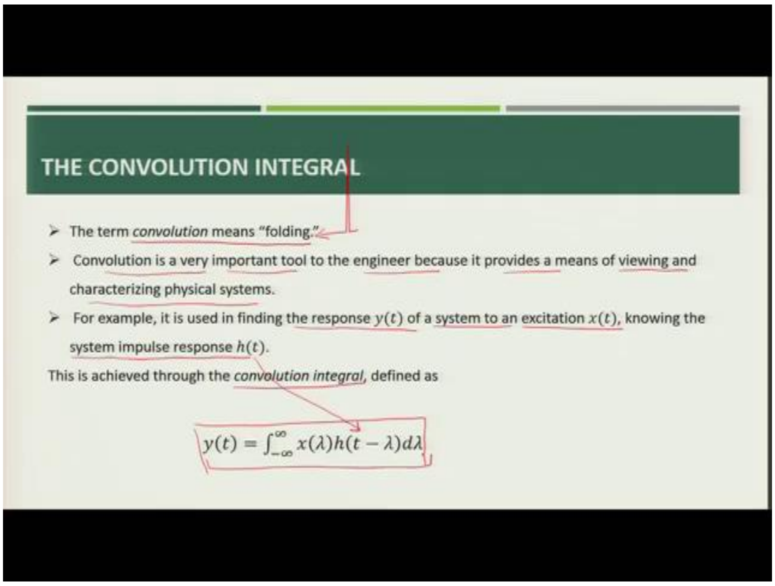
Basic Electric Circuits
Professor Ankush Sharma
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Module 7: Circuit Analysis Using Laplace Transform
Lecture 33: Convolution Integral

Namashkar. In today's session we will discuss about convolution integral, so we will see how we can utilise convolution integral in our circuit analysis. Let us first understand, what is the convolution integral?

(Refer Slide Time: 0:36)



The term convolution means folding, so folding means when you take the mirror image of a particular function, means you are folding across the y axis. The term convolution integral has come from that concept and what does it mean we will see when we will discuss the convolution integral in detail. And this integral is very important tool for the engineers because it provides the means of viewing and characterising the physical systems.

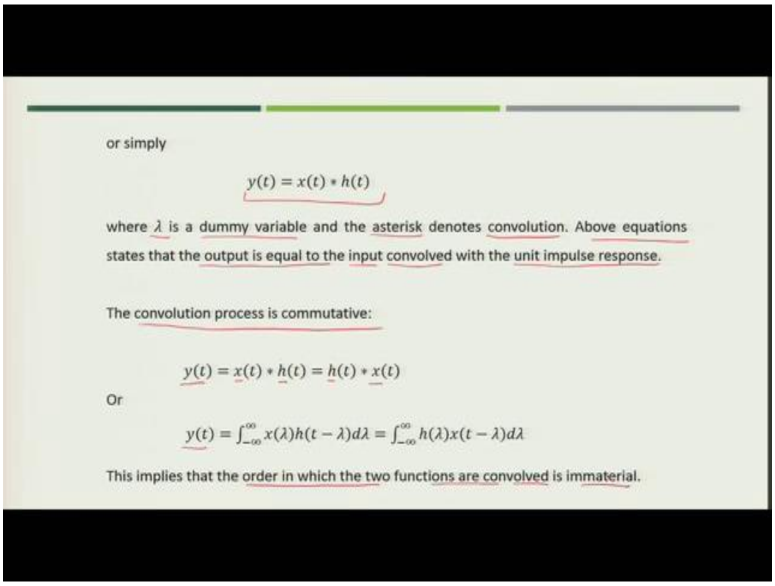
For example, if you have the impulse response of a particular system that is $h(t)$, which we discuss in our previous lectures, so if you know the impulse response, you can find the response $y(t)$ for any system with the excitation of $x(t)$. This is very important and very strong integral to find out the response of a system, the impulse response of which is known.

The convolution integral is defined as,

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

$h(t)$ is your impulse response, λ is a dummy variable and $x(\lambda)$ is the excitation of $x(t)$. When you take the convolution of $x(t)$ you will get the response $y(t)$ and this integral is called as a convolution integral.

