Basic Electric Circuits
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Module 06
Laplace Transform and its Application
Lecture 30
Inverse Laplace Transform

Namashkar. In the last class, we were discussing about the various properties of Laplace transform. So, in today's class, we will discuss about the Inverse Laplace Transform.

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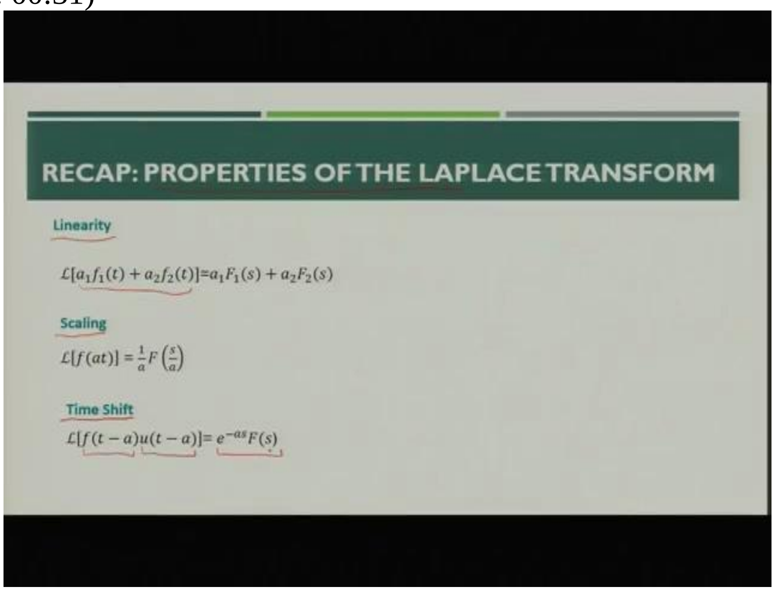

So, let us start the discussion of today's lecture. Before going into the inverse Laplace transform, let us understand what we discussed in the last class related to the properties of the Laplace transform. First we discuss about linearity where, $\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$. Scaling property, where, $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$. Now, in case of time shift if your signal is shifting by some time, say a, the Laplace transform of that function would be $e^{-as} F(s)$.

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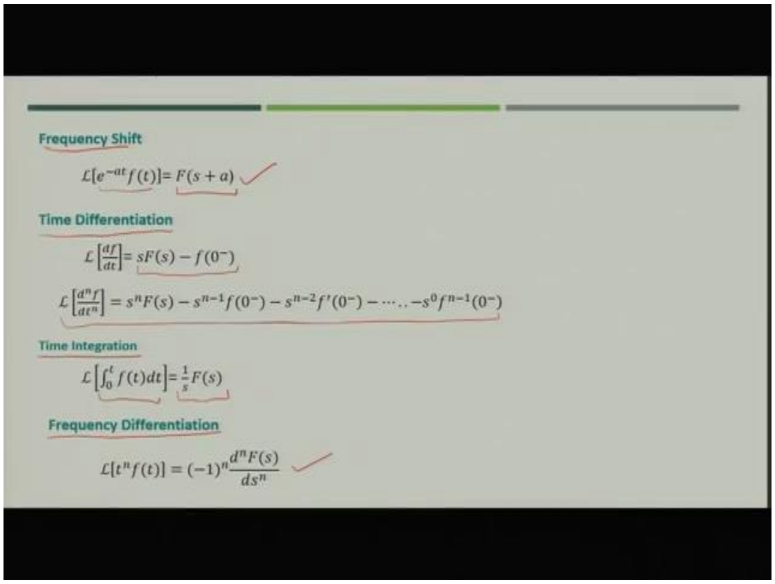

Now, then, we discussed about frequency shift. So, it means that, if we have a function $e^{-at}f(t)$, the Laplace transform of that would be $F(s+a)$. That means that wherever we have s in case of the Laplace transform of $f(t)$, we will replace it by $s+a$. Then we discussed about the time differentiation, means that the derivative of f, $\frac{df}{dt}$, then $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$.

So, we can generalize also by saying that the nth derivative of the function, then the Laplace transform of nth derivative of f would be,

$$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots ..-s^0f^{n-1}(0^-)$$

So, this will give you the Laplace transform for nth derivative of our function.

Then, we discussed about the time integration. If you have a function, $\int_0^t f(t)dt$ and you are asked to find the Laplace transform of the integral, it is Laplace transform would be $\frac{1}{s}F(s)$, where $F(s)$ is the Laplace transform of function $f(t)$.

Now, frequency differentiation, if you have a function $t^nf(t)$, so its Laplace transform would be $(-1)^n\frac{d^nF(s)}{ds^n}$. So, particularly this function and this, the Laplace Transform, in case of frequency shift and frequency differentiation will be used most frequently when we will do the inverse Laplace transform. So, all those, the properties which we have discussed, you must keep in mind because these will be used frequently in the, calculation of inverse Laplace transform.

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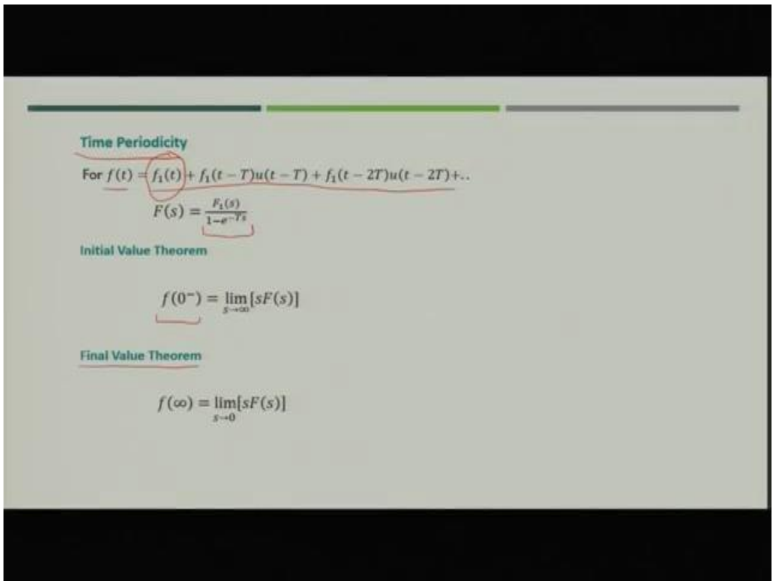

Now, next the time periodicity, we discuss that if the function can be given by a periodic series $f_1(t) + f_1(t-T)u(t-T) + f_1(t-2T)u(t-2T)+$... you can represent our periodic function, as a combination of various functions. And the first function is the, the value of this periodic function at first time period. And then others will be like compilation of the first function with time shift conditions. So, this is time shifted by T then time shifted by 2T and so on.

