Basic Electric Circuits

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Module 6

Laplace Transform and its Application

Lecture 29

Properties of the Laplace Transform

Namaskar, so in last class we were discussing about the few major properties of the Laplace transform. In this class also we will continue our journey on discussing the few important properties of the Laplace transform.

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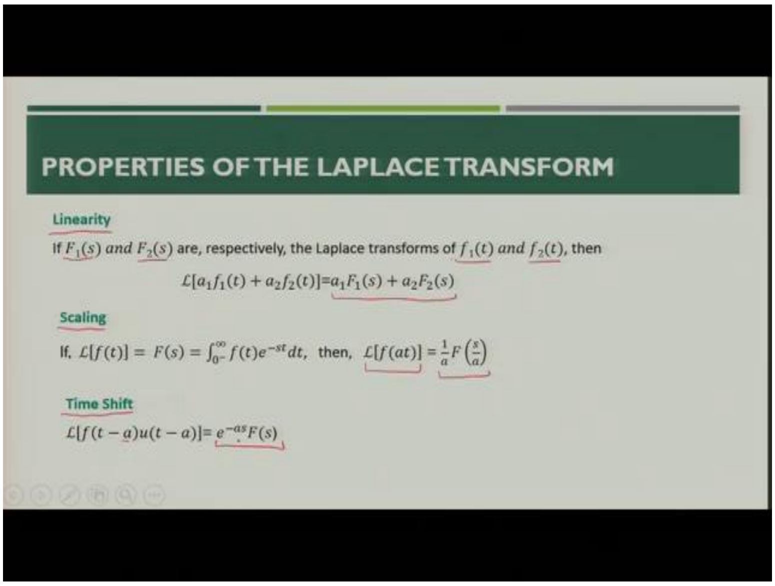

Let's see what we discussed in the previous class related to properties of the Laplace transform. We discussed about linearity, so in that we demonstrated that if the Laplace transform of $f_1(t)$ and $f_2(t)$ are $F_1(s)$ and $F_2(s)$. So, their linear combination will be like, $a_1 f_1(t) + a_2 f_2(t)$. If you take the Laplace transform of the linear term you will get,

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)]=a_1 F_1(s) + a_2 F_2(s)$$

Similarly, for scaling, the Laplace transform of $f(at)$ will be

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Now, in case of time shift if your signal is shifting by some time, say a, the Laplace transform of that function would be $e^{-as}F(s)$. So, these three we discussed in the last class.

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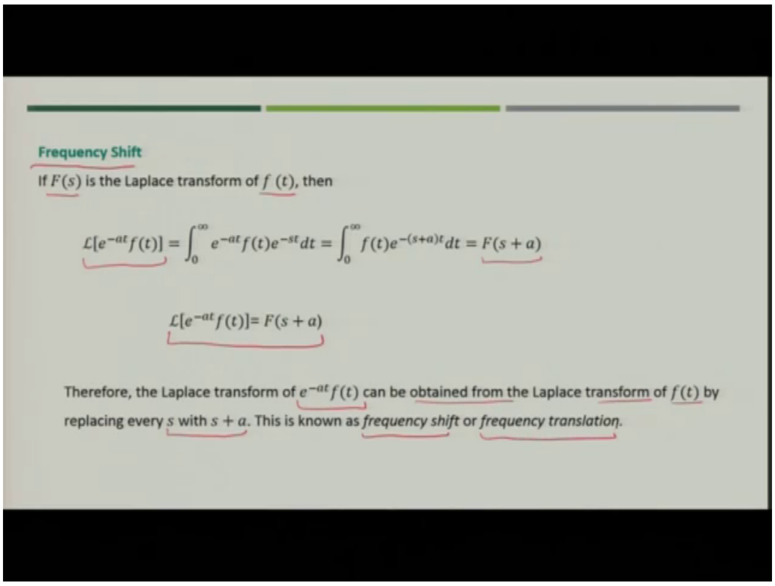

Now let us talk about the frequency shift. Now $F(s)$ is the Laplace transform of $f(t)$ then Laplace of $e^{-at}f(t)$ would be,

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

$$\mathcal{L}[e^{-at}f(t)]= F(s+a)$$

So you can say that in case of frequency shift, $\mathcal{L}[e^{-at}f(t)]$ can be obtained from the Laplace transform of $f(t)$ by replacing every s with $s+a$. So, this particular property is called as frequency shift or frequency translation.

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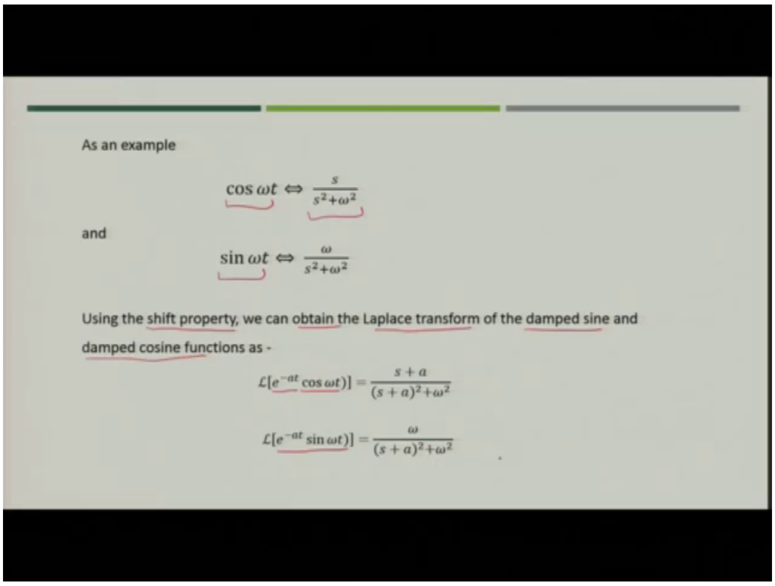

