Basic Electric Circuits
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Module 7
Circuit Analysis Using Laplace Transform
Lecture – 31
Laplace Transform of Circuit Elements

Namaskar, so in this week we will see how we can utilize the Laplace transform into circuit analysis.

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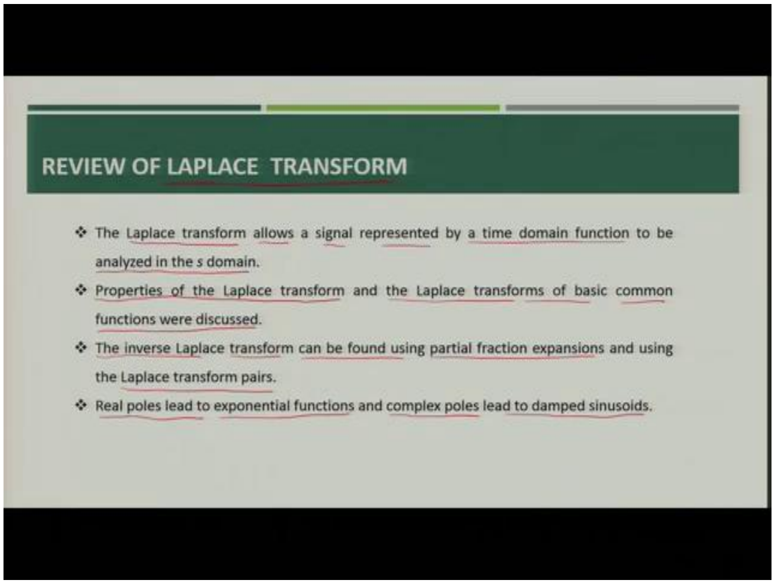

So, let us start our discussion before going into the detail lets first understand what we discussed about the Laplace transform. So, basically the Laplace transform allows a signal represented by a time domain function to be analyzed in the s domain. So, basically if you have signal which have some function in time domain that can be represented in s domain for analysis. Now, properties of the Laplace transform and the Laplace transform of basic common functions were discussed in the previous classes.

Then we discussed the inverse Laplace transform and we said that the inverse Laplace transform can be found using partial fraction expansion or using Laplace transform pairs technique. And then we also discussed that real poles lead to exponential functions and complex poles leads to damped sinusoids. And we also discussed the initial and final value theorems also through which we can find out the initial and final conditions required for circuit analysis. So, these are the key points which we discussed when we are discussing about the Laplace transform.

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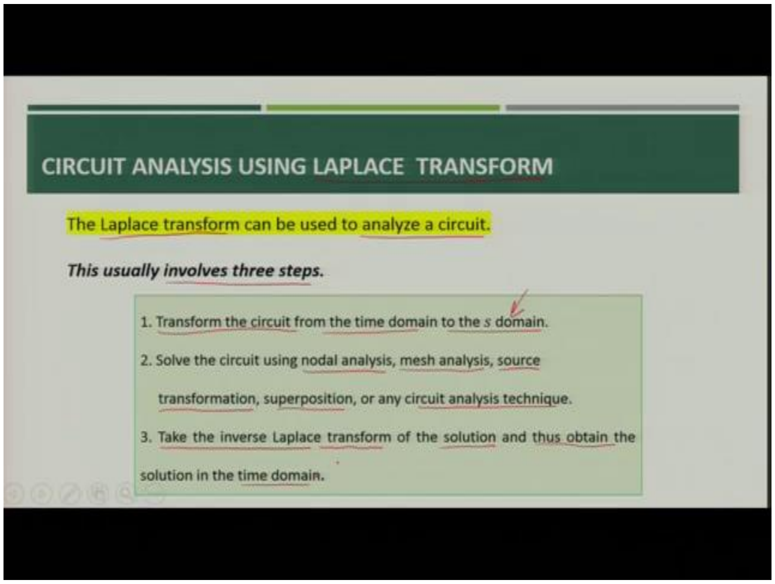

Now, let us talk about how we will do the circuit analysis using Laplace transform. So, the Laplace transform is one of the best techniques for analyzing an electrical circuit. It involves three major steps, one is that you transform the circuit from time domain to the s domain, it means that whatever the circuit elements you have in the electrical circuit all will be first transformed into s domain.

And then we will solve the circuit using the circuit analysis techniques which we discussed in the previous classes those are like nodal analysis, mesh analysis or may be source transformation, superposition, or any circuit analysis techniques which we discussed this includes your Thevenin's and Norton's equivalent's also.

