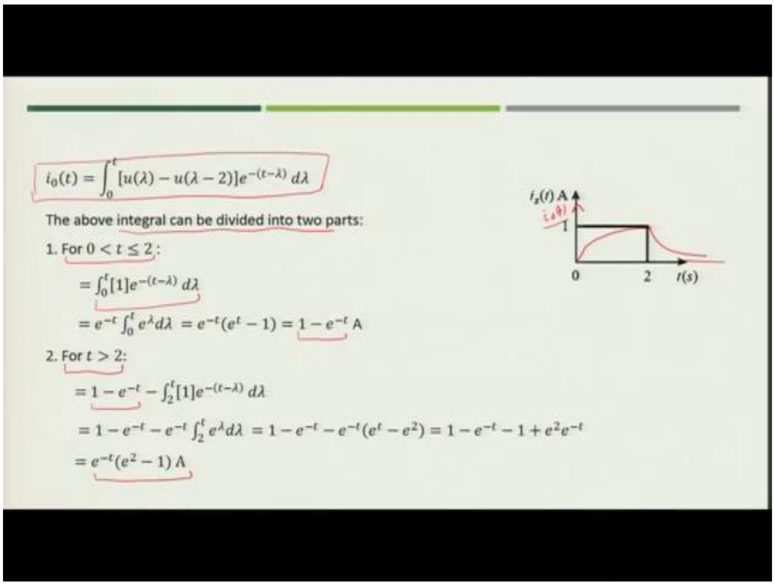

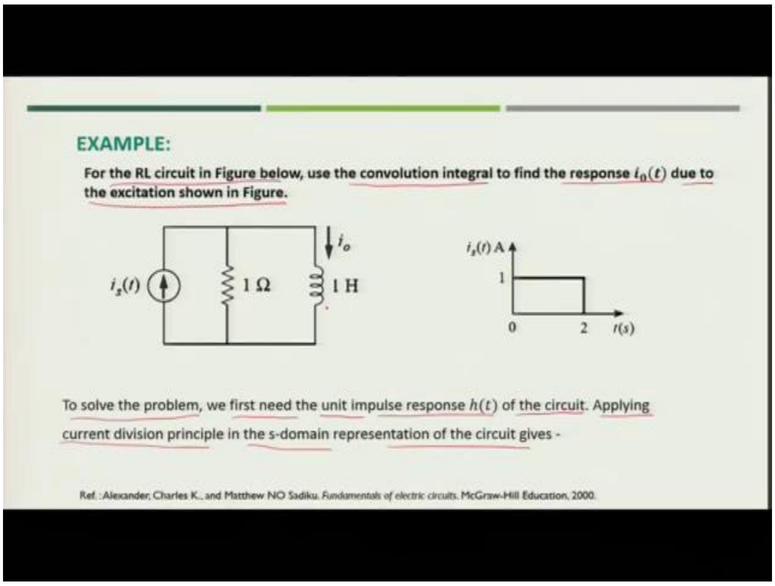

So we will put the value of $i_s(\lambda) = u(\lambda) - u(\lambda - 2)$. So then when we multiply by $e^{-(t-\lambda)}d\lambda$ that is again the value which we computed so we have now the convolution integral ready to be solved. Now here if you see this particular integral you can divide this integral into two parts for time t ranging between zero to two this particular value will be zero because it will be one when t is greater than two.

So fro time t ranging between zero to two this will be zero so finally we are left with only only this will will which will become one and $e^{-(t-\lambda)}$ because this the $u(\lambda) = 1$ for t greater than zero so when you will solve you will get the value of this i_0 t between time t, zero to two as $1 - e^{-t}$.

Now for time t greater than two this will anyway remain there so when you separate these two terms you can simply say that integral of $u(\lambda)e^{-(t-\lambda)}d\lambda$ is nothing but what we calculated in the previous step, so you can simply put the value as $1 - e^{-t}$ and then next is $u(\lambda - 2)$ function now we know that this will this will have value as one when the time t is ranging between two to t so we will change the integral value from two to t this value will become one and then $e^{-(t-\lambda)}d\lambda$.

Now if you solve this and simplify you will get the value of this particular current is $e^{-t}(e^2 - 1)$.