Now, if you are asked to find out the Laplace transform of $f(t)$, you will say $F(s) = \frac{F_1(s)}{1-e^{-Ts}}$, where $F_1(s)$ is the Laplace transform of $f_1(t)$. We also discussed initial value and final value theorems. We calculated that if we want to find out the initial value of a function, the initial value can be directly given by $\lim_{s\to\infty}[sF(s)] = f(0^+)]$. Similarly, in case of final value theorem, we discussed that the value of function at infinity will be $f(\infty) = \lim_{s\to 0}[sF(s)]$.

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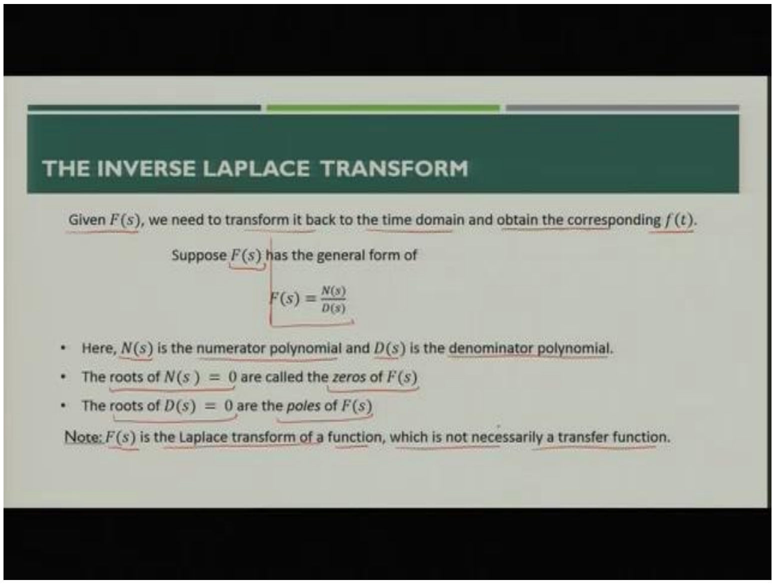

Now, these we have discussed in the previous class. So, let us start the discussion about the inverse Laplace transform. In this case, we are given the value of *F(s)* and now we need to transform it back to the time domain and obtain the corresponding value of *f (t)*. So, if you have *F(s)* in the general form like $F(s) = \frac{N(s)}{D(s)}$, where *N(s)* is the numerator polynomial and *D(s)* is the denominator polynomial.

Now, roots of *N(s)* equal to 0 are called the 'zeroes of transfer function *F(s)*' and roots of *D(s)* equal to 0 polynomial are called the 'poles of function, transfer function *F(s)*'. Now, we must keep in mind that *F(s)* is Laplace transform of one function which cannot necessarily be a transfer function. So, this we must keep in mind, so that we are not confused with the transfer function and the Laplace transform.

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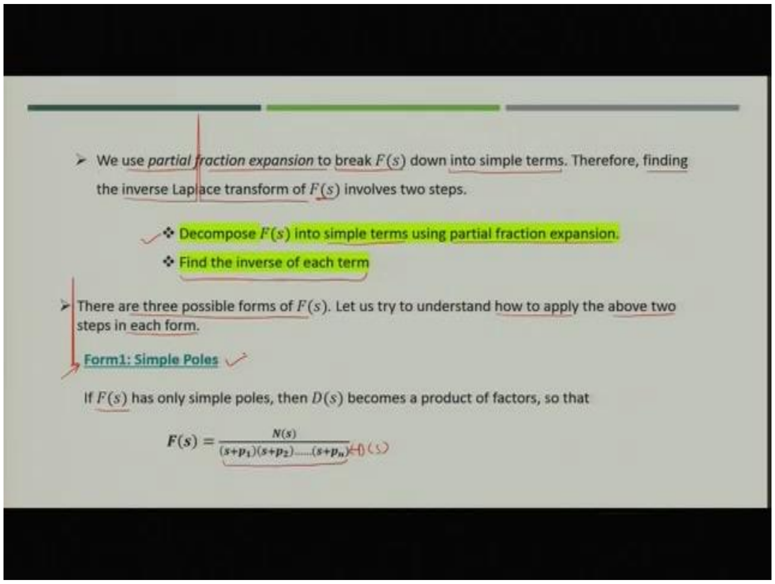

To find out the, the value of function *f(t)*, we use partial fraction expansion method to break the function *F(s)*. So, when you break the function *F(s)*, you will break into simpler terms, so that you can easily find the inverse Laplace transform of *F(s)*. So, what are the steps are involved in the computation of inverse Laplace transform. We use two steps. One is, we will first decompose the function *F(s)* into simple terms using partial fraction expansion. And then we find the inverse Laplace transform of each term, which we have found in the first step. Now, there are three possible forms of the Laplace transform.

First form is when you have simple poles. If you have a Laplace transform *F(s)*, which is represented as a numerator polynomial divided by denominator and where you can have the denominator as shown in equation (1). The denominator polynomial is in the form of product of factors $(\boldsymbol{s}+\boldsymbol{p_1})(\boldsymbol{s}+\boldsymbol{p_2})\ldots\ldots(\boldsymbol{s}+\boldsymbol{p_n})$. Then you can use the simple pole approach to find out the Inverse Laplace Transform.