The second step will be that whatever the circuit we have formed in s domain we will utilize the circuit analysis technique to solve the circuit in s domain and finally, when we get the answer in s domain, we will use inverse Laplace transform for the solution and finally we will obtain the solution in in time domain. These are the three major steps we will follow when we do the circuit analysis. So, first thing which we have to understand is that how we can convert the various circuit elements from time domain into s domain.

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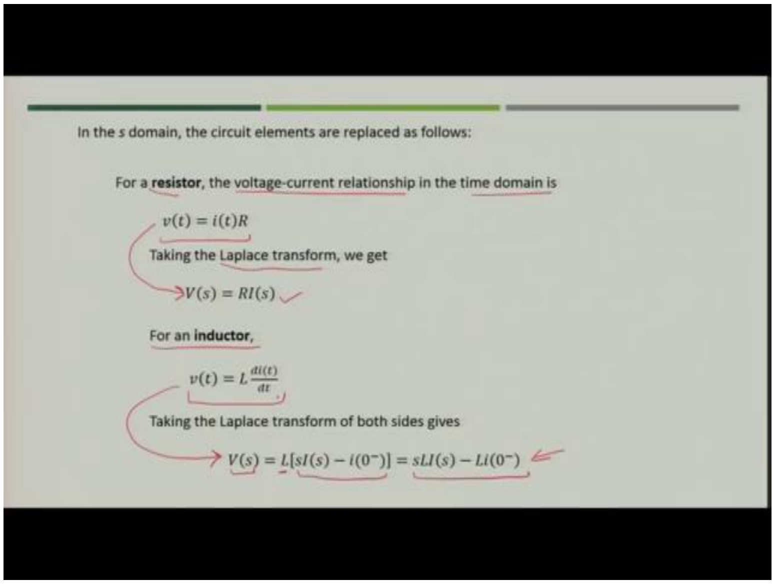

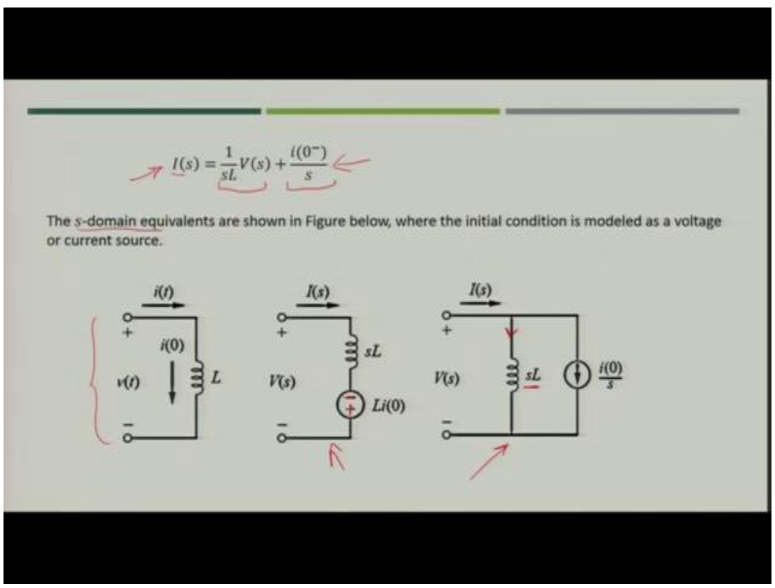

Let us first see the three key elements which we generally have in our circuit those are resisters, inductors and capacitors. How can we transform the three elements into the s domain? For resistor the voltage current relationship in time domain, we know that it is given by $v = iR$ as per ohms law. Now, if you take the Laplace transform we get the time domain equation getting converted into s domain.

So, $v(t)$ will become $V(s)$, R is a constant quantity and there is no change, and $i(t)$ will be converted into $V(s)$. This we get when we convert resistor into frequency s domain.

Now, for inductor as we know that the voltage across inductor can be given by this equation that is $L\frac{di}{dt}$. Now, we will use time differentiation technique which we discussed in the previous classes when we were discussing about the various properties of Laplace transform. We will convert this time domain equation for inductor into Laplace domain that is s domain.

So, $v(t)$ will become $V(s)$. Now, L will remain same being a constant quantity, but in case of $\frac{di}{dt}$ we will use time differentiation technique. So, we will get Laplace transform of $\frac{di}{dt}$ as $sI(s) - i(0^{-})$. So, we can say that the Laplace transform of inductor voltage can be given by $sLI(s) - Li(0^{-})$.