Now let's take an example. We know,

$$\cos wt \Leftrightarrow \frac{s}{s^2 + w^2}$$

and

$$\sin wt \Leftrightarrow \frac{w}{s^2 + w^2}$$

Now if we use shift property we need to obtain the Laplace transform of the damped sinusoids that is damped sine and cosine functions means we are multiplying $\cos wt$ or in $\sin wt$ with e^{-at}.

Then its Laplace transforms can be given by simply replacing s with $s + a$. So,

$$\mathcal{L}[e^{-at}\cos wt)] = \frac{s + a}{(s + a)^2 + w^2}$$

$$\mathcal{L}[e^{-at}\sin wt)] = \frac{w}{(s + a)^2 + w^2}$$

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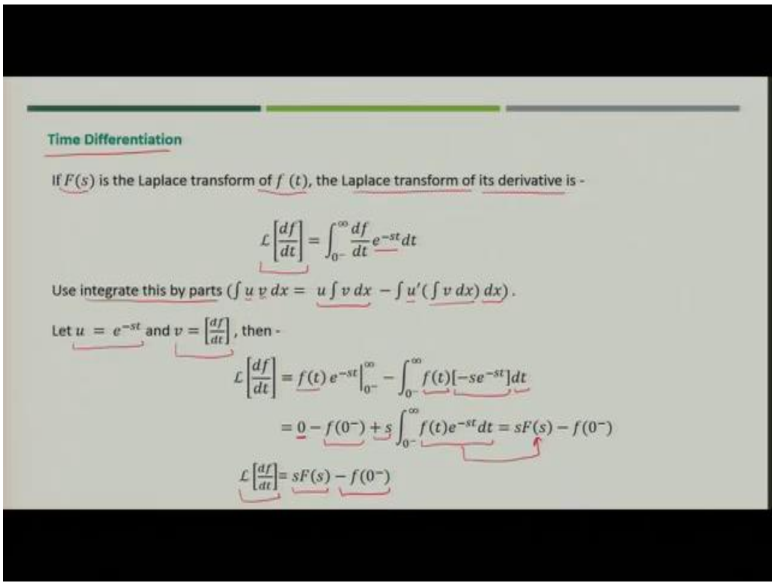

Now let's talk about the time differentiation if $F(s)$ is the Laplace transform of $f(t)$ then we need to find out the Laplace transform of its derivative, i.e., $\mathcal{L}\left[\frac{df}{dt}\right]$. So, we use the standard definition of Laplace transform as,

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st} dt$$

Now, how you will solve this particular type of integral? We will use the integration by parts method. So, the same property we will use for finding out the integral of this equation. So here let $u = e^{-st}$, $du = -se^{-st}\, dt$, and $dv = (df/dt)\, dt = df\,(t)$, $v = f\,(t)$. Then

$$\mathcal{L}\left[\frac{df}{dt}\right] = f(t)e^{-st}|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)[-se^{-st} dt$$

$$= 0 - f(0^-) + s\int_{0^-}^{\infty} f(t)e^{-st} dt = sF(s) - f(0^-)$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$$

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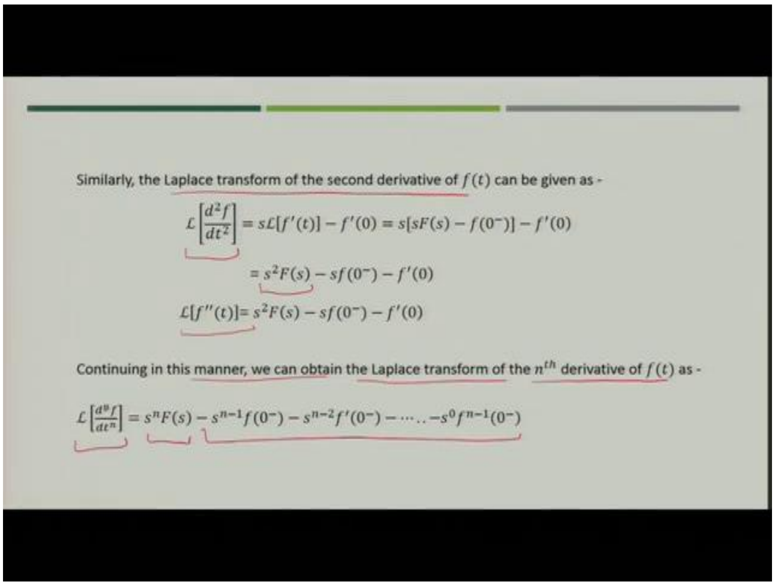

So now similarly, you can take the second derivative of *f (t)* also. So, you will get $\mathcal{L}\left[\frac{d^2f}{dt^2}\right]$?