(Refer Slide Time: 2:39)



This can be represented as,

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

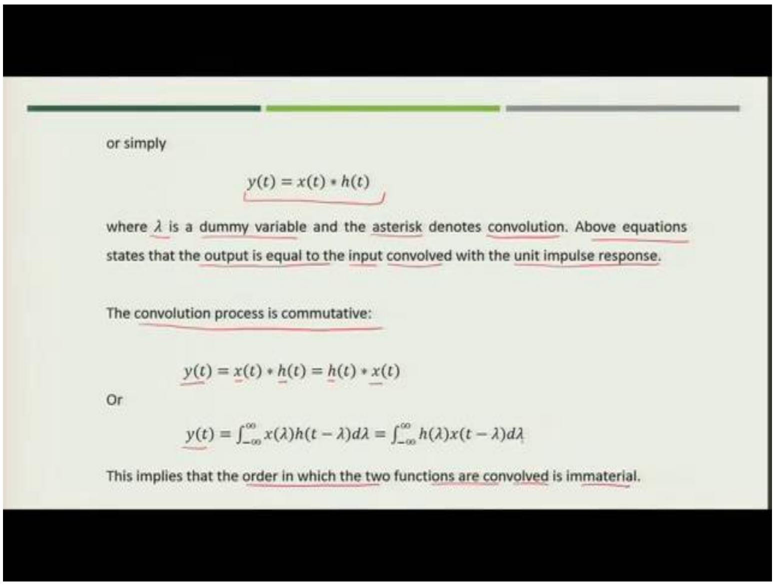
So, we can say that basically the $y(t)$ is convolution of x and h functions and the lambda which we discussed in the previous slide was dummy variable and this would be the convolution. Basically, the asterisk will represent the convolution of two functions. Now the above equation states that the output is equal to input convolved with the unit impulse function.

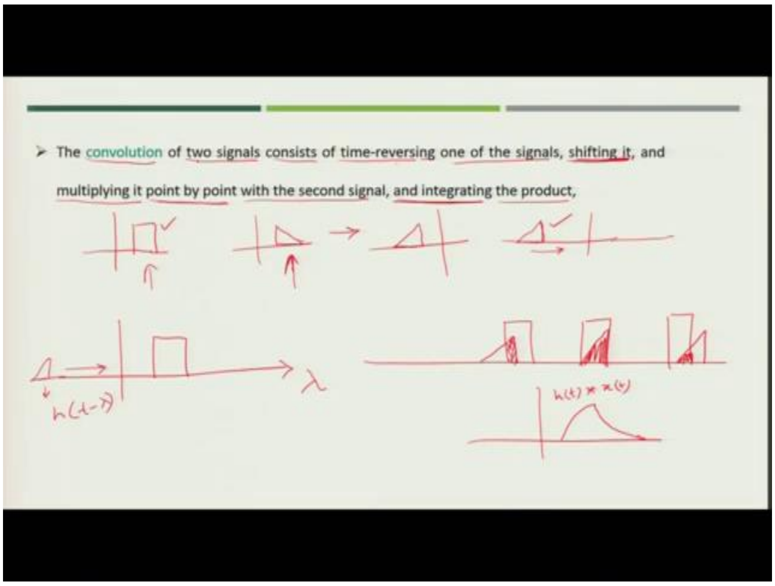
So, we can find out the response of the system when we know the unit impulse response. Now the convolution process is commutative, that means if you interchange the positions of x and h, the output is not going to change that means convolution of x and h will be same as convolution of h and x. In the integral form what you will represent,

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

That means that the order in which two functions are convolved is immaterial.

(Refer Slide Time: 4:30)





Now the term convolution, how you will define, the two signals which consists of time reversing of one of the signals, shifting it and multiplying it point by point with the second signal then and integrating the product then the output would be the convolution of two signals. So how we will visualise, let us take an example so that you can understand the concept.

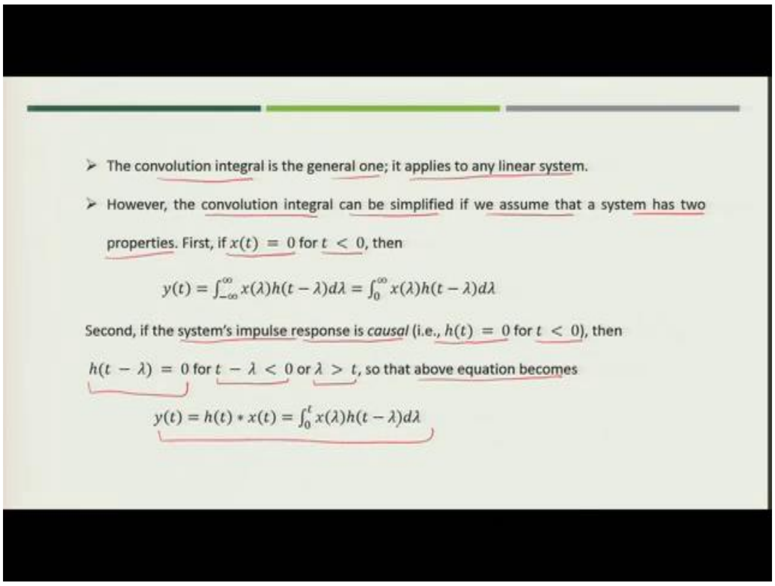
Let us say that there are two functions, one is let us say is function like this and second is function like this, now when we say the convolution of two signals means time reversing of one of the signal means you are folding that signal so when you, suppose if you fold that signal, what you will do, you will take the mirror image of that signal. This will become like this, shifting it, shifting it means you can shift it at some location because it will vary with respect to time and we will keep on shifting this signal across the access and we will try to find out the multiplication of this signal and this signal point by point and then integrate it so that you can get the product.