So at this point the the current is exponentially rising so if you are asked to plot the I not t also on this particular signal so for time t ranging between zero to two it is exponentially raising so you response would be like this, and for t greater than two this term is anyway constant so the response should be govern by e^{-t} means for t greater than two it would be exponentially decaying so the I naught t for time t greater than two would be exponentially decaying so this would be the response of current I naught for the input signal applied as Is. So now you can understand how you can solve the circuits with the help of convolution integral.

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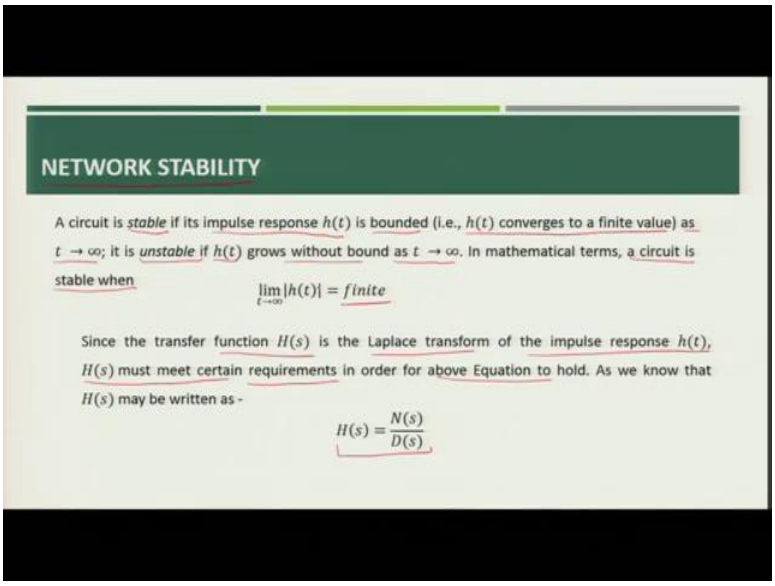

Now let us move on to another concept called network stability, here the circuit is stable if its impulse response is bounded, means the $h(t)$ converges to the finite value when $t \to \infty$ and if it is unstable if s t grows without bounds s $t \to \infty$. So, in mathematical terms you can say that the circuit is stable when,

$$\lim_{t \to \infty} |h(t)| = finite$$

Now since the transfer function $H(s)$ is the Laplace transform of the impulse response $h(t)$ this is what we discussed, so $H(s)$ must meet the certain requirement for this particular equation to hold. So, we know that we the transfer function of the circuit can be written as $\frac{N(s)}{D(s)}$ that is numerator and denominator. And let us see what the criteria for this is $H(s)$ to follow the stability criteria.

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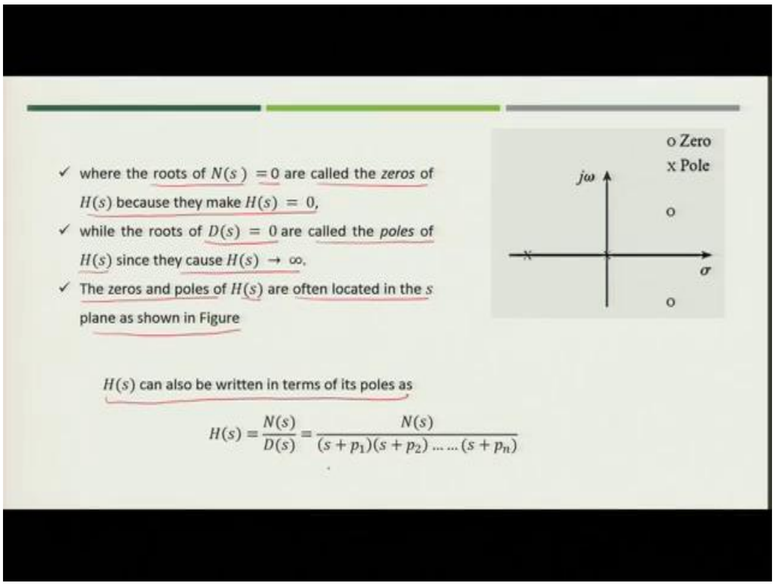

So,

- ✓ where the roots of $N(s) = 0$ are called the *zeros* of $H(s)$ because they make $H(s) = 0$,
- ✓ while the roots of $D(s) = 0$ are called the *poles* of $H(s)$ since they cause $H(s) \to \infty$.
- ✓ The zeros and poles of $H(s)$ are often located in the s plane. Now zeros and poles of $H(s)$ are often located in the s plane as shown in the figure so generally the poles would be in the left half plane and zeros can be at any location weather here or there or maybe at the imaginary axis.