So, this we get in the s domain next is how represent that equation in the circuit. We know that if you have $V(s) = L[sI(s) - i(0^{-})] = sLI(s) - Li(0^{-})$ at the same time you can also calculate the value of current $I(s)$ with the help of the same equation. So, you will rearrange the equation in such a way that you get $I(s) = \frac{1}{sL}V(s) + \frac{i(0^{-})}{s}$. So, now we have the inductor voltage as well as inductor current in s domain.

Now, we will convert these equations into an equivalent circuit. So, if you see in time domain we have a inductor which is carrying some current I at any time t with initial current as i at 0 and voltage across inductor is v at any time t. So, this is the time domain representation of inductor voltage and current. Now, if we have to convert them into s domain $v(t)$ will become $V(s)$ and $i(t)$ will become $I(s)$.

Now, for the inductor L what we will do? We know that $V(s) = sLI(s) - Li(0^{-})$. To represent the minus sign the polarity of fictitious voltage source that is that will represent the initial condition for inductor will become l at time t is equal to 0 and the polarity will be opposite.

So, the equation for voltage in s domain for the inductor will become sL that is the inductor in series with voltage source equal to l at l into I at 0 and with negative polarity on top and positive in the bottom. So, this you can see that will represent the equation which we just derived while we were transforming time domain into s domain.

Similarly, for current. Now, you know that if the voltage across inductor is $v(t)$ will be represented as $V(s)$ in s domain and similarly, current $i(t)$ will be represented as $I(s)$. Now, if you see this equation so from this you can see that current $I(s)$ is converted, is divided into two parts so we will use that equation to find out the equivalent circuit in case of we want to find out current $I(s)$.

So, from first the value what we get is $i(0)/s$. So, this will the value of a current source that will represent the initial condition that is initial current flowing through the inductor. So, $i(0)/s$ the direction will be downward because it is positive so, current$I(s)$ will give you the same direction for fictitious current source which you will add form representing the initial condition of current flowing through the inductor.

Now, next the value of current flowing through the inductor. So, this would be nothing but $\frac{1}{sL}V(s)$. So, it means that this value of inductor will be sL. So, in this way we can represent the inductor either for in the form of voltage $V(s)$ or in the form of current $I(s)$. So, these are required whenever you are solving the circuit equation. This would be usually utilized when you are using the Kirchhoff voltage law and this can be utilized when you are utilizing Kirchhoff current law for converting the circuits.

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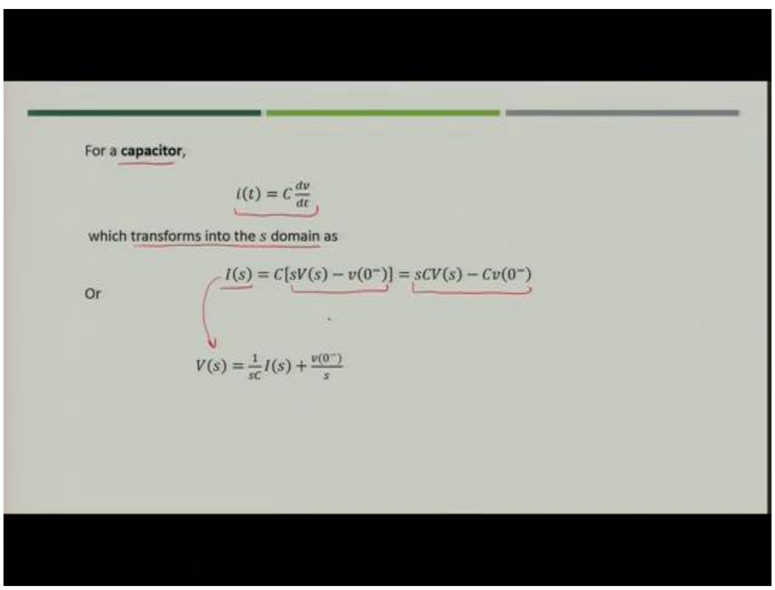

Now, next let us talk about the capacitor so, in case of capacitor we know current at any time t can be given by $i(t) = C\frac{dv}{dt}$. So, using the equation in time domain we will transform this into s domain. When we convert it into s domain it will become

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

Now, in the same way if you rearrange the, this this equation or this expression in terms of voltage

$$V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s}$$

So, these two would be the governing equations for capacitor in s domain. So, we can represent as we did in case of inductor, we can do in case of capacitor also and represent these two equations in the circuit.