How we will do, so let us say that you are putting up signal here, this particular signal is moving, so it is moving from this side to let us say this is axis lambda so you are shifting, this may be like if you say this is $h(t-\lambda)$ then you shift this function so what you have defined, $\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$. Next you have to take the product of two and then sum up because integration is nothing but the summing up the product of these two signals over a particular time period.

When you shift, let us take the axis and see how it is getting mixed, so at one particular time the signal would be like this so you will have total product and the sum of those product which is the area like this and then after some time this will become like this, right. And after some time, it may leave and will become like this. What is happening now that the signal is shifting because you are shifting it so the signal is shifting, we will only shift one, second one is stationary, so we are shifting with respect to the value lambda. A this axis it is shifting and we are trying to find out the value of the product at each and every point and ten you are integrating means you are summing up all the products and then you are again you are taking another location and finding out the product.

Finally, if you see and if you trace the product and the integration of these two functions, so it may look like this, so this would be considered as a convolution of these two signals. This is the basically the output which you will get when you apply convolution integral between two functions. We will discuss more in detail when we proceed more in this lecture and we will try to understand with help of few more examples so that you are clear about the meaning of convolution integral.

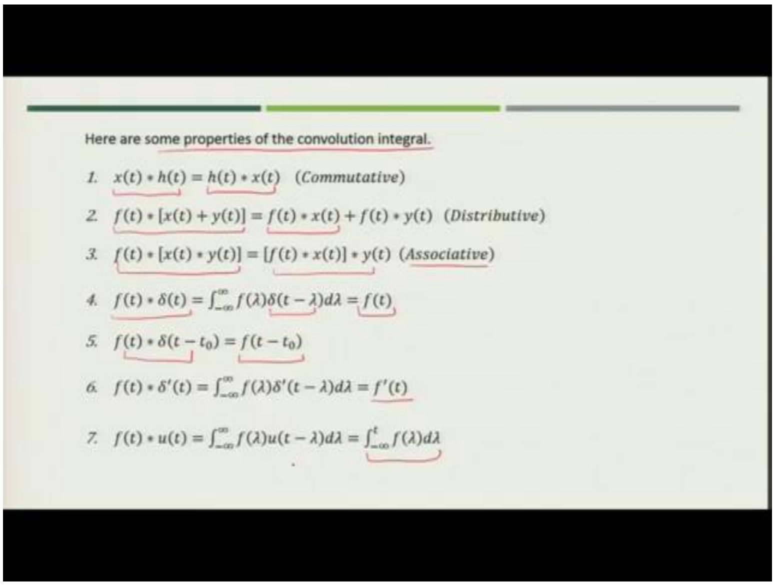
(Refer Slide Time: 9:11)



Now the convolution integral is the general one, it applies to any linear system. But the convolution integral can be simplified if we assume that the system has two properties. So, what are those two properties. First is that $x(t)$ is equal to zero for t less than zero, it means that your integration which is now from minus infinity to infinity will reduce to integration between zero to infinity for the convolution integral and second property what we can use is that systems impulse response is causal, that means $h(t) = 0$ for anytime t less than zero.

So, if this property holds, $h(t - \lambda) = 0$ for $t - \lambda < 0$ or $\lambda > t$. The above equation will now become the convolution, $y(t) = h(t) * x(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda$ because for any value $\lambda > t$ this will become 0. When we say that the functions are following these 2 properties, first function is 0 for t less than 0 and second is causal means it is 0 for $\lambda > t$ then we get the reduced integral.