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s\mathcal{L}[f'(t)] - f'(0) = s[sF(s) - f(0^-)] - f'(0)$$

$$= s^2F(s) - sf(0^-) - f'(0)$$

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0^-) - f'(0)$$

If you continue in this manner you can create more generic Laplace transform that is for *n*th derivative of *f (t)* which you can write as,

$$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots..-s^0f^{n-1}(0^-)$$

The value of derivative that is first derivative at zero and so on up to the value of derivative n minus 1 at derivative at zero. So, this is your genetic expression for finding out the Laplace transform of $\frac{d^nf}{dt^n}$.

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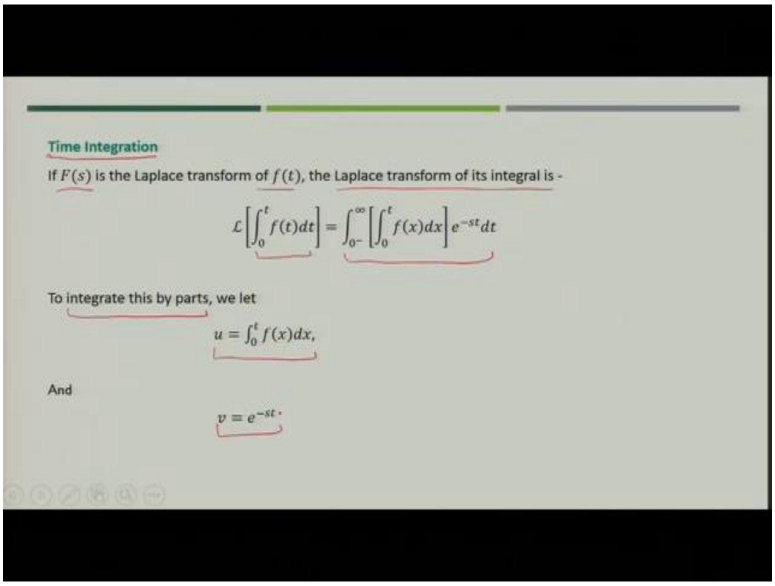

Now let us talk about the time integration. Now *F(s)* is the Laplace transform of *f (t)* then Laplace transform of its integral is,

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \int_{0^-}^{\infty}\left[\int_0^t f(x)dx\right]e^{-st}dt$$

To integrate this by parts, we let

$$u = \int_0^t f(x)dx, \qquad du = f(t)dt$$

And

$$dv = e^{-st}dt, \qquad v = -\frac{1}{s}e^{-st}$$

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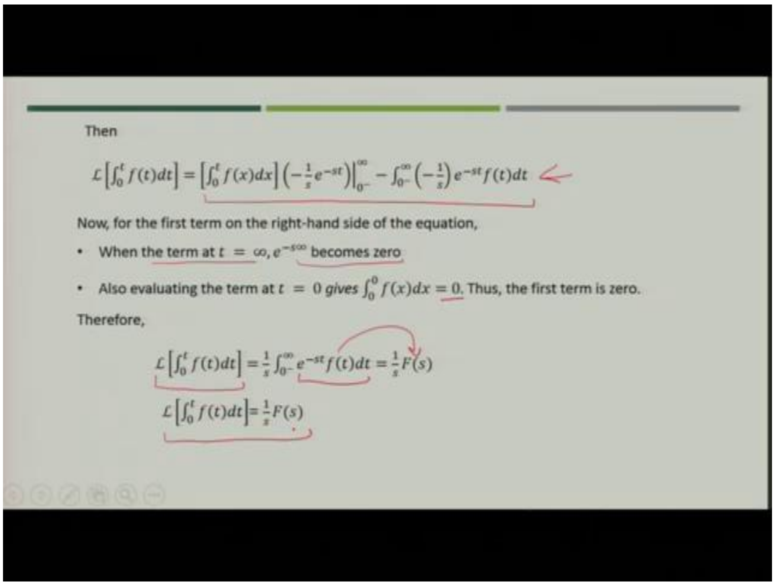

So, when you use the integration by parts property you will get the function

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \left[\int_0^t f(x)dx\right] dt \left(-\frac{1}{s}e^{-st}dt\right)\Bigg|_{0^-}^{\infty} - \int_{0^-}^{\infty}\left(-\frac{1}{s}\right)e^{-st}f(t)dt$$

Now if you see carefully this particular expression what you will see when the term t tends to infinity? Let us talk about the first expression, first term which you have in this expression.

For the first term on the right-hand side of the equation, evaluating the term at $t = \infty$ yields zero due to $e^{-s\infty}$ and evaluating it at $t = 0$ *gives* $\frac{1}{s}\int_0^0 f(x)dx = 0$. Thus, the first term is zero. Finally what we are left with is,

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}\int_{0^-}^{\infty} e^{-st}f(t)dt = \frac{1}{s}F(s)$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