Now

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\dots\dots(s+p_n)}$$

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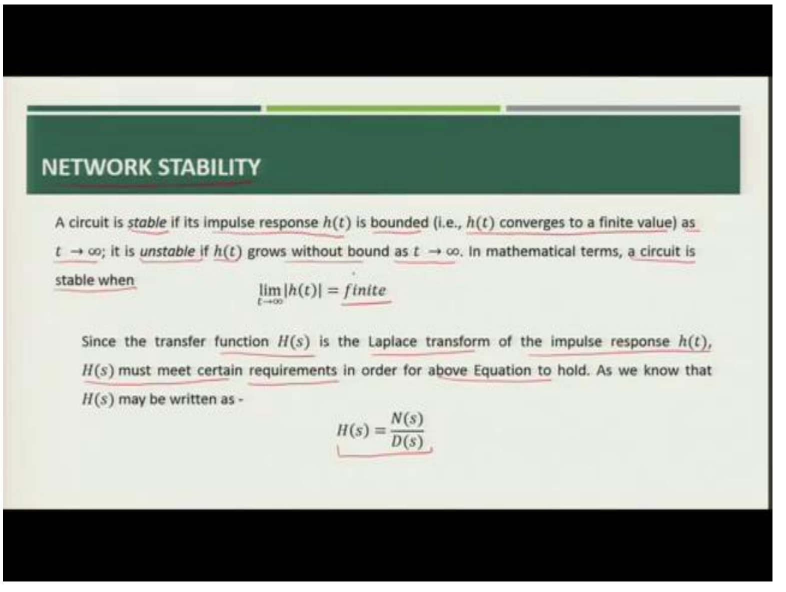

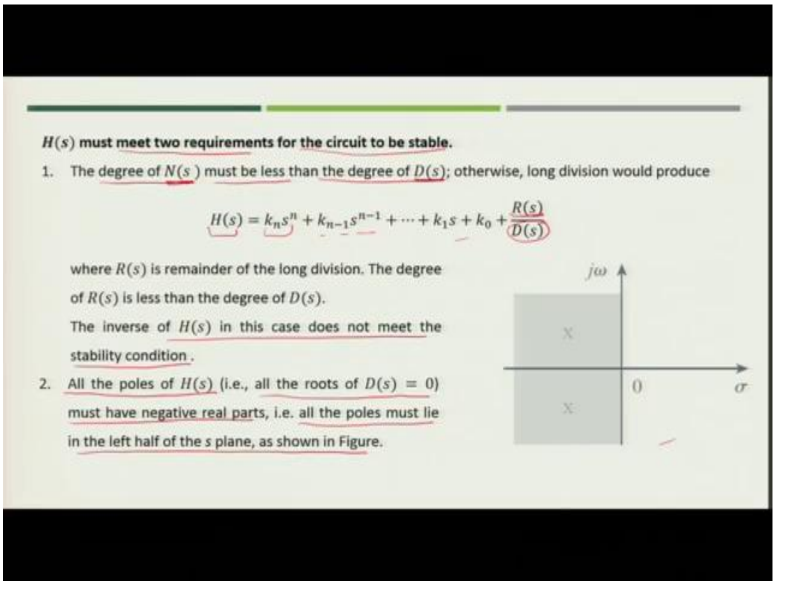

Now what are the criteria, so there are two basic requirement for the circuit for be stable first is that degree of $N(s\ )$ must be less than the degree of $D(s)$ if it is not then what will happen if you represent

$$H(s) = k_n s^n + k_{n-1} s^{n-1} + \cdots + k_1 s + k_0 + \frac{R(s)}{D(s)}$$

This would be like one expression for those values which are like if the degree of numerator is more than degree of denominator plus some residual where the degree of residual is less than denominator.