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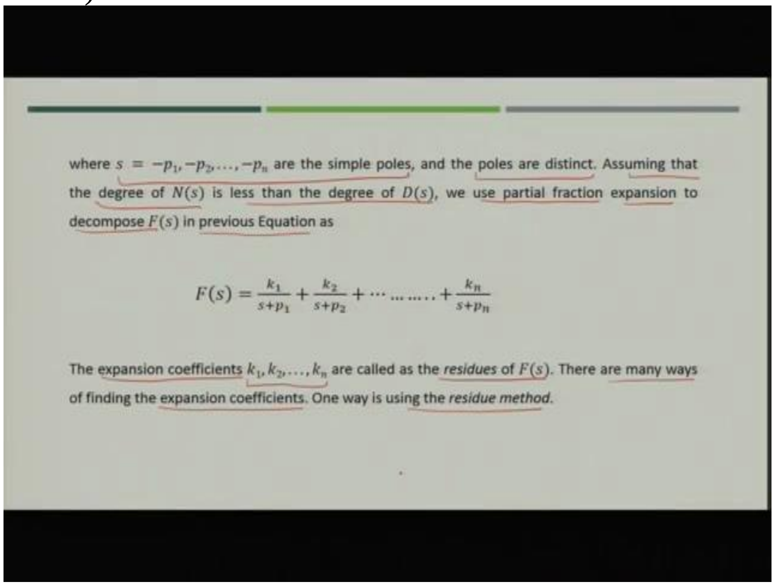

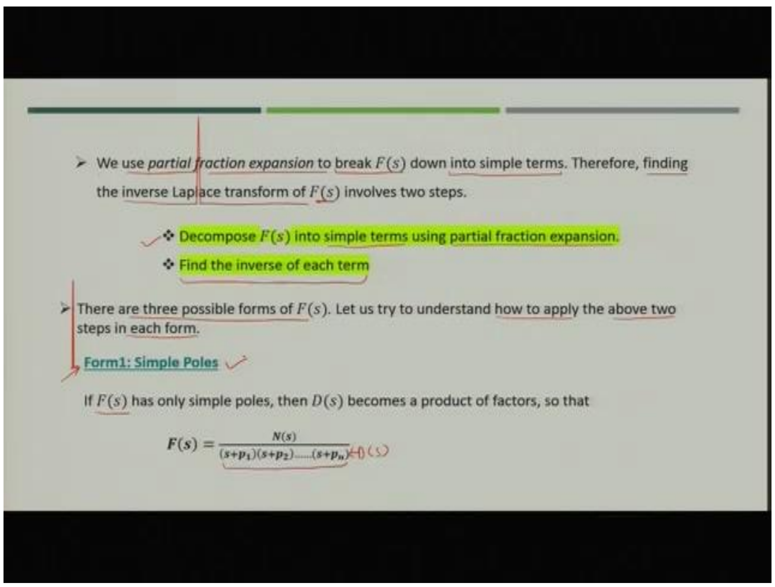

Now, what are the p1, p2 and pn. So, these are the simple poles of the Laplace Transform function. Now, these poles are distinct. Now, if we assume that the degree of *N(s)* is less than the degree of *D(s)* that means the highest degree of s in denominator is greater than the highest degree of s in numerator we can apply the partial fraction expansion method to decompose the Laplace Transform which was shown in the previous equation.

So, this is the Laplace Transform and we want to decompose it. How will decompose? We can represent this function as decomposed terms like

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots \ldots \ldots . . + \frac{k_n}{s+p_n}$$

where expansion coefficients $k_1, k_2, \ldots, k_n$ are known as the *residues* of *F(s)*. Now, there are many ways through which you can find these values of expansion coefficients. The one, the most popular technique is residue method.

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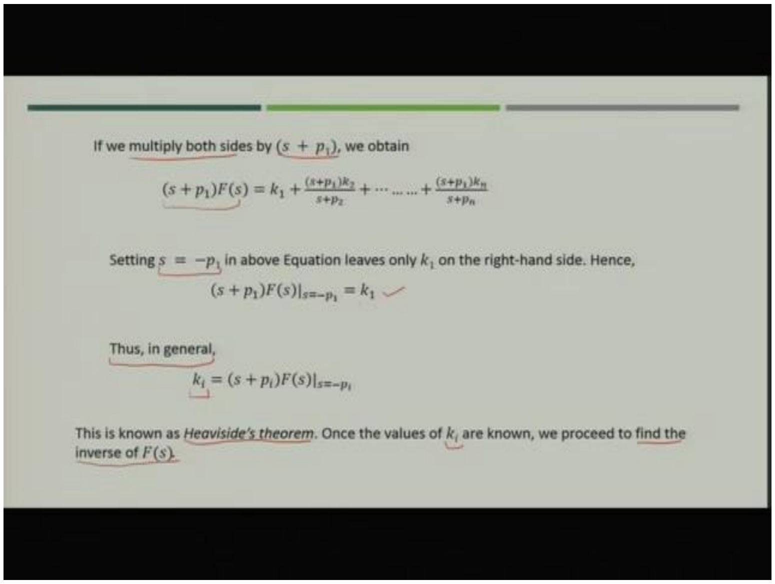

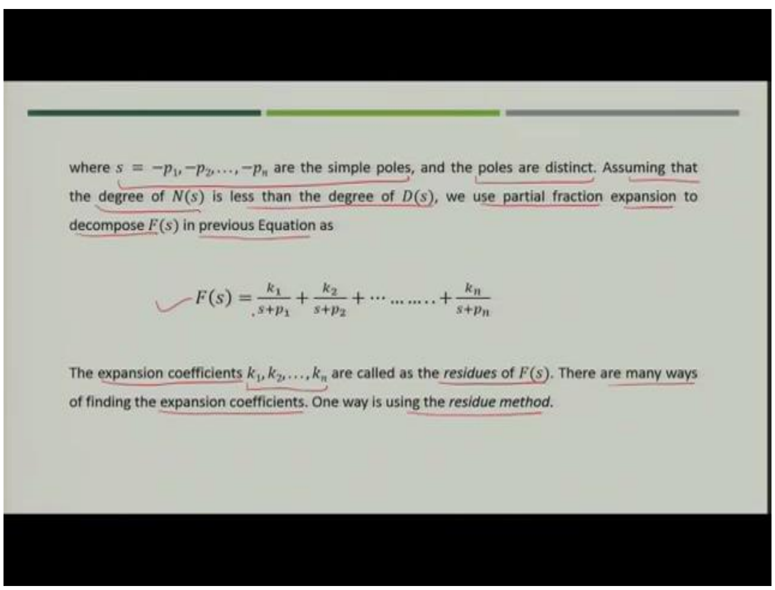

Now, in this, what we will do, we will multiply both sides by s plus p1. So, if you see the expression for Laplace Transform, which we decompose into various terms, so that you can solve it. The first term has the denominator *(s* + *p*$_1$*)*. So, what we will do first, we will multiply both side by *(s* + *p*$_1$*)*. So, when you multiply, this will become

$$(s+p_1)F(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \cdots \ldots \ldots + \frac{(s+p_1)k_n}{s+p_n}$$

Now, if you set the value of $s = -p_1$ at both sides, the right side terms other than the k_1 will become 0.