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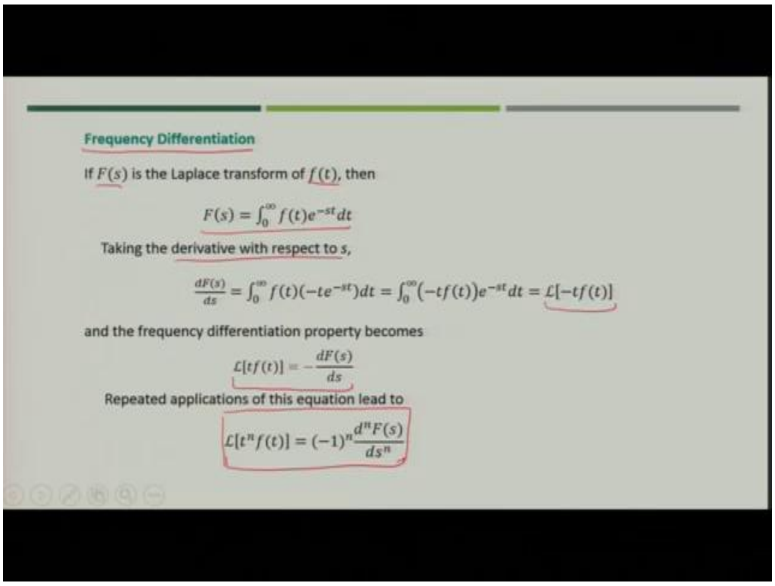

Now let’s talk about the frequency differentiation. If *F(s)* is the Laplace transform of *f (t)*, then

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Taking the derivative with respect to *s*,

$$\frac{dF(s)}{ds} = \int_0^\infty f(t)(-te^{-st})dt = \int_0^\infty \big(-tf(t)\big)e^{-st}dt = \mathcal{L}[-tf(t)]$$

So,

$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$

For more generic you can write like,

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$$

This expression will give you the frequency differentiation property.

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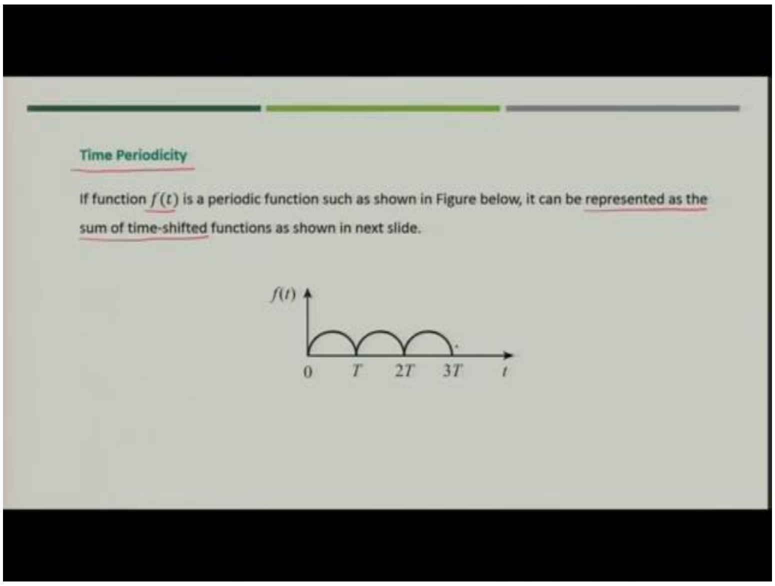

Now let's talk about the time periodicity, if the function *f (t)* is a periodic function which is shown in the figure it is periodic with period t. So, it can be represented as the sum of time shifted functions, how? Let us see in the next line, so if you have this function you can divide this periodic function into three parts.

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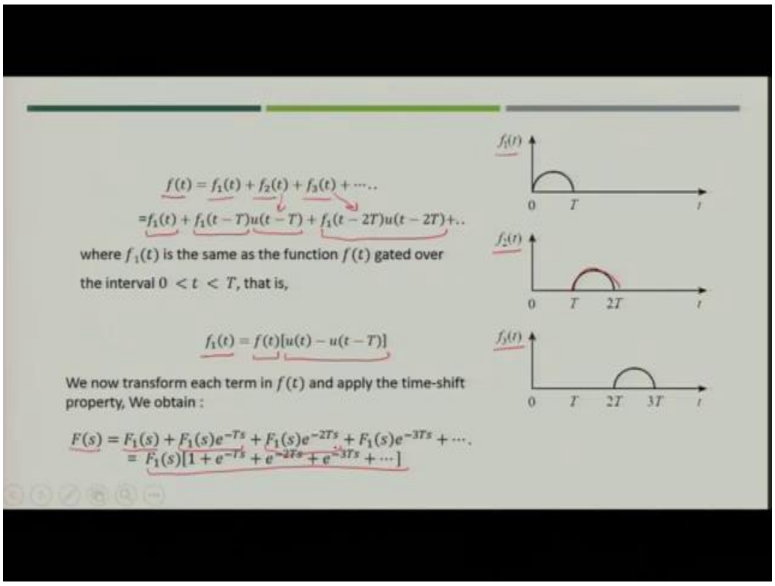

So, one is say $f_1(t)$ which have a period between 0 to t. Second is $f_2(t)$ which has a period between t to 2t and third is having the period between 2t to 3t. Now if you are compiling the function

$$f(t) = f_1(t) + f_2(t) + f_3(t) + \cdots ..$$

You can write the above expression in terms of the step functions as,

$$f(t) = f_1(t) + f_2(t) + f_3(t) + \cdots..$$

$$= f_1(t) + f_1(t-T)u(t-T) + f_1(t-2T)u(t-2T) + ..$$

Now if you are asked to find out the Laplace transform of the periodic function that is $F(s)$ what you will write? You will write the Laplace transform of function $f_1(t)$ that is $F_1(s)$. Then we will use the time shift property and we will write the Laplace transform of the second term that will become $F_1(s)e^{-Ts}$. For third term it will become $F_1(s)e^{-2Ts}$ and so on. You will finally get the expression,