(Refer Slide Time: 11:03)

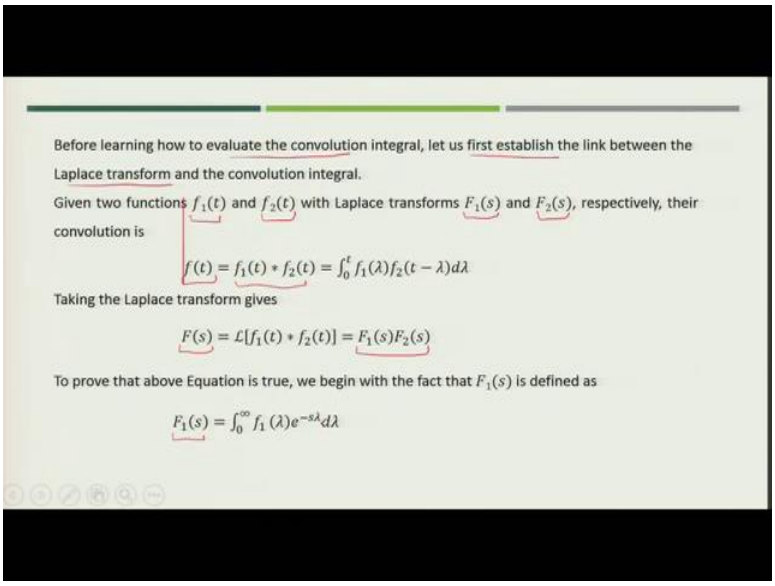


Now let us see few of the properties of convolution integral, we are not going to derive it because it is not in our like scope of the current subject, but we have to keep in mind that there are few properties which we will use when we progress in this particular course. First property is convolution of h and x is commutative, i.e., $x(t) * h(t) = h(t) * x(t)$, means you can exchange the locations of h and x and the convolution integral will hold. Second is distributive property, means $f(t) * [x(t) + y(t)] = f(t) * x(t) + f(t) * y(t)$.

Now third is associative property, $f(t) * [x(t) * y(t)] = [f(t) * x(t)] * y(t)$. Since it is associative you can use either of the method for getting the convolution integral. Now next is the property when the function is convolved with unit impulse function, $f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\lambda)\delta(t-\lambda)d\lambda = f(t)$.

Similarly, if you are convolving with $\delta(t-t_0)$ that is time shift impulse function, you will get the output as $f(t-t_0)$. Next the convolution of $f(t)$ with the derivative of unit impulse, $\delta'(t)$, you will get the output as $f'(t)$. Now if you are taking convolution of f with unit step, the output would be $f(t) * u(t) = \int_{-\infty}^{\infty} f(\lambda)u(t-\lambda)d\lambda = \int_{-\infty}^{t} f(\lambda)d\lambda$. These would be the major properties which we will keep on using when we use the convolution integral in the various circuit concepts.

(Refer Slide Time: 14:19)



Now before learning how we can evaluate the convolution integral, let us first understand how the convolution integral is linked with the Laplace transform. So, suppose if you have two functions like $f_1(t)$ and $f_2(t)$, the Laplace transform of those two time varying functions are $F_1(s)$ and $F_2(s)$, then if you take the convolution of these