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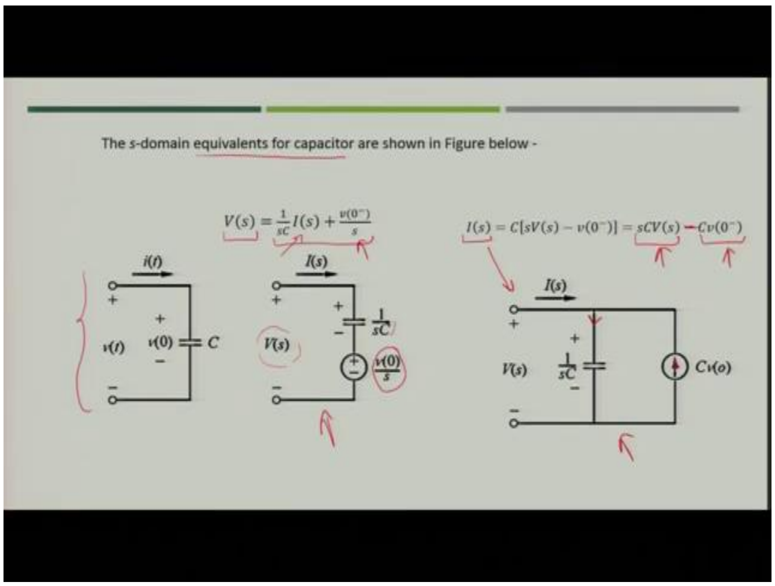

So, let us see how we will represent the equivalents of capacitor in s domain. So, this is the (equal) equivalent representation of capacitor in time domain where you have voltage across capacitor as $v(t)$, initial voltage across capacitor is $v(0)$.

Now, when we are asked to find out the voltage $V(s)$ in s domain across the capacitor so, we will use this equation and you can simply see that it contains two terms and both are since it is the voltage so, both terms can be a voltage terms and since it is a positive it means that a positive voltage source that is plus polarity on the top and negative at the bottom will be the value so, this will be the fictitious voltage source that is v at time is equal to 0 by s which will represent the initial condition that is the voltage across capacitor at time is equal to 0.

Now, $\frac{1}{sC}$ will be the value of the capacitor here because when you say the current is $I(s)$ flowing through the circuit so $\frac{1}{sC} I(s)$ will be the drop of across capacitor plus the voltage that is (v) v0 at v at time t equal to 0 and by s and when you sum up you will get the value at v in s domain. So, this will be required when you are having the Kirchhoff voltage law to be applied.

Now, next you can also represent the current $I(s)$ in terms of these two terms when you these two terms these are nothing but the current terms because $I(s)$ is the current. $I(s)$ is divided in two parts, so when we represent it in circuit the first term you will see the values 1 upon Sc and when you divide $\frac{1}{sC}$ by $V(s)$ you will get $sCV(s)$ which will be nothing but the current flowing through this leg of the circuit.

Next is C into V at time t=0 is again a current term which is a constant value because this will represent the voltage across capacitor at time t=0 and since, this having a negative sign the direction of current would be opposite of the direction of the current $I(s)$. So, in this you can represent equivalently the equation for Is and this you can utilize whenever you see the circuit to be analyzed with the help of Kirchhoff's current law. So, these voltage and current equations you can use whenever it is required to solve the circuit according to your convenience.

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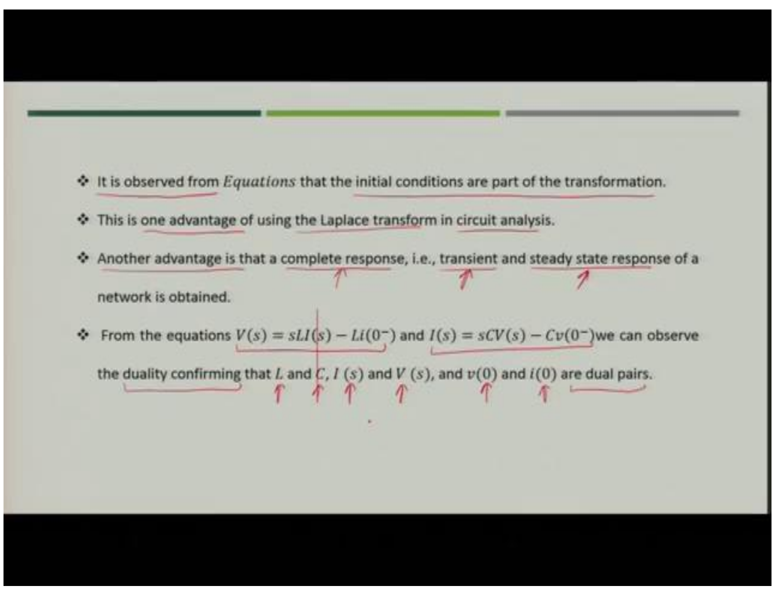

Now, from these equations we observe that the initial conditions are the now part of the transformation so we need not to worry about how we will incorporate with the initial conditions in s-domain because the s-domain equations take care of initial conditions also. Now this is the one of the biggest advantages of using Laplace transform in circuit analysis.