$$F(s) = F_1(s) + F_1(s)e^{-Ts} + F_1(s)e^{-2Ts} + F_1(s)e^{-3Ts} + \cdots.$$

$$= F_1(s)[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots]$$

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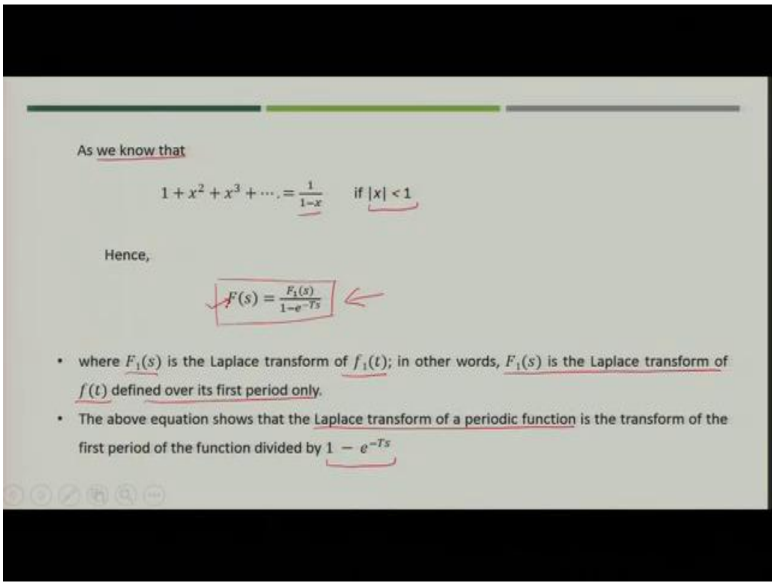

So now if you sum up the series

$$1 + x^2 + x^3 + \cdots. = \frac{1}{1-x}$$

if $|x| < 1$. So now if you use this particular property you can simply write

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}}$$

So, what we can observe from this? We can observe that $F_1(s)$ is the Laplace transform of $f_1(t)$ that means it is the Laplace transform of function defined over its first period only. Now what we get? If you want to find $F(s)$ that is the Laplace transform of complete periodic function we have to divide $F_1(s)$ by $1 - e^{-Ts}$ then we will get the Laplace transform of complete periodic function.

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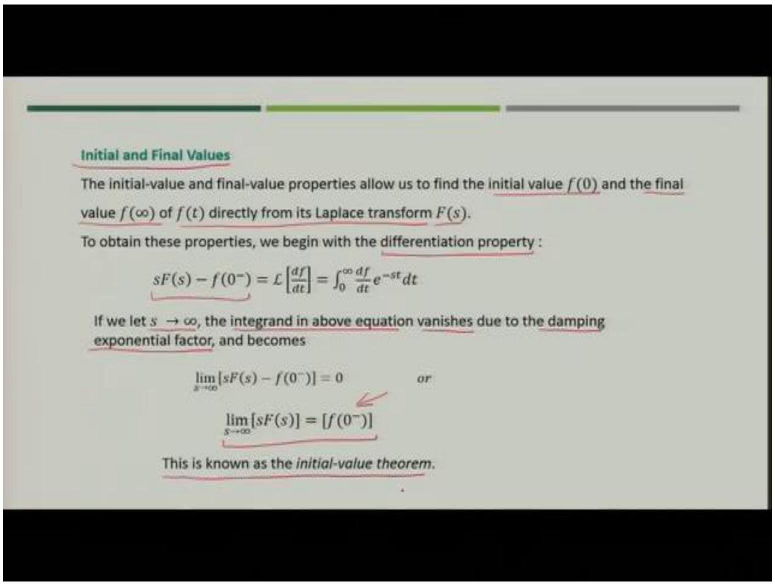

Now let us talk about two important properties, the initial value and the final value theorems. These two theorems are very important in finding out the initial and final values of function. So that means this allow us to find the value of function at time t is equal to 0 and the value of function at infinity. And we can find out these two values directly with the help of Laplace transform of $f(t)$ that is $F(s)$.

How we will get? Let us try to understand. Let us take the differentiation property of the function that is the $\mathcal{L}\left[\frac{df}{dt}\right]$ so what we write? We write $\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st} dt$ is nothing but $sF(s)$ minus the value of function at zero. Now if we let $s \to \infty$ then the integrand in the above equation will vanish because of the damping exponential factor. So, if you put the value s as infinity we get $\lim_{s\to\infty} [sF(s) - f(0^+)] = 0$.

So, the whole of the expression to the right of $\mathcal{L}\left[\frac{df}{dt}\right]$ will become zero. So when it will become zero you can simply write it as $\lim_{s\to\infty} [sF(s) - f(0^+)] = 0$ or you can write because these value will be a constant value, so you can simply write $\lim_{s\to\infty} [sF(s)] = f(0^+)]$. This particular property is known as the initial value theorem.