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda \qquad (6)$$

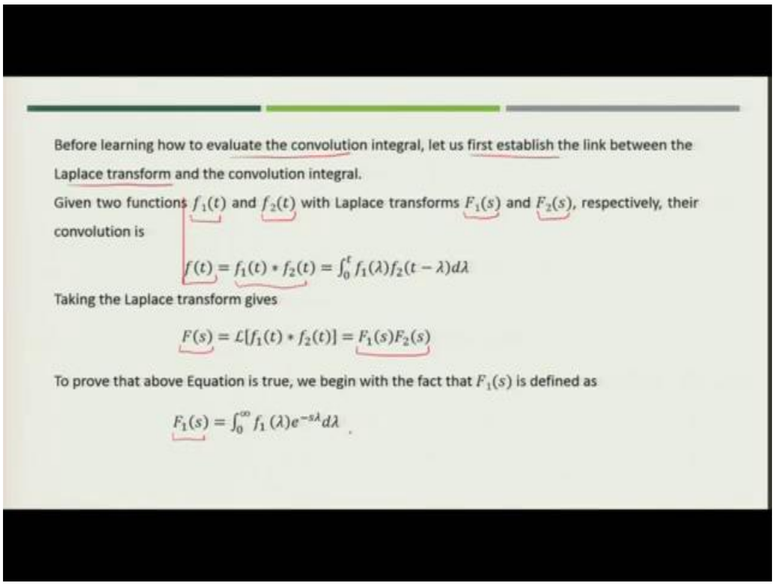
Taking the Laplace transform gives

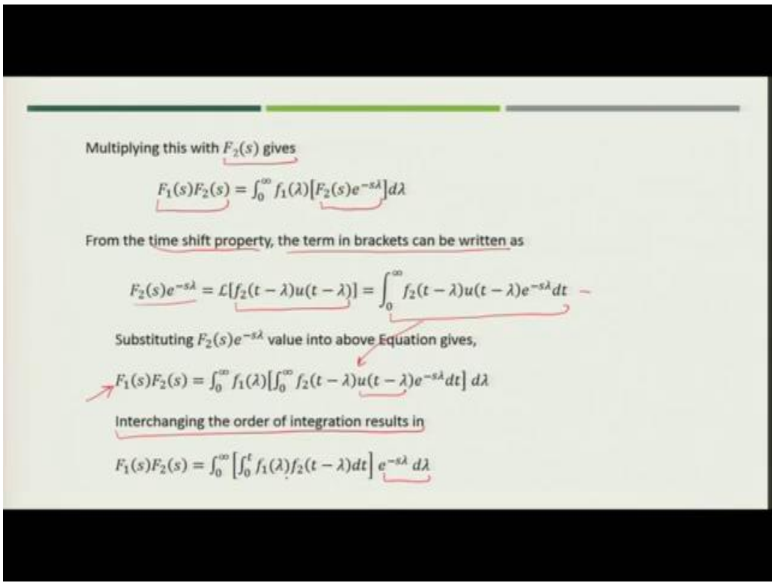
$$F(s) = \mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

It means the Laplace transform of the convolution of $f_1(t)$ and $f_2(t)$ is simple multiplication of $F_1(s)$ and $F_2(s)$. Now let us see how we can justify this concept whether it is true or not so let us see that there is a function $f(t) = f_1(t) * f_2(t)$, so you will define it like $\int_0^t f_1(\lambda) f_2(t-\lambda) d\lambda$. Now if you take the Laplace of function $f_1(t)$,

$$F_1(s) = \int_0^\infty f_1(\lambda) e^{-s\lambda} d\lambda.$$

(Refer Slide Time: 16:06)





Multiplying this with $F_2(s)$ gives

$$F_1(s)F_2(s) = \int_0^\infty f_1(\lambda)[F_2(s)e^{-s\lambda}]d\lambda$$

Now if you use the time shift property because if you see this holds you can say that when you use then time shift property this can be converted as

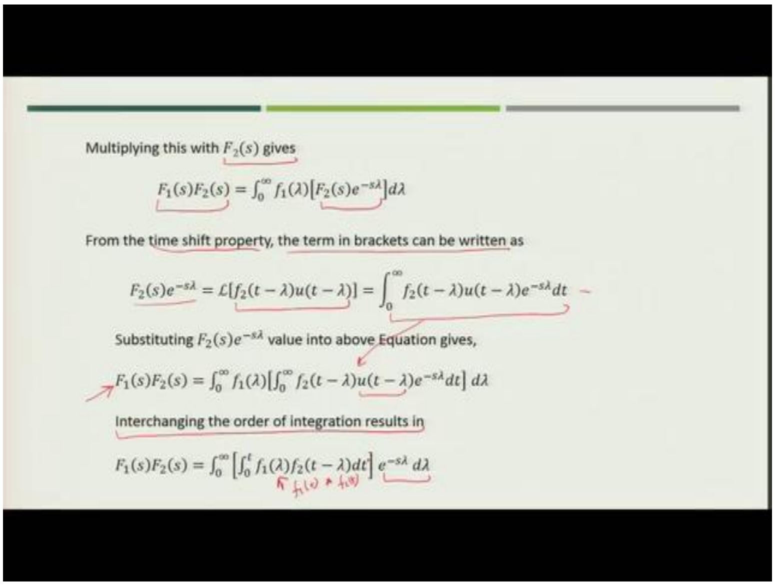
$$F_2(s)e^{-s\lambda} = \mathcal{L}[f_2(t-\lambda)u(t-\lambda)] = \int_0^\infty f_2(t-\lambda)u(t-\lambda)e^{-s\lambda}dt$$