So, you can simply say that

$$(s+p_1)F(s)|_{s=-p_1} = k_1$$

The whole product of   this would be calculated at $s = -p_1$. So, when you solve, you will get the value of k_1. So, now in more general form, you can simply write for any factor

$$k_i = (s+p_i)F(s)|_{s=-p_i}$$

So, this will give you the value of constant term k_i. Now, this particular property is called the 'Heaviside Theorem'.

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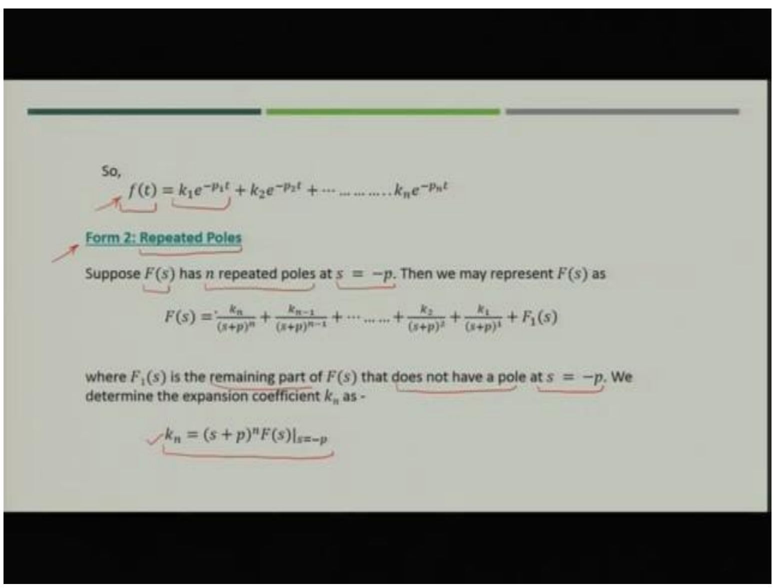

Now, once the value of k_i are known, we can easily find out the inverse Laplace transform for *F(s)*. How? Because now we know that it is decomposed into this form. Now, if you see this, this is nothing, but the Laplace transform of $k_1e^{-p_1t}$. So, this is nothing but frequency shift property. So, if you recollect the frequency shift property, you can simply find out the value of inverse Laplace transform of first term.

So, this will become $k_1 e^{-p_1 t}$ and similarly, for other times also you can find out the inverse Laplace transform. So finally, when you club all of them, you will get the inverse Laplace transform of the function *F(s)*. That means the inverse Laplace transform of *F(s)* is

$$f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots \ldots \ldots \ldots . . k_n e^{-p_n t}$$

Now, second case is that when you have repeated poles, means suppose if the Laplace Transform *F(s)* has *n* repeated poles at $s = -p$. Then what you can write, you can write simply,

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots \ldots \ldots + \frac{k_2}{(s+p)^2} + \frac{k_1}{(s+p)^1} + F_1(s)$$

The last $F_1(s)$ is the remaining part of $F(s)$, which does not have pole at $s = -p$. This is the remainder of the, the Laplace Transform, which is not represented in this form. Now, as we did in the previous case here also, we can find out the value of k_n. So, k_n would be given as,

$$k_n = (s + p)^n F(s)|_{s=-p}$$

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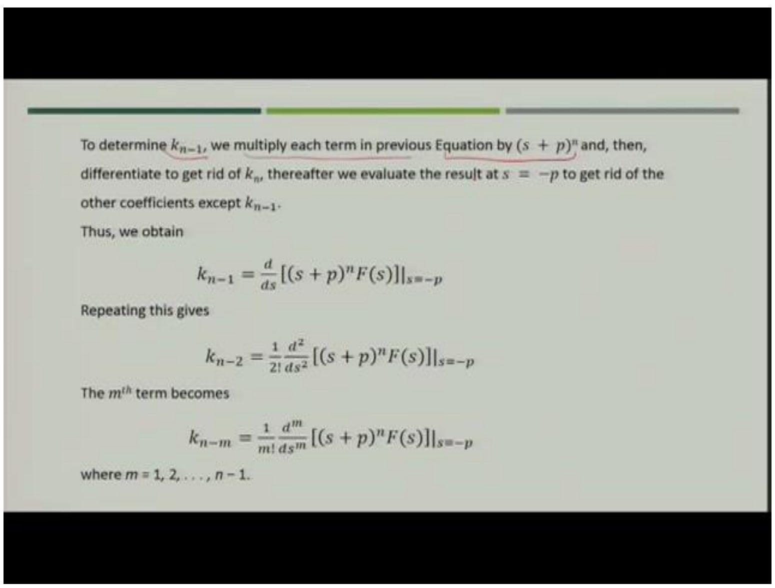