Now, another advantage is that this will give you a complete response which will include transient and steady state response of the circuit. So, if you remember when we are discussing about the step response of maybe first order, second order circuit we converted the solution into natural response and the first response and the found both response separately and finally we sum then up to get the final response. But in this case, we need not to worry about having two solutions analyze separately and then summing up because the Laplace transform and the circuit analysis using Laplace transform will directly give you the complete response of the circuit.

Now, if you see the closely these two equations that is $V(s) = sLI(s) - Li(0^-)$ and $I(s) = sCV(s) - Cv(0^-)$. When you observe then you can confirm the duality because if you see both equations you will see that L is dual of C, $I(s)$ is dual of $V(s)$ and $v(0^-)$ is dual of current $i(0^-)$. So, these create the dual pairs.

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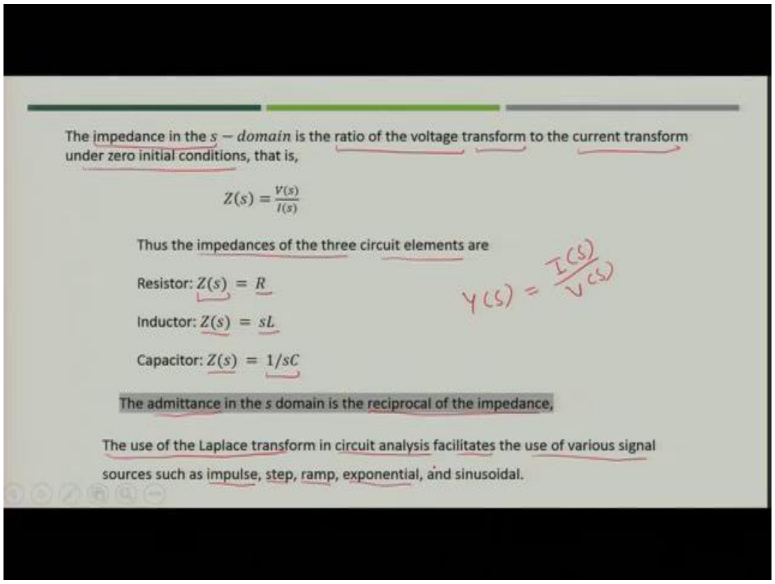

Now, when we talk about the impedance in s-domain so impedance would be nothing but the ratio of voltage Laplace transform and current Laplace transform under zero initial conditions. So we can write impedance $Z(s) = \frac{V(s)}{I(s)}$. So, the impedance of these three circuit elements which we discussed will become $Z(s) = R$ for resistance, for inductive it will be $Z(s) = sL$, for capacitor the $Z(s) = 1/sC$.

Now, if you want to find out the admittance, admittance would be the reciprocal of the impedance. So, in case of $Z(s)$ if you want to find out the admittance that is $Y(s) = \frac{I(s)}{V(s)}$. And similarly, the values of like conductance the inverse of inductor and capacitor will change. Now the use of Laplace in circuit analysis will facilitate the use of various signal sources.

So, like impulse response, step response, ramp and exponential source response can be utilizing for analyzing the circuit. So now, let us take some examples so that we can understand how we will utilize the circuit elements and their Laplace transform in the overall circuit analysis.

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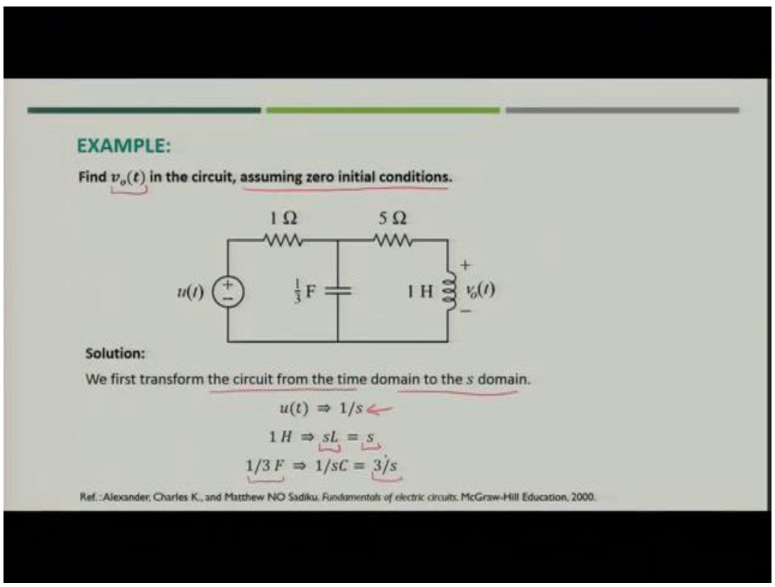