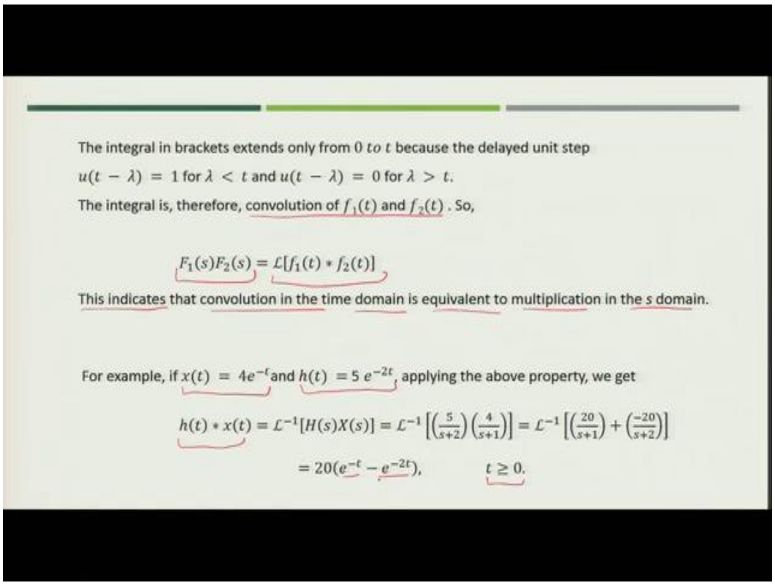
So,

$$F_1(s)F_2(s) = \int_0^\infty f_1(\lambda)\left[\int_0^\infty f_2(t-\lambda)u(t-\lambda)e^{-s\lambda}dt\right] d\lambda$$

Now if you see this will hold until $t > \lambda$. When the $\lambda > t$ this function will become zero, so what you can write in terms of integral. First, we interchange the order of integration so now it will become zero to infinity and in the second integral we will say that it is ranging between zero to t because of the unit step time shift function and we will say that it is f1 lambda into by f2 t minus lambda dt and e to the power minus s lambda d lambda we will take out for the second integral. So you can rearrange the this particular expression in this particular form. Now if you see this term this is nothing but the convolution of $f_1(t)$ and $f_2(t)$.

(Refer Slide Time: 19:05)



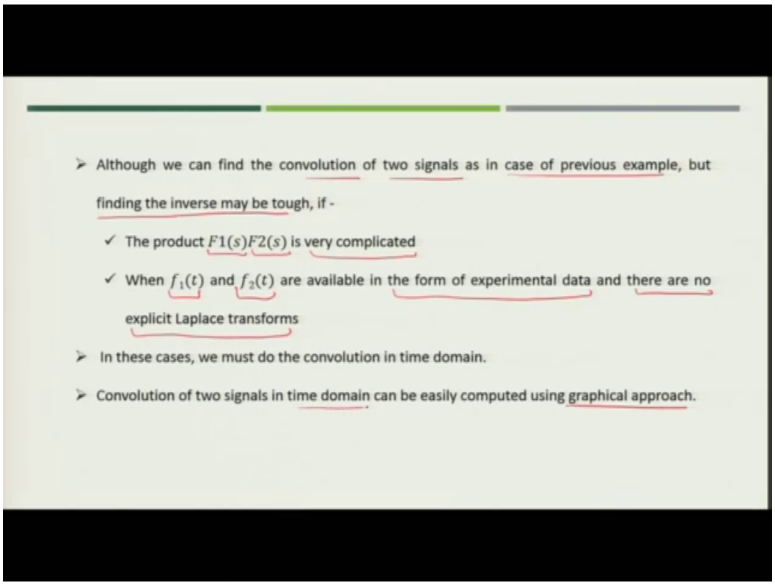


If you see this, you will simply represent it as f1 maybe t convolution of f2 t and then the outer integral is nothing but the Laplace of the convolution of f1 and f2. So you can simply write $F_1(s)F_2(s)$ is nothing but Laplace of the convolution of f1 and f2. So, you can simply say that the convolution in time domain is equivalent to the multiplication in s domain. This is how you get when you rearrange the terms and justify that the convolution is, convolution of f1 and f2 is multiplication in s domain.

Now let us take one example so that you can understand how you will utilise this particular concept. Let us see that $x(t) = 4e^{-t}$and $h(t) = 5\,e^{-2t}$ and when we apply the above property the convolution of h and x is the inverse Laplace transform of $H(s)X(s)$. What is $H(s)$? $H(s)$ becomes $\frac{5}{s+2}$ because it is Laplace transform of $h(t)$. Similarly, $X(s)$ is nothing but Laplace transform of $x(t)$ so you can simply write it as $\frac{4}{s+1}$.

Now you if you rearrange the term within the square bracket will $\left(\frac{20}{s+1}\right) + \left(\frac{-20}{s+2}\right)$. If you take now the inverse Laplace transform of the term which is there in the square bracket, it will become $20(e^{-t} - e^{-2t})$. So, now you can say the convolution of $x(t) = 4e^{-t}$and $h(t) = 5$ e^{-2t} is $20(e^{-t} - e^{-2t})$. This would be when $t \geq 0$. So, with this you can simply see that when you apply the Laplace transform of the convolution you can easily get the answers.