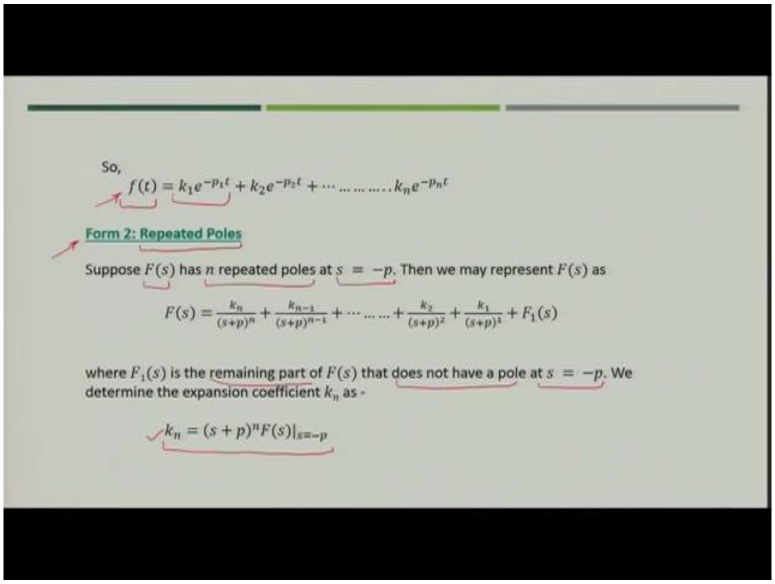

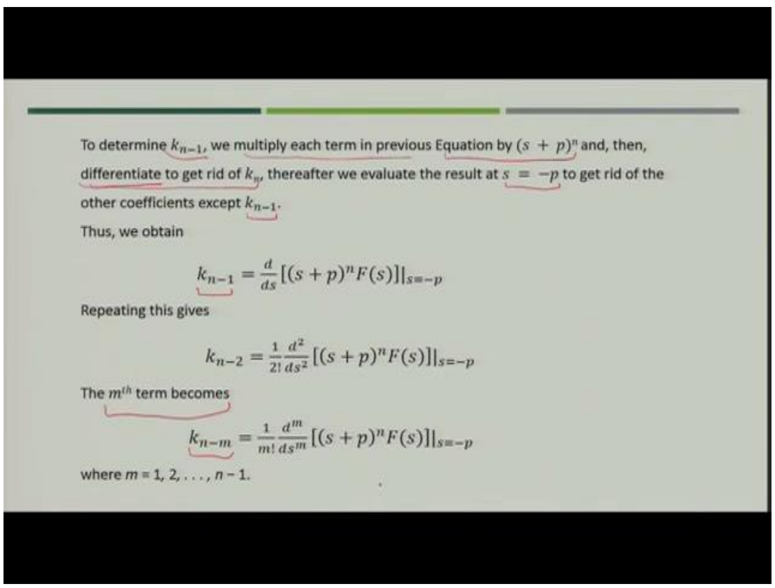

Now, how to find the value of k_{n-1}? For finding out the value of k_{n-1}, we must multiply each term by $(s + p)^n$ as we did in the previous case. Now, in this case, you will have k_n as a constant term, but we need to find out k_{n-1} because k_n we have already calculated. So, what we can do, we can get k_n, if you differentiate the whole expression.

So, we get,

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)]|_{s=-p}$$

Similarly, for others also you

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]|_{s=-p}$$

The *m*th term becomes

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)]|_{s=-p}$$

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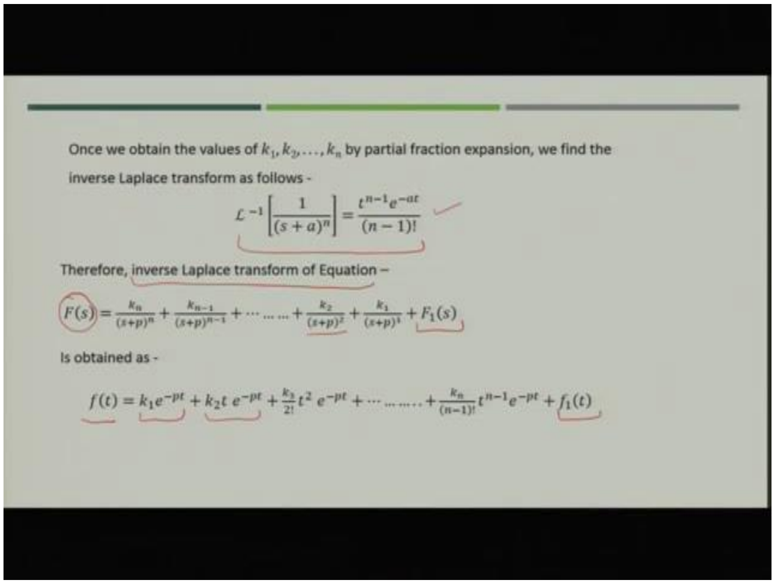

So, now you have all the constants in your hand. Now, if you remember the inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!}$$

This is nothing but with the help of frequency, differentiation property, you can find out the value of inverse Laplace transform of $\frac{1}{(s+a)^n}$.

So, if you compile the individual terms, for first one, you can simply write $k_1 e^{-pt}$, for second you can write $k_2 t\, e^{-pt}$ and for third term similarly, you can write $\frac{k_3}{2!}t^2\, e^{-pt}$ and so on. So, for the kth term you will, for the nth term, you will write $\frac{k}{(n-1)!}t^{n-1}e^{-pt}$ plus $f_1(t)$. That is the inverse Laplace transform of reminder of the function that is $F_1(s)$. So, with this way, you can find out the inverse Laplace transform or function $f(t)$.

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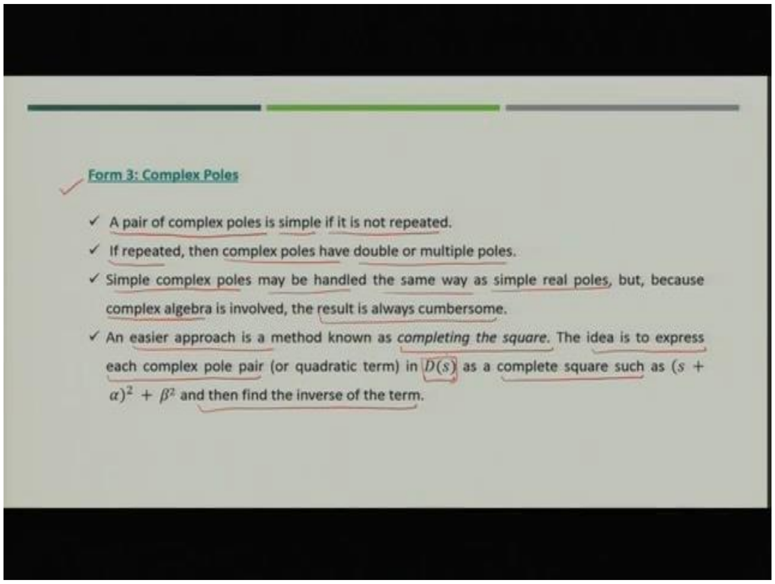

Now, we have a third case, where we have complex poles. Now, pair of complex poles is simple, if it is not repeated and if it is repeated, then complex poles have double or multiple poles. Now, simple complex poles maybe handled the same way, we handle the simple real poles. But since we have a complex Algebra involved, the result may be a little bit cumbersome.

So, we will use an alternative approach to find out the inverse Laplace transform in case we have complex poles. So, the approach is the method which is known as completing the square. Now, what does it mean? This is the idea to express each complex pole pair in the denominator as a complete square such as $(s + \alpha)^2 + \beta^2$. So, whatever we have in the denominator, we can represent them as set of complete square plus remainder of the term which are there in the denominator. Then we can easily find the inverse Laplace transform of the function $F(s)$.