So, when you divide $\frac{N(s)}{D(s)}$ and you simplify you can represent $H(s)$ as sum special plus some residual divided by denominator. Now if you see this particular expression where degree of $R(s)$, the remainder of the long division, is less than the degree of $D(s)$ so this will anyway be stable when t tends to infinity but if you see this particular segment the this segment will cause the $H(s)$ to infinity when you put value of s tends to infinity.

Because of this the $H(s)$ will be unstable, so the first requirement is that this for system for be stable the degree of $N(s)$ should be less than the degree of $D(s)$. So, when you take the inverse of the $H(s)$ it will not meet the stability condition, $\lim_{t\to\infty}|h(t)| = finite$ So when you have s in the numerator as an independent quantity this will not allow $H(s)$ to be finite the inverse of the inverse Laplace of $H(s)$ to be finite when t tends to infinity.

So that is why degree of $N(s\ )$ must be less than the degree of $D(s)$. Now the second condition is that all poles of $H(s)$ that is all roots of the denominator equation that is $D(s)$ equal to zero must have negative real parts. That means that all poles must lie in the left half of the s plane if you see the s plane the poles must lie in the left half only. okay. So, these are the two conditions which we have to follow as a requirement for h s to of the circuit to be stable.

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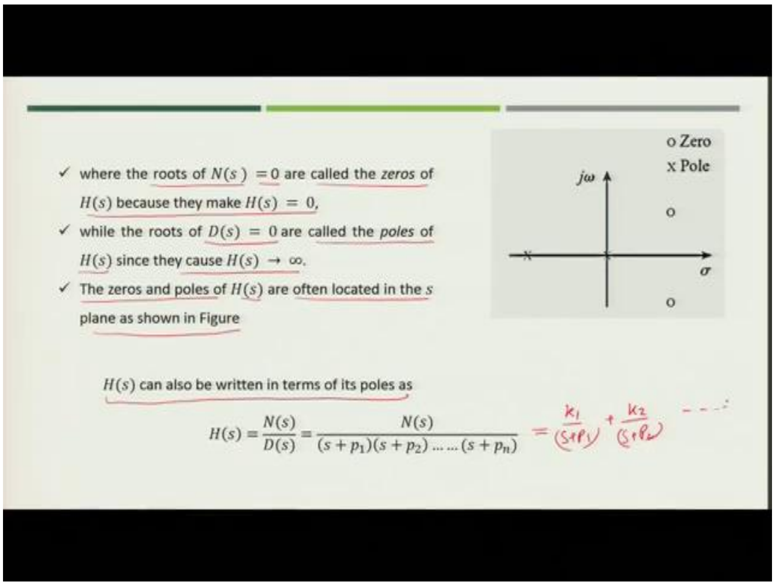

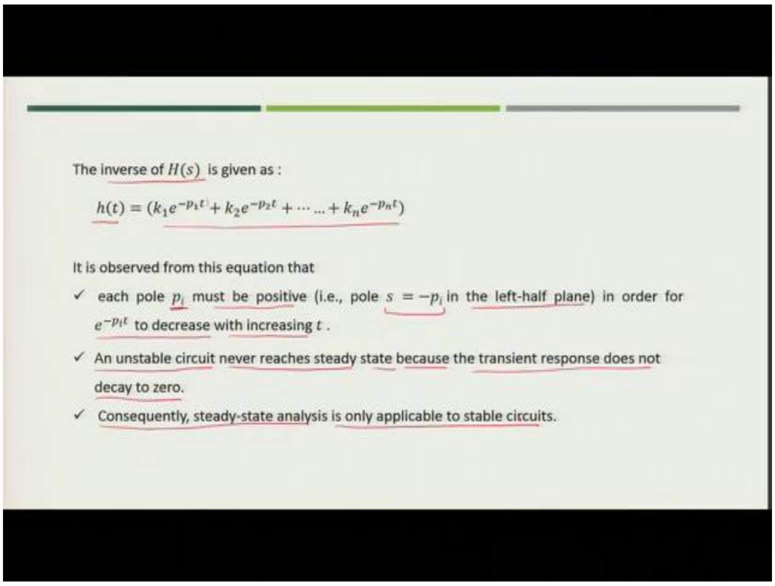

Now as we know that we can represent $H(s)$ in this form so if you take the inverse Laplace transform

$$h(t) = (k_1e^{-p_1t} + k_2e^{-p_2t} + \cdots \ldots + k_ne^{-p_nt})$$

Now if you observe this particular equation you will see that the value of each pole pi must be positive which will make the pole that is $s \ = -pi$ in the left half plane. Then you will see that if this is positive, this particular expression will be e^{-p_it} , which would be exponentially decaying which makes sure that the $h(t)$ is bounded means it is conversing to a particular value with the increase in time t.