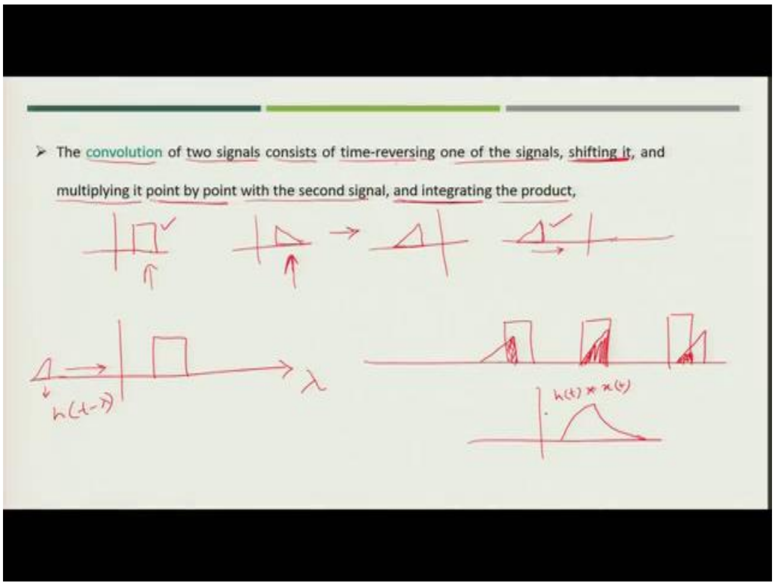
(Refer Slide Time: 21:50)

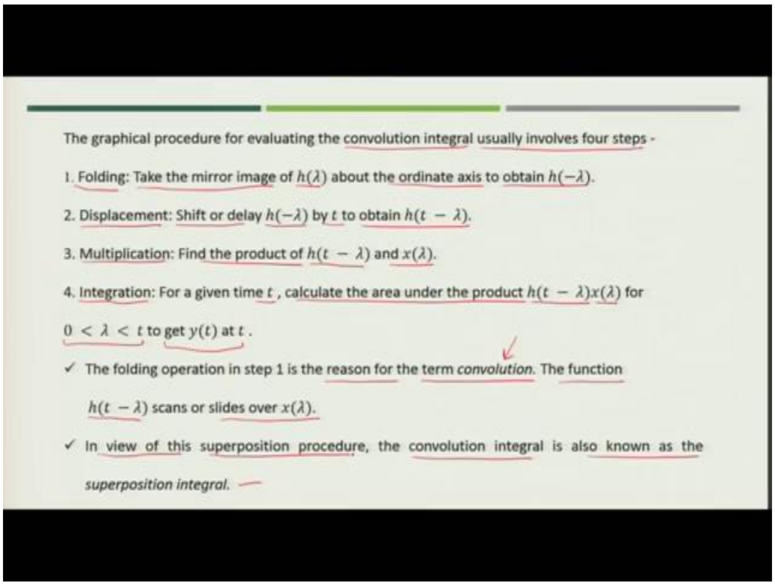


Now there is another point which we must remember with respect to this property which we have just now calculated. Because although this convolution of two signal which we saw in the previous cases looks simple but sometimes finding the inverse maybe tough, why because the moment when you have $F_1(s)$ and $F_2(s)$ the product of both is complicated then it would be difficult to utilise the concept of convolution using Laplace transform. Then what you will do, you will use some alternate technique.

Another case is that when $f_1(t)$ and $f_2(t)$ are available in the form of some experimental data and there is no explicit Laplace transform of these two functions available. When this particular condition happens you will not be able to get the Laplace transform of $f_1(t)$ and $f_2(t)$ and then getting the convolution of f1 and f2 using the Laplace transform product, would be difficult. So, what you have to do, you have to use the alternate approach that is called the graphical approach to find the convolution of two signal in time domain.

(Refer Slide Time: 23:20)