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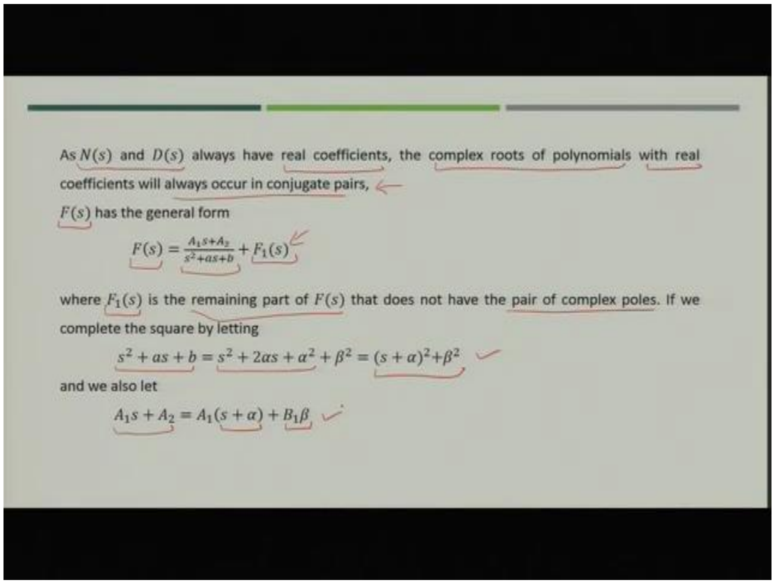

What does it mean, like how we will solve it? Let us take one simple example, so that you can understand how we can proceed for this kind of condition. Since *N(s)* and *D(s)* always have real coefficients. So, the complex roots of the polynomial with real coefficients will always occur in conjugate pairs. So, we will keep this in mind and try to solve the, try to find out the inverse Laplace transform, *F(s)*.

Suppose the inverse Laplace transform *F(s)* is given to you and you want to find out its inverse Laplace transform. *F(s)* can be decomposed into two parts. First is

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s)$$

where you do not have any complex poles. So, $F_1(s)$ is the remaining part of $F(s)$, which does not have pair of complex poles.

Now, what you must do next, in the denominator, you have term $s^2 + as + b$. You need to complete the square in the denominator. So, we add few terms in the polynomial and subtract few of them in the polynomial and rearrange in such a way that it looks like $s^2 + 2\alpha s + \alpha^2 + \beta^2$.

So, if you can rearrange the polynomial in such a way, you can simply say that this particular polynomial can be represented as $(s + \alpha)^2 + \beta^2$. Similarly, in the numerator also, you can arrange the polynomial in such a way that $A_1 s + A_2 = A_1(s + \alpha) + B_1 \beta$.

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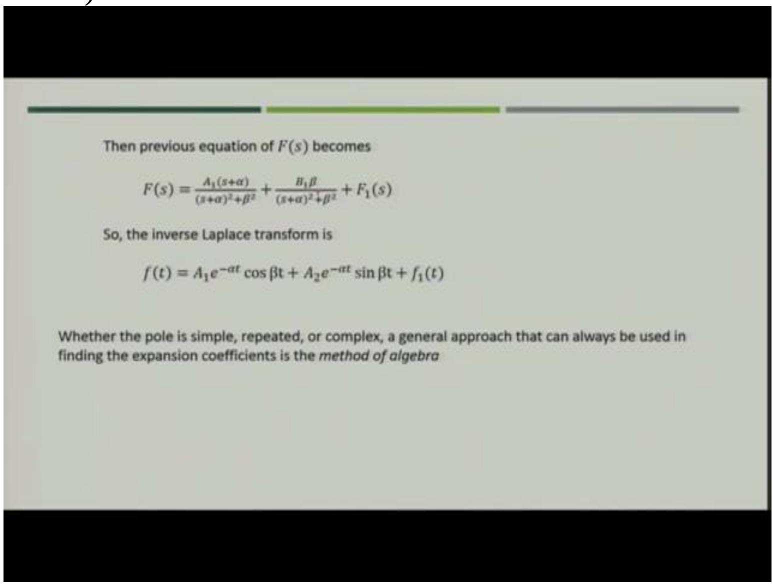

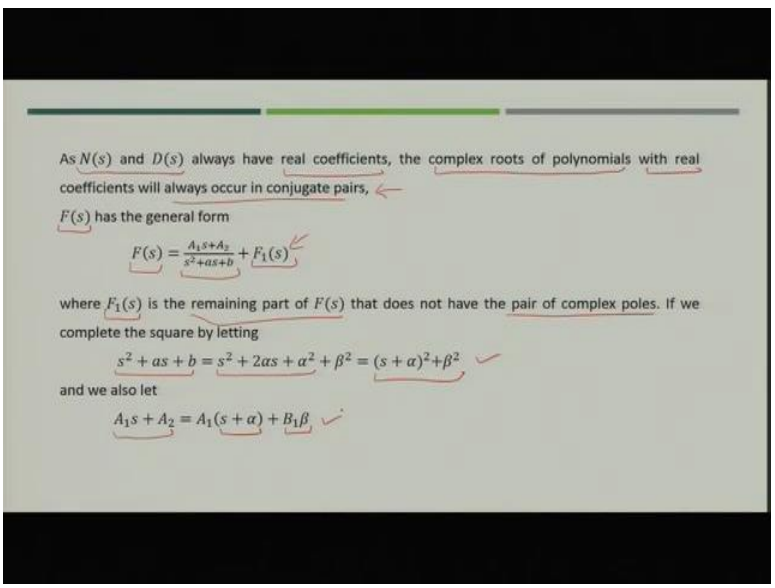

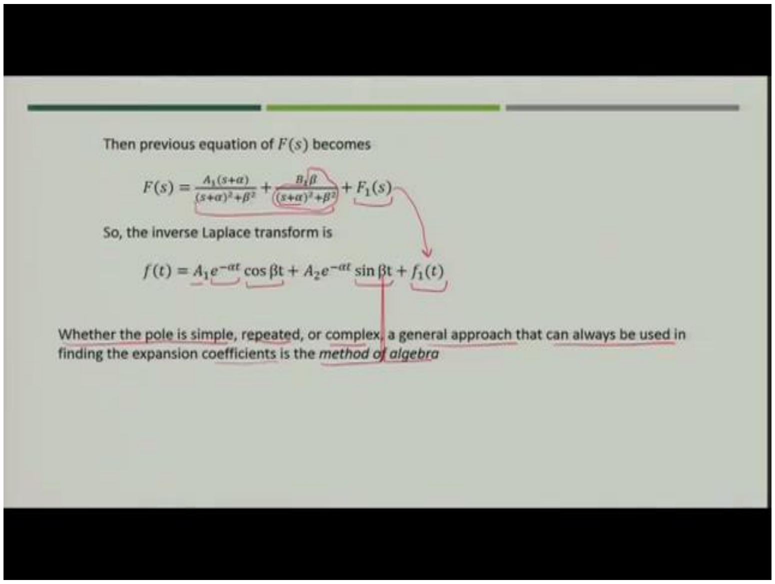

So, when you rearrange numerator as well as denominator, what you get, you can simply represent the function

$$F(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{B_1\beta}{(s+\alpha)^2+\beta^2} + F_1(s)$$