Now in unstable circuit it will never reach to its steady state value because the transient response does not decay to zero. If p is negative instead of p being positive it is exponentially rising term .

Then what will happen this will never reach to its steady state. It is not exponentially decaying, it is exponentially rising. In that case transient response will not decay to zero. So, the study state analysis is only applicable to the stable circuits.

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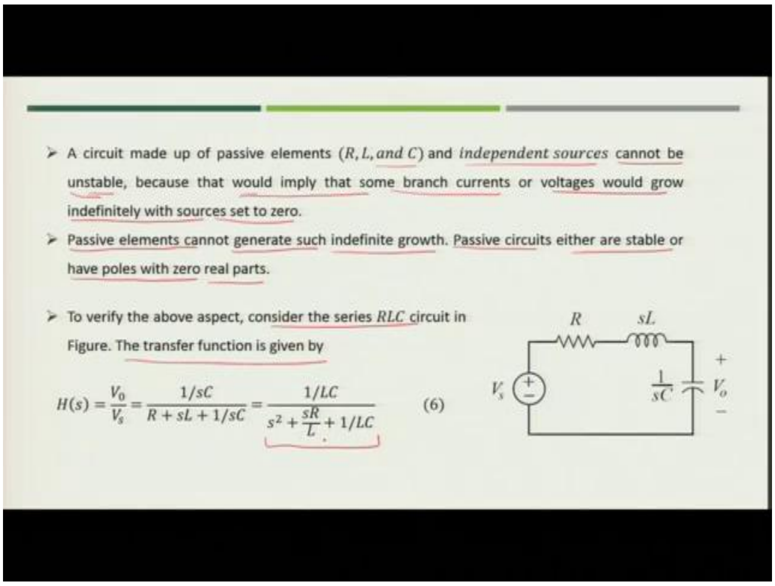

Now let us see how we can co-relate with the our electrical circuits now if the circuit is made up of passive elements that is RLC and independent sources are part of the circuit this system cannot be stable, cannot be unstable because this if it is unstable then it would employ that sum branch current of voltage would grown indefinitely when sources are set to zero which does not happen practically in the circuits which are having RLC components because R is the component which is responsible to decay the response.

So, the passive elements cannot generate such indefinite growth so passive circuits either are stable or have poles with zero real parts so what does it mean we will as discuss when we will see this example let us see this particular case, if you have a series RLC circuit the transfer function of series r l c circuit can be given as h s is equal to v not by v s.

If you compile the ratio,

$$H(s) = \frac{V_0}{V_S} = \frac{1/sC}{R+sL+1/sC} = \frac{1/LC}{s^2+\frac{sR}{L}+1/LC}$$

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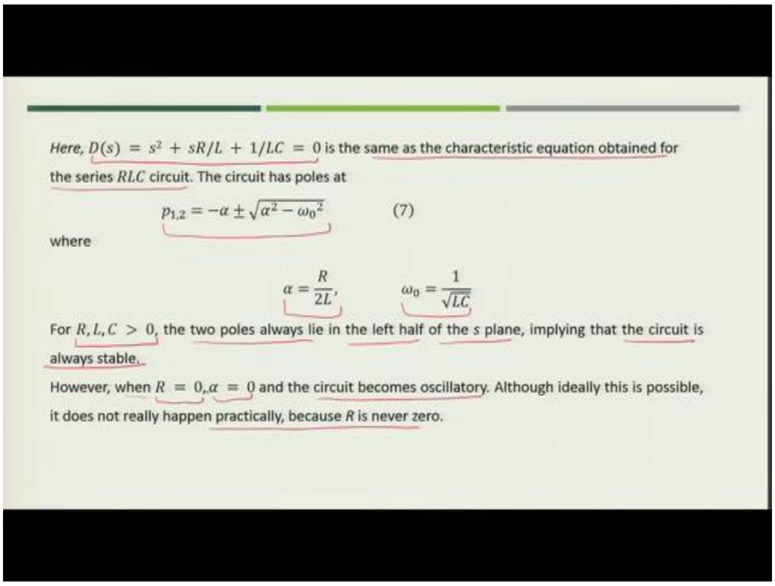