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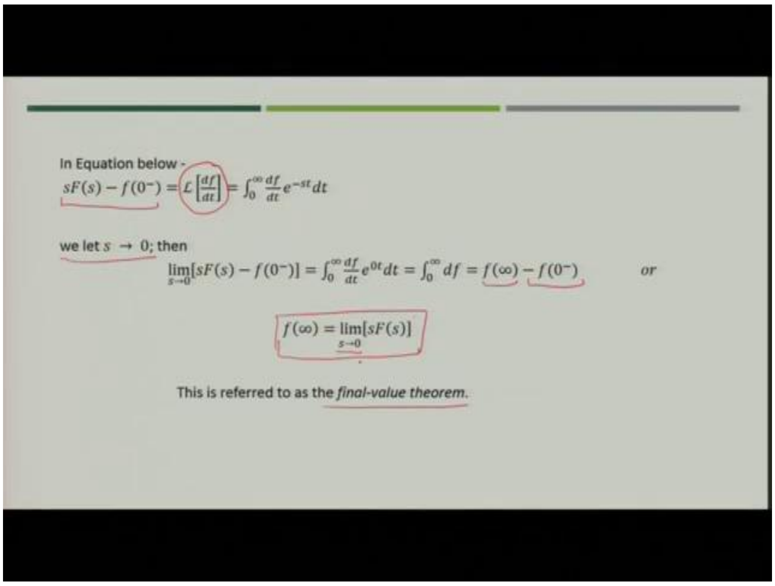

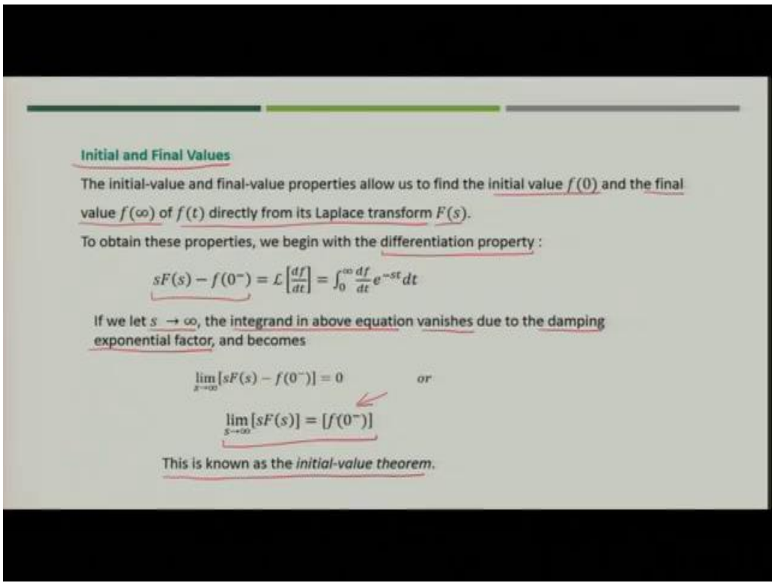

Now in case of final value theorem, we take the same property, i.e,

$$sF(s) - f(0^+) = \mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st} dt.$$

Now if $s \to 0$ what we can write?

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \int_0^\infty \frac{df}{dt} e^{0t} dt = \int_0^\infty df = f(\infty) - f(0^-)$$

This will become the value of function at infinity minus value of function at zero. Now you have value of function at zero at both sides, those both will cancel out. So, finally what you will get? The value of function at infinity is

$$f(\infty) = \lim_{s\to 0}[sF(s)]$$

So, if you compare both of them here when s tends to infinity the value of function $sF(s)$ will give you with the initial value. While in case of final value f infinity limit will tends to 0 and you will find the value of function $sF(s)$ when limit s tends to 0. So you can simply remember it by understanding that when s tending to infinity means the value which you will get will be at origin and when s tends to 0 the value which you will get for the function is at infinity.

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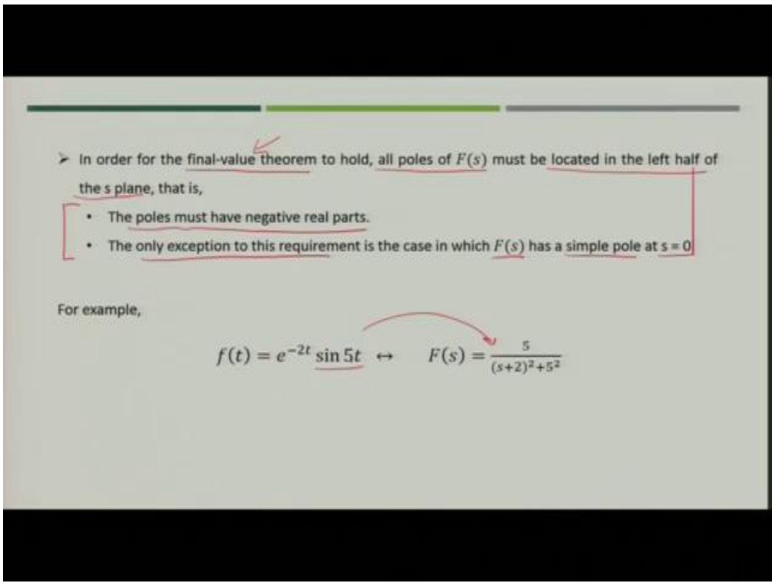

So, these two are very important theorems but the final value theorem has its own limitation. What are those limitations? The final value theorem will be valid only when all poles of F(s) must be in the left half of the s plane. What do you mean by poles? Let us say this function,

$$f(t) = e^{-2t}\sin 5t \quad \leftrightarrow \qquad F(s) = \frac{5}{(s+2)^2 + 5^2}$$

You will simply take the denominator and equate it to 0 and you try to find out the roots of this expression. Since it is a second-order equation you will get two roots of s. So those two roots will be nothing but the poles of $F(s)$. You have to observe whether the value of those roots are having any negative real part or not. If poles are having negative real part than only you can apply the final value theorem in the function $F(s)$. The only exception in this case is the case when $F(s)$ simple pole at s equal to 0. Means if this expression is having another component like one by s then you will have one pole at s is equal to 0.