Now, if you see the first term, if you remember when we, when we were discussing about the Laplace transform, we discuss that $\cos wt \Leftrightarrow \frac{s}{s^2+w^2}$

So, using that expression for $\cos wt$ that is you can find out the inverse Laplace transform because, if you see this expression, this is nothing but the frequency shift property plus the cos βt term. So, what you can simply write, you can simply write, the inverse Laplace transform of this would be $A_1e^{-\alpha t}$ because here, you have $s+\alpha$. We will use frequency shift property and you will say that it is $A_1e^{-\alpha t}$ cos βt. Similarly, for this case also, you can write $A_2e^{-\alpha t}$ sin βt because here also you will see, that this particular segment is nothing but the Laplace transform of sin βt plus the frequency shift because here you have $s+\alpha$.

So, you will simply write $A_2e^{-\alpha t}$ sin βt $+ f_1(t)$ which is the inverse Laplace transform of $F_1(s)$. So, when you have complex poles, you can rearrange the expressions in such a way that it can be arranged in a particular fashion. Then you can easily find out the inverse Laplace transform.

So, whether the pole is simple, repeated or complex, the general approach that can always be used in finding the expression is the method of Algebra. That means that whatever did in Algebra, the same can be applied here also, if you want to find out the, the constant components in any case of the Laplace Transform, which we discussed till now.

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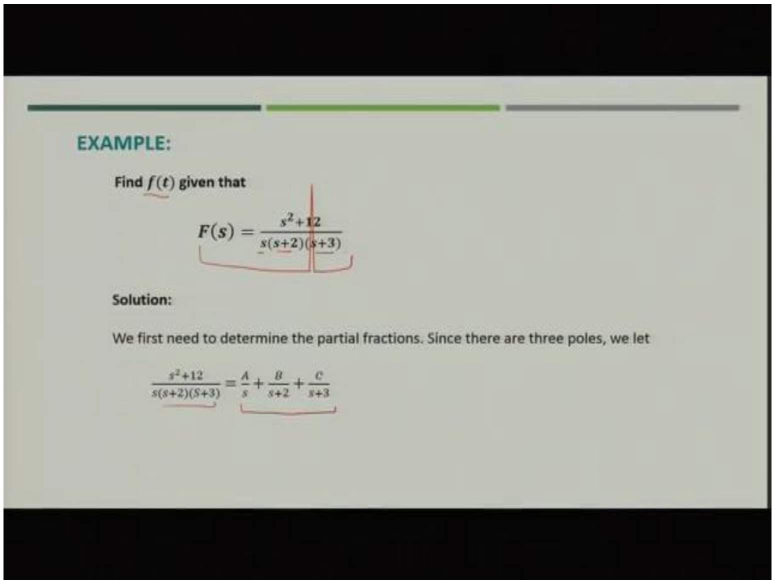

So, to understand these concepts, let us understand with the help one example, so that you are more clear about how to proceed with finding out the inverse Laplace transform. So, let us take one example. We have been given a Laplace transform $\boldsymbol{F(s) = \frac{s^2+12}{s(s+2)(s+3)}}$.

So, here you can see in the denominator, you can clearly distinguish all three poles. That is poles are at s is equal to 0, at s equal to minus 2, at s is equal to minus 3. So, this expression can be represented as

$$\frac{s^2 + 12}{s(s + 2)(S + 3)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{s + 3}$$

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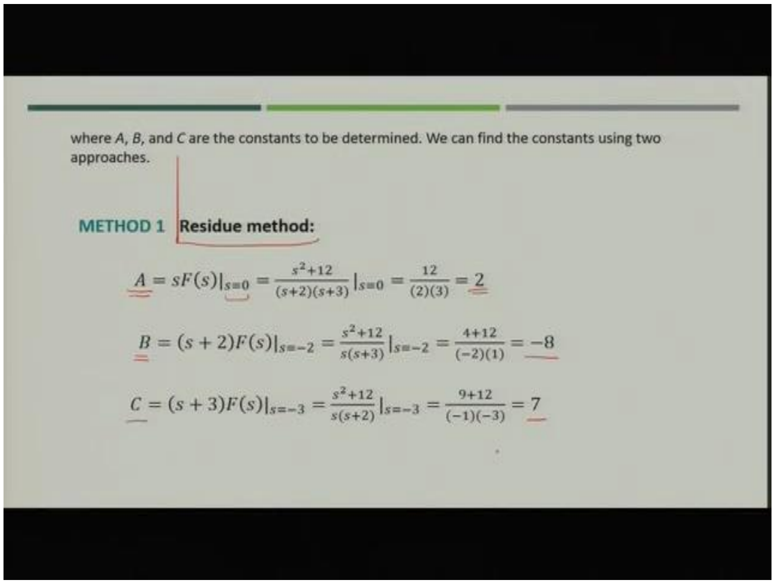

Now, next task is that you need to find out the value of the unknown constants which are A, B, C. So, we can use the residue method, which we discussed just now. So, in this case we will first multiply the Laplace transform with s and equate it for s is equal to 0.

$$A = sF(s)|_{s=0} = \frac{s^2 + 12}{(s+2)(s+3)}|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s+2)F(s)|_{s=-2} = \frac{s^2 + 12}{s(s+3)}|_{s=-2} = \frac{4+12}{(-2)(1)} = -8$$

$$C = (s+3)F(s)|_{s=-3} = \frac{s^2 + 12}{s(s+2)}|_{s=-3} = \frac{9+12}{(-1)(-3)} = 7$$