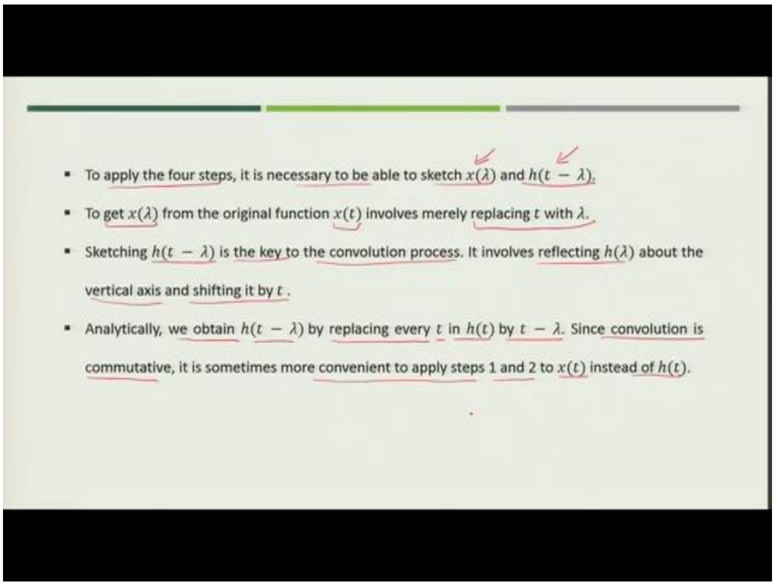


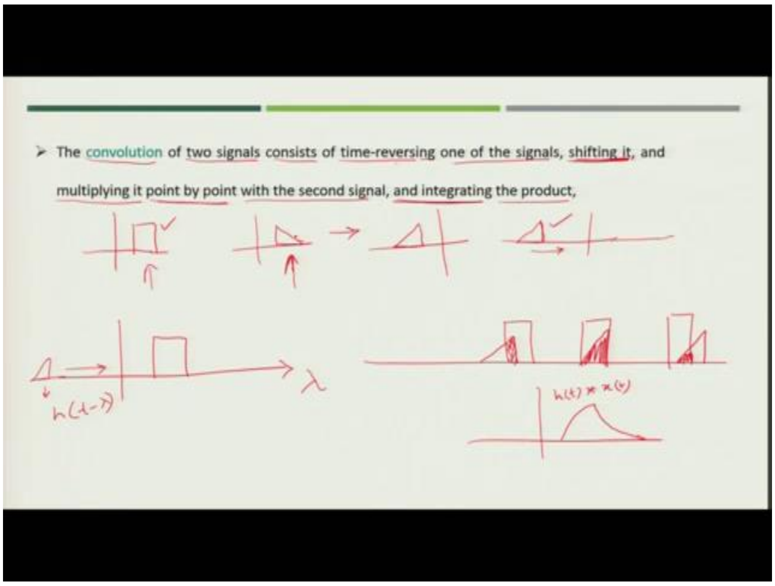
What is the graphical procedure, graphical procedure says that the four steps which we are describing below will be followed to find out the convolution integral of two functions. First is folding, folding means you will take the mirror image of $h(\lambda)$ about the ordinate axis and obtain the value of $h(-\lambda)$. Now you must displace the function so that means shift or delay $h(-\lambda)$ by t to obtain $h(t - \lambda)$.

Now next task is that you must multiply, find the product of $h(t - \lambda)$ and $x(\lambda)$ and then integrate for a given time t and calculate the area under the product $h(t - \lambda)x(\lambda)$ for $0 < \lambda < t$ to get $y(t)$ at t and then you will get finally the value of $y(t)$. Now the folding operation in step one is the reason of the term convolution. So, this is what we discuss in the first slide that folding that is nothing but taking the mirror image is the reason of calling this integral as convolution integral.

Now the functions $h(t - \lambda)$ scans or slides over $x(\lambda)$, so this what we saw in the example which we were discussing that when you slide over the given signal and you take the product of each term and then you sum it up so that you get the value of integral. It means that you will utilise the shifting function which will scan or slide over the $x(\lambda)$ to get the final integral. So, in view of this the super position procedure because this also super position of function $h(t - \lambda)$ over $x(\lambda)$, the convolution integral is also known as the super position integral. So, you can now see that the convolution integral can also be called as the super position integral because it contains the super position procedure also.

(Refer Slide Time: 26:10)





So, to apply the four steps, it is necessary to be able to sketch $x(\lambda)$ and $h(t - \lambda)$. For the graphical approach we have to keep in mind that we can sketch both of the functions and means we can plot both of the function on x y axis and then get the $x(\lambda)$ from the original function $x(t)$, this involves merely replacing t with a dummy variable λ. Then sketching $h(t - \lambda)$ is the key to the convolution process because it involves reflecting $h(\lambda)$ about the vertical axis and shifting it by t.

So analytically we obtain $h(t - \lambda)$ by replacing every t in the $h(t)$ by $(t - \lambda)$. Now since the convolution is commutative it is sometimes more convenient to apply step one and two to $x(t)$ instead of $h(t)$. So, we must look at the functions x and h and we can with the help of just understanding the sketch of x and h, we can easily find out which one we have to fold and slide over the other one.

So, like in the previous case what we did that we folded this particular function, if we fold we will not be able to differentiate between original and the folded function and we will not be able to explain it so that is why I folded this function. Similarly, when you calculate the convolution using the graphical approach you can see which one is the best for the folding. This you have to always keep in mind when you are using the graphical approach. So now we will take one example so what we can do, we can stop our discussion today at this point and we will continue our discussion on convolution integral in the next lecture also. Thank you.