Let us take one example where we need to find out $v_o(t)$ across the inductor and we assume that there is no initial current through the inductor or initial voltage across the capacitor, so it will have the 0 initial condition. Now, what we must do first? First, we have to transform the circuit from time domain into s-domain, so the source which is unit step function when you will convert into s-domain it will become $1/s$, inductor will become the inductor is sL and value of L is 1 so it will become s. The $1/3\ F$ farad capacitor will have the value as $3/s$ in s-domain.

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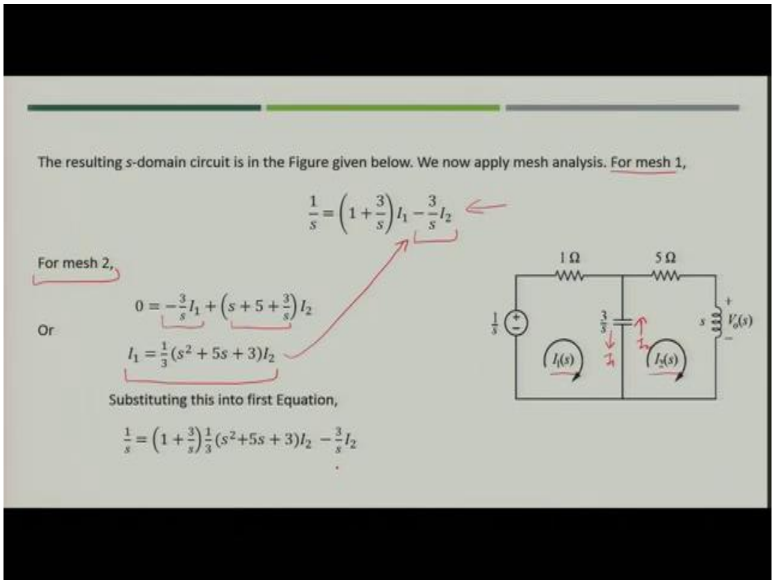

So now we have all these elements in s-domain we can represent this circuit in s-domain now. So source will now become $1/s$ there will be no change in the resistances so these will remain same, capacitor has now become $3/s$ and inductor has become s and we need to find out first the voltage across the inductor in s-domain and then we will convert it into time domain.

Now, if you see the circuit you can utilize the mesh analysis so let us say that the current $I_1(s)$ is in the first loop, $I_2(s)$ is in loop 2. For mesh 1 you can write the equation as

$$\frac{1}{s} = \left(1+\frac{3}{s}\right)I_1 - \frac{3}{s}I_2$$

For mesh 2, what will happen? You will sum up all the impendences so it will become $s+5+\frac{3}{s}$. For I_2 you will get the $s+5+\frac{3}{s}$ and then since I_1 is in opposite direction of I_2 for capacitor you will get another term for capacitor that is $-\frac{3}{s}I_1$.

So now, when you simplify this you will the get the value

$$I_1 = \frac{1}{3}(s^2+5s+3)I_2$$

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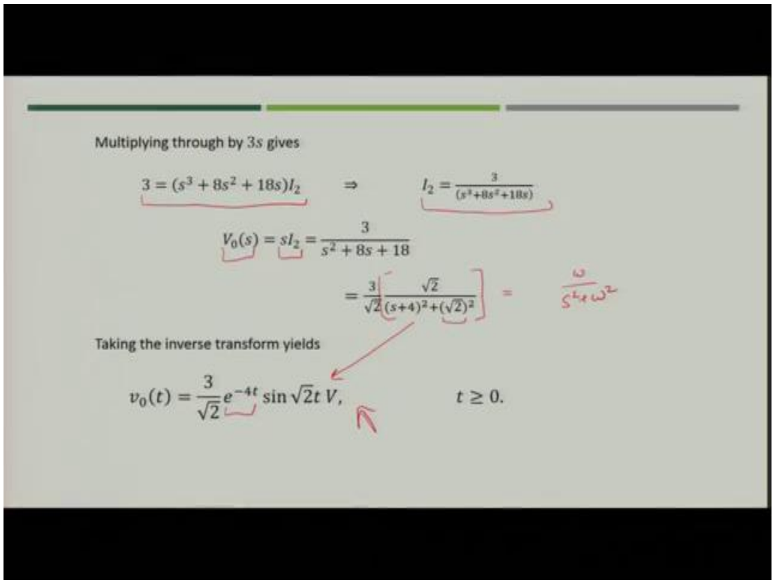