So that is the only exception in finding out the final value theorem. So, these two things you have to always keep in mind, when you are asked to find the final value of the function. Now let us take the example,

$$f(t) = e^{-2t}\sin 5t \quad \leftrightarrow \quad F(s) = \frac{5}{(s+2)^2 + 5^2}$$

So, the final value theorem if you apply the value of f at infinity will be

$$f(\infty) = \lim_{s\to 0}[sF(s)] = \lim_{s\to 0}\frac{5s}{s^2 + 4s + 29} = 0$$

This is what you can see from the function f t also because you have a exponentially decaying component with the function. So final value will always be zero.

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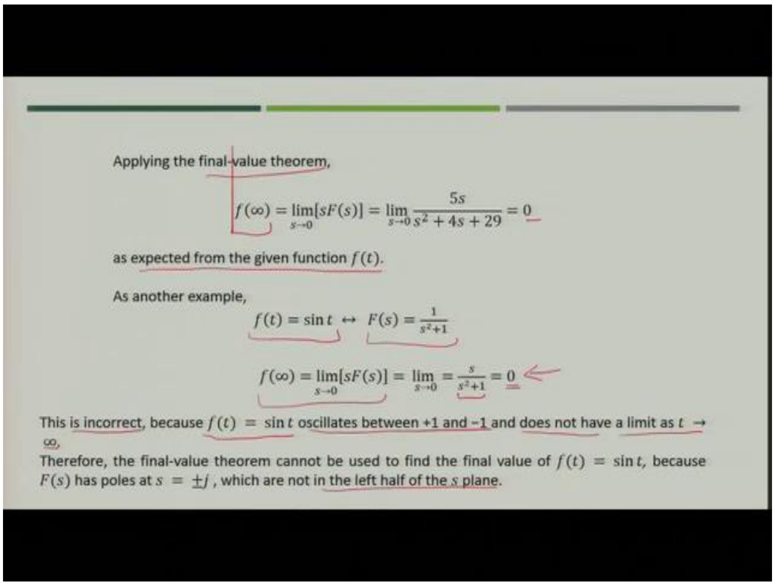

Now let us take another example, let us take the function

$$f(t) = \sin t \;\leftrightarrow\; F(s) = \frac{1}{s^2 + 1}$$

Now if you apply final value theorem

$$f(\infty) = \lim_{s\to 0}[sF(s)] = \lim_{s\to 0} = \frac{s}{s^2+1} = 0$$

But this is wrong, why? Because the function f t that is sin t will always oscillate between plus 1 and minus 1. And it will not have limit as t tends to infinity.

Now the final value theorem cannot be used in this case because if you take the poles that is the roots of this particular second-order equation. You will see the poles has the value $s = \pm j$ which are not in the left half of s plane. These two poles do not have any negative real value, which means that in this case you cannot apply the final value theorem. So, you have to always keep in mind where you can apply the final value theorem where you cannot use the final value theorem.

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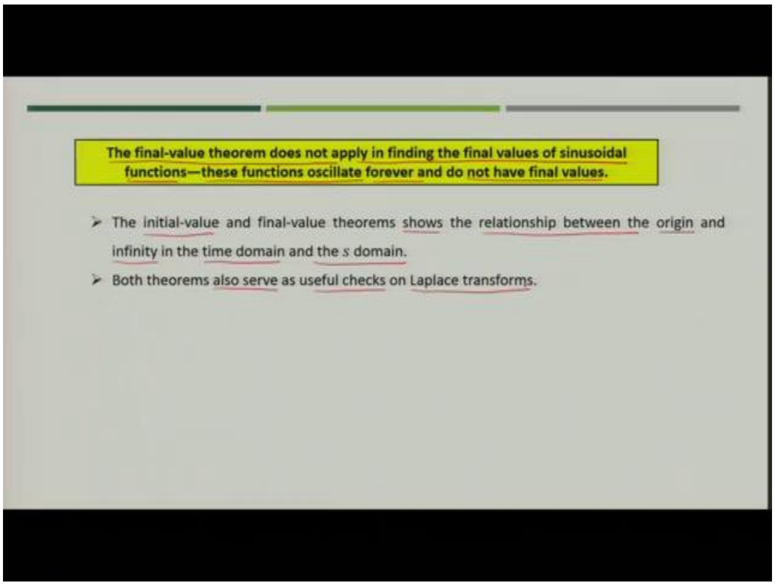

You can summarize, that the final value theorem does not apply in finding the final values of sinusoidal functions. Because these functions are oscillatory in nature and does not have the final values. Now initial values and final value theorems shows the relationship between origin and the infinity in the time domain as well as in the s domain. These two theorem also serve as a useful check on Laplace transform.