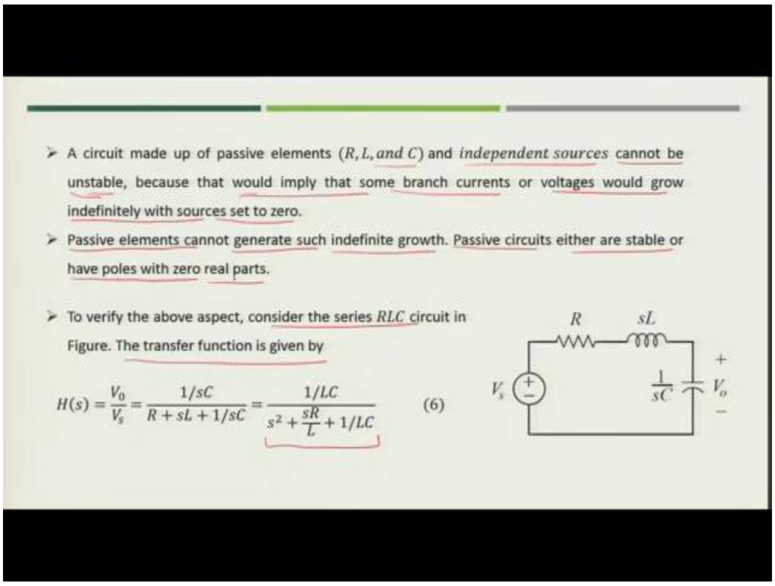

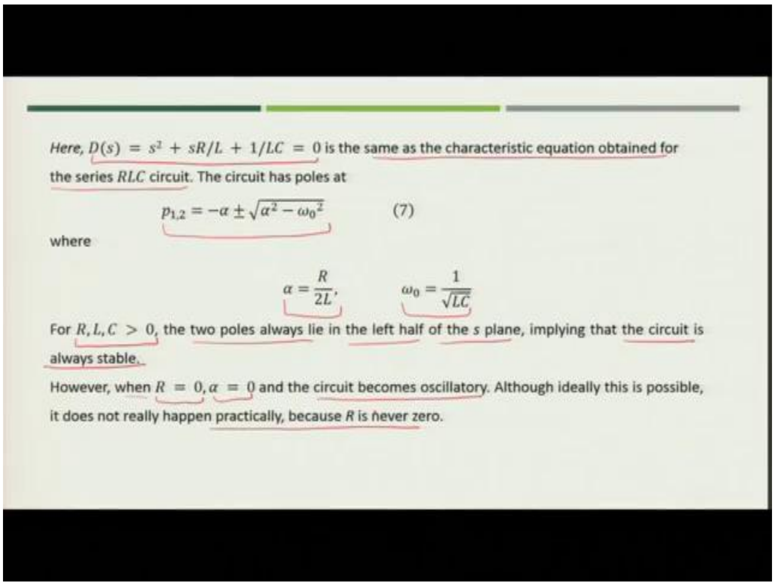

So the denominator expression would be $D(s) = s2 + sR/L + 1/LC = 0$. Now this is same as the characteristic equation which we obtain in these case of series r l c circuit so what we can say now, we can say that the this particular circuit will have two poles and the value would be $p_{1,2} = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}$ where $\alpha = \frac{R}{2L}$, and $\omega_0 = \frac{1}{LC}$, and this we have already seen when we were discussing the series r, l, c circuit.

So now if you see this expression for r l and c greater than zero two poles will always lie in the left half of s plain. This implies that the circuit is always stable. However, when you see $R = 0, \alpha = 0$ the circuit becomes oscillatory although ideally it is possible, but it does not happen practically because r will never be zero.

This case when you have $R = 0, \alpha = 0$ this particular condition will apply the passive circuits will have holes with zero real parts so when our have poles lying on the imaginary axis then you will see that real part is zero and the response would be oscillatory.

So, with this we can close our today's session and we will continue our discussion on this particular aspect in the next session also thank you.