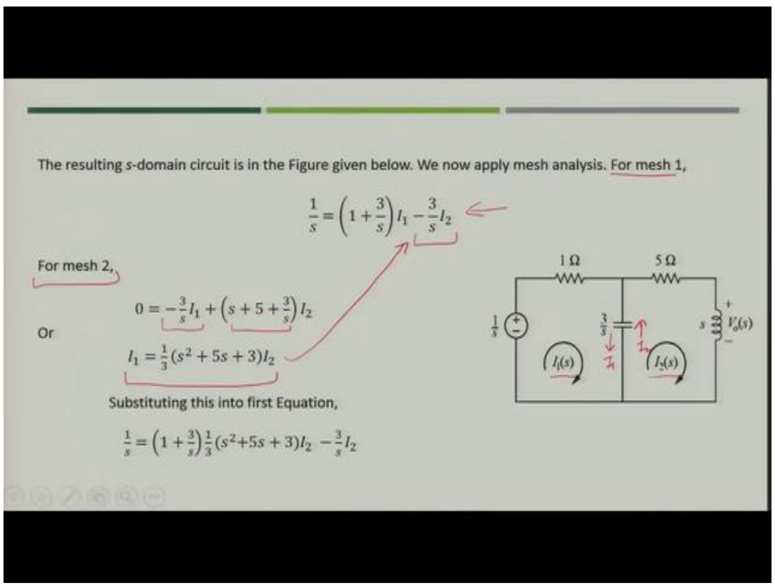

Now, if you simplify

$$3 = (s^3 + 8s^2 + 18s)I_2 \qquad \Rightarrow \qquad I_2 = \frac{3}{(s^3 + 8s^2 + 18s)}$$

Now, to solve it you will utilize the creating the square technique. So, you will first create the square of these terms, so it will become $(s+4)^2 + (\sqrt{2})^2$. Now, in numerator you can multiply by $\sqrt{2}$ and in denominator you can have $\sqrt{2}$. So, if you see this component what you can see?

You can see this is nothing but the Laplace transform of $e^{-4t}\sin\sqrt{2}t$. Hence, with this you will get the voltage response across the inductor.

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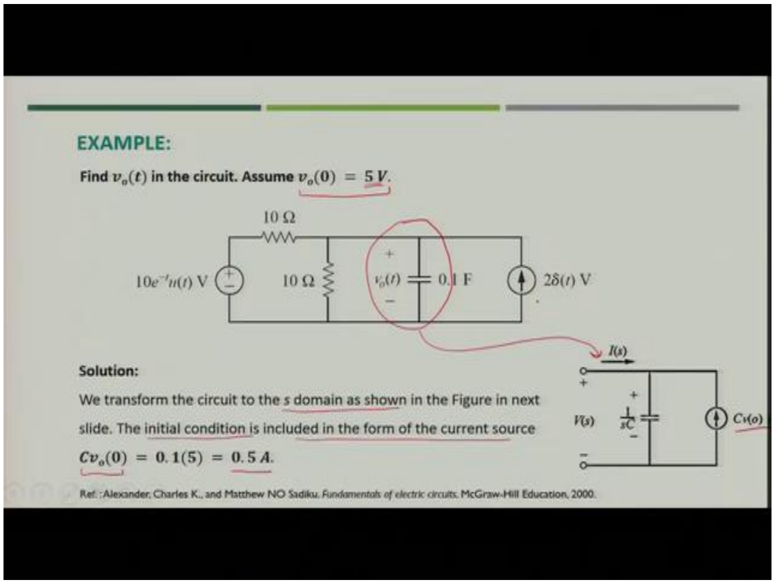

Now, let us see another case, when you have the initial condition that is voltage across the capacitor and the value is 5 volts. We will first transform the circuit into s-domain. But we have to first see that initial condition which is given will have the current source $Cv_o(0)$ because if you see these all are connected in parallel it means that Kirchhoff's current law would be most suitable option.

So now when you see Kirchhoff's current law means you must convert this capacitor into equivalent s domain. So now this particular term will be converted into $1/sC$ and in parallel with you will have current source value of $Cv_o(0)$, $v_o(0)$ is given as 5 volt, C is given 0.1 F. Hence,

$$Cvo(0) = 0.1(5) = 0.5\,A$$

Now, using this particular circuit you can transform the complete circuit in the S-domain.