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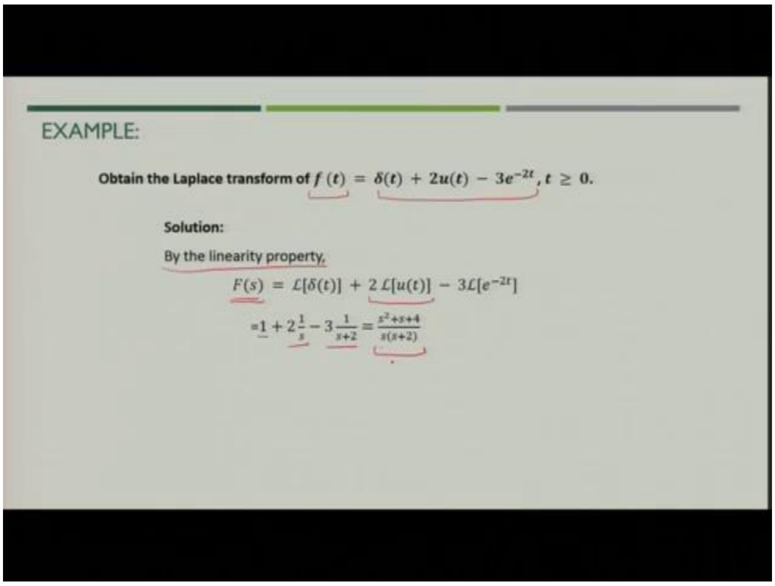

So now let us see few of the example, so that we can understand what we discuss till now more clearly. Let us see there is function f t which is a combination of unit in pulse, unit step and the exponential function. You will use linearity property here & find out the value of the Laplace transform of f t. So, let us apply the linearity property. By the linearity property,

$$F(s) = \mathcal{L}[\delta(t)] + 2\,\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}]$$

$$=1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2+s+4}{s(s+2)}$$

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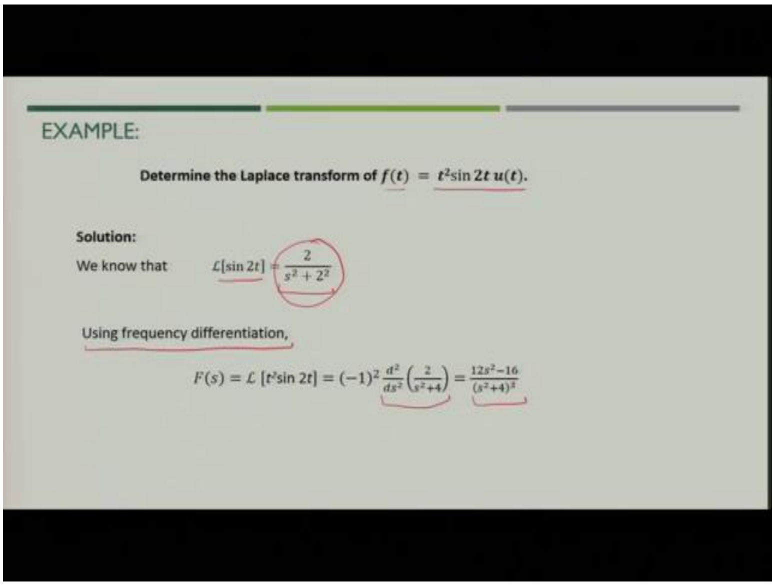

Now let us take another example, if function ***f (t) = t*²sin 2*t u(t)***. So now what we know is that the

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 2^2}$$

Now what you have to do? You must use the frequency differentiation because you have a t square item. You have to double differentiate this particular term, that is,

$$F(s) = \mathcal{L}\ [t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) = \frac{12s^2-16}{(s^2+4)^3}$$

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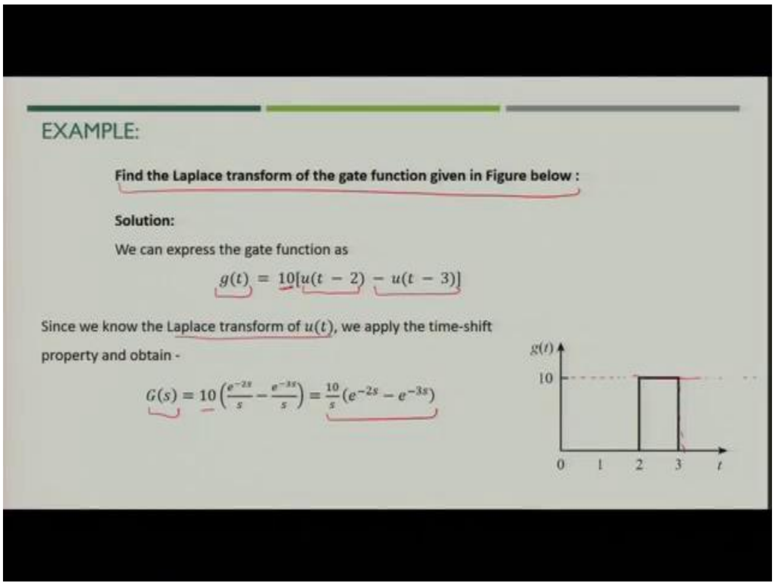

Now next is if we are asked to find out the Laplace transform of gate function. So, what we can see here? This is a gate function which starts from two and completes at three. You can say that this particular function is combination of two unit step function, how? You can say this function $g(t) = 10[u(t-2) - u(t-3)]$.

You can just take the Laplace transform of both unit step function delayed by their respective time period. The Laplace transfer will become

$$G(s) = 10\left(\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right) = \frac{10}{s}\left(e^{-2s} - e^{-3s}\right)$$

So now with these, we can close our today’s session. In this session discussed about the properties of the Laplace transform. And the next class we will discuss about the calculation of inverse Laplace transform thank you.