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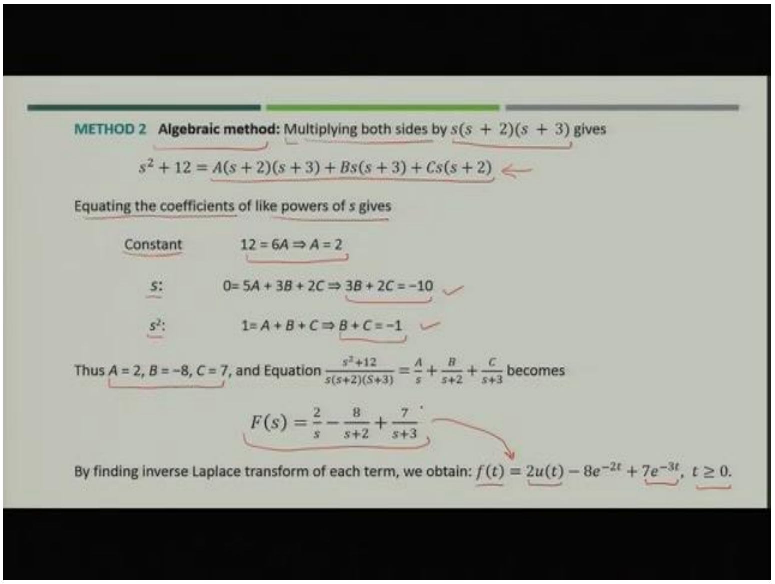

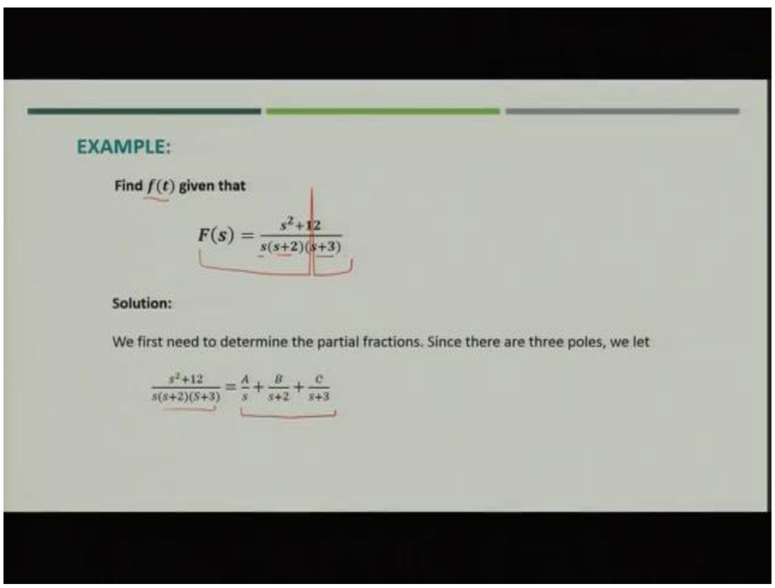

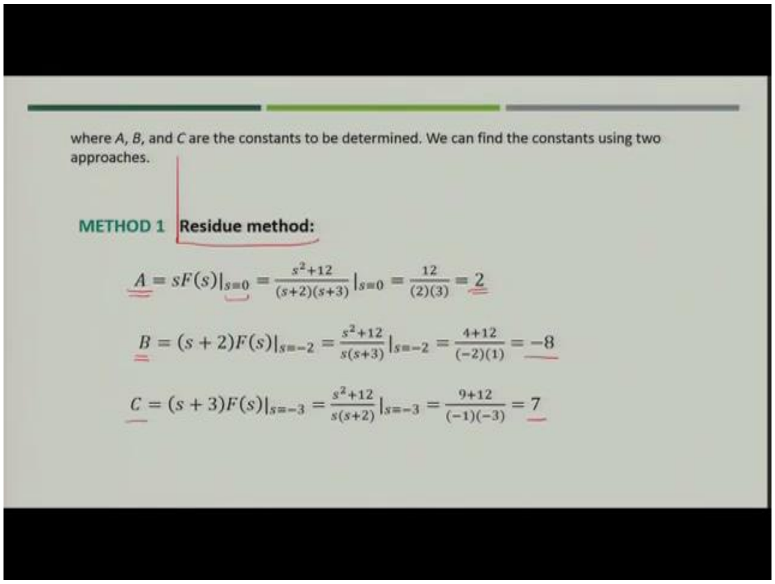

So, with this you can easily find out the inverse Laplace transform. Another method is that you can use simple algebraic method, which you generally use for solving the general expressions. So, here we will apply those technique to find out the inverse Laplace transform. So, what we can do, we can multiply both sides by $s(s + 2)(s + 3)$.

So, what we get? So, if you see the expression here, $s^2 + 12$ is there in the numerator and in the denominator, we have $s(s + 2)(s + 3)$. So, if you multiply the both sides by this, what you get, you will say that the $s^2 + 12 = A(s + 2)(s + 3) + Bs(s + 3) + Cs(s + 2)$.

So, if you see here, if you multiply both sides by $s(s + 2)(s + 3)$, so what you will get? Denominators will cancel out and the numerator, what you get is $s^2 + 12 = A(s + 2)(s + 3) + Bs(s + 3) + Cs(s + 2)$. So, this is what we got from the multiplication.

Now, what next you have to do, you have to just equate the coefficients of like powers of s. So, for constant terms, if you equate the only constant terms, you will simply get the value of $A = 2$. When you equate, only the terms having s, you can, you will get the value of expression. You will get the equation that is $3B + 2C = -10$ and when you equate the terms of s^2, you will get another equation that is $B + C = -1$.

Now, you have two equations and two unknowns, you can solve and you will get the value again as $A = 2$, $B = -8$, $C = 7$. So, either you use this method or the previous method, which we discussed that is the residue method, you will get the same results.

And finally for the, given Laplace transform $F(s)$, you can get the inverse Laplace transform by simply decomposing, you can easily say that the inverse Laplace transform of Fs is, $f(t) = 2u(t) - 8e^{-2t} + 7e^{-3t}$ provided the time $t \geq 0$, because this is applicable for $t \geq 0$.

So, now you can easily understand that how you can proceed with finding out the inverse Laplace transform. So, with this we can close our today's session, where we discuss about how to proceed with finding out the inverse Laplace transform. So, next week we will continue our utilization of Laplace transform in analyzing the various circuits.

So, remember if we, when we discussed the first order, second order circuits, we saw that there were, various second order and first order differential equations, which are sometimes difficult to solve. So, we will use the Laplace transform techniques and see how we can easily solve those kind of equations, when we see the second order or first order circuits, thank you.