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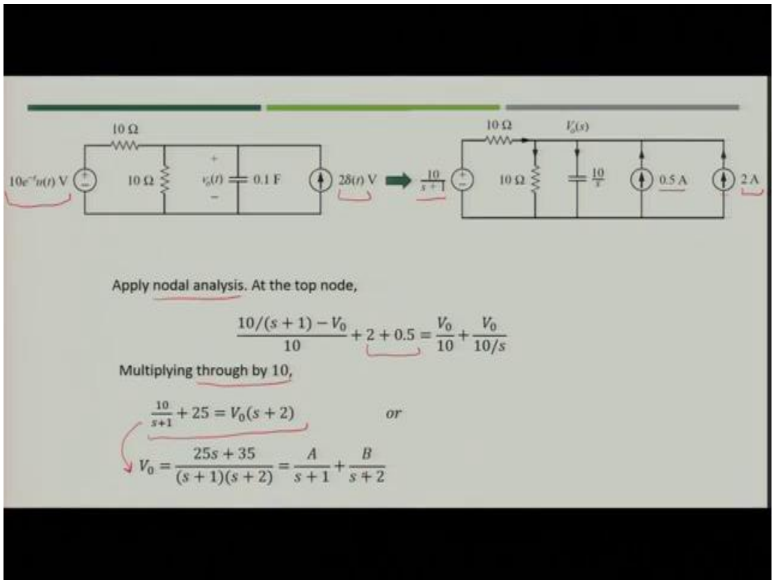

So it will be converted in s domain as shown in the figure. The unit step function multiplied by the exponential means you are getting the time shift. So, the Laplace transform of the voltage source will become $10/(s+1)$ and then resistances will remain same, the capacitor now will get converted into $10/s$, and in parallel width you will have one current source having 0.5 ampere as value because this will represent initial condition. And then you have current sourcewhich on converting into Laplace transform will become 2 ampere.

Now, what you have to do? You have to apply the Kirchhoff current law at the top node. So, you will see 1, 2 and 3 are the currents which are going inside the node and 2 are coming out of the node. So, when you compile incoming and outgoing currents you can say that

$$\frac{10/(s+1)-V_0}{10}+2+0.5=\frac{V_0}{10}+\frac{V_0}{10/s}$$

Here, the outgoing current is $\frac{V_0}{10}$ and the here outgoing current will be $\frac{V_0}{10/s}$. Now, if you multiply both sides by 10 and rearrange you will

$$\frac{10}{s+1}+25=V_0(s+2) \qquad or$$

$$V_0=\frac{25s+35}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$$

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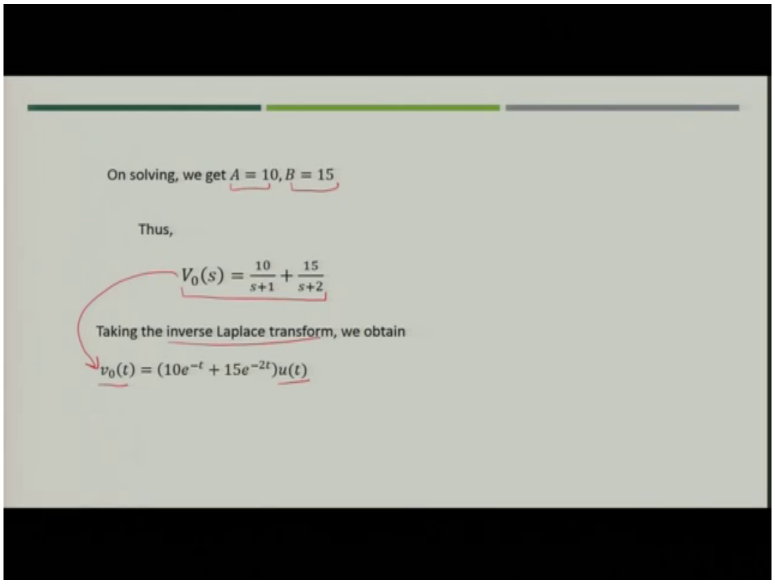

When you solve you can get the value of $A = 10,\ B = 15$ and

$$V_0(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

This can easily be written in terms of inverse Laplace transform you will get

$$v_0(t) = (10e^{-t} + 15e^{-2t})u(t)$$

So now, with this we can close our today's session. So, in this we discussed about various circuit elements and how you can convert them into Laplace transform and we will talk about now the transfer function in the next class, thank you.