Basic Electric Circuits
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Module 6 Laplace Transform and its Application
Lecture 28 Definition of the Laplace Transform

Namaskar! So, in the last class we were discussing about the step response of series RLC circuit and the parallel RLC circuit and we saw that when we solve these type of circuits we get second order differential equation as a equation to solve. Sometimes these types of equations are difficult to solve when the circuit is more complex. So, we will try to understand another technique that is the Laplace transform which is easier to understand and we when we apply Laplace transform in these type of circuits it is more easier to solve the first order and second order circuit. So before going into the details, let's first try to understand what is Laplace transform.

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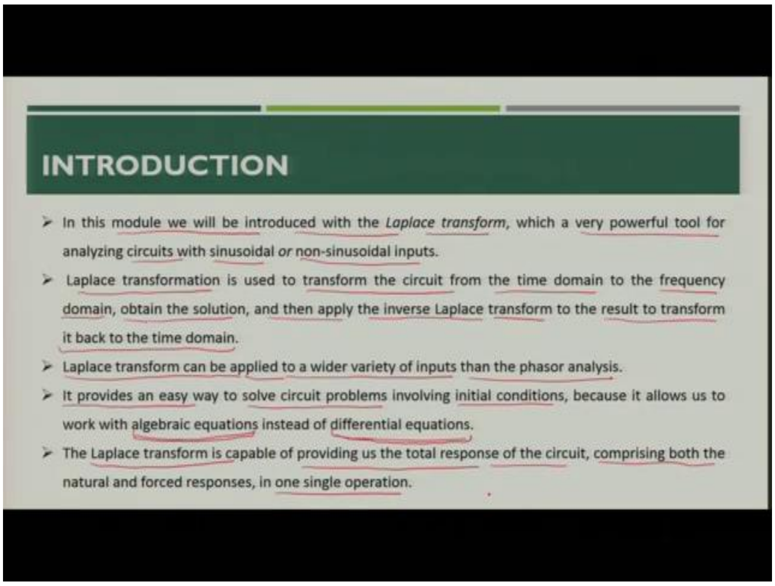

So in this module we will be introduced with the Laplace transform and this is very powerful tool for analyzing the circuit with sinusoidal or maybe the non-sinusoidal inputs. Now, Laplace transform is used to transform the circuit from the time domain to the frequency domain and obtain the solution. So, when you use the Laplace transform you will first convert the circuit from time domain to frequency domain. Obtain its solution and then you will apply again the inverse Laplace transform and the result will be transformed back in the time domain. Now Laplace transform can be applied to a wider variety of inputs than the phasor analysis. So, in the initial lectures we discussed what is phasor and how we will solve the circuit with the help of phasor analysis.

But phasor analysis has its own limitations because it cannot be applied in very wide variety of inputs. So that is why the Laplace transform is easier and more powerful than the phasor analysis in solving this type of circuits. So, it provides an easy way to solve the circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations. So, when we were discussing the RLC circuits we saw that there were differential equations compiled and we were trying to solve those differential equation. Differential equation solution becomes little bit complex while when we compare with the algebraic equations.

The algebraic equations are easier to solve. The advantage of having the Laplace transform is that the differential equations which are coming in the circuit can be easily converted into the algebraic equations and we can solve it easily. So, Laplace transform is capable of providing us the total response of the circuit, which comprising of both the natural as well as forced responses and that comes in one single operation.

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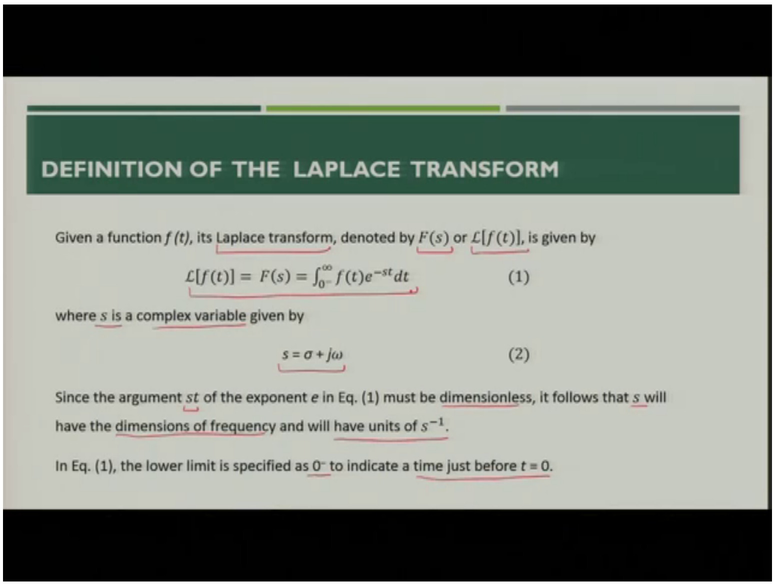

Now, how we will define the Laplace transform? Laplace transform is denoted by *F(s)* or you can say $L(f(t)) = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$. What is s? s is a complex variable and is given by $s = \sigma + j\omega$. Now, since the argument *st* of the exponent should be dimensionless that means that if you see *st*, t is the time domain, so s would be the dimension of frequency. So that the combination *st* can become dimensionless. So, s will have a unit of inverse of second.

Now, in equation (1), which you just saw that is the definition of your Laplace transform the limit which is 0 minus indicates that we are talking about the time just before t equals to 0. So, this makes sure that we are incorporating the phenomenon like switching on and switching off the circuit. That means that the singularity functions like step response or impulse response are incorporated in this definition.

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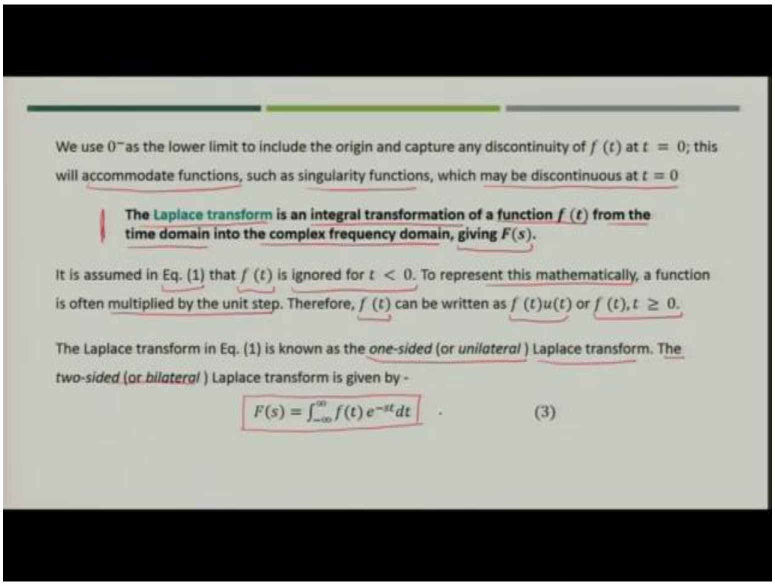

So, this will accommodate the functions such as singularity functions, which may be discontinuous at time t is equal to 0. So, what we can say about the Laplace transform? We can say that Laplace transform is nothing but an integral transformation of a function *f(t)* from the time domain into the complex frequency domain and it is given by *F(s)*. Now, if you assume that in equation (1), *f(t)* is ignored for time t less than 0, that means what you are doing? You are representing this mathematically as a function multiplied by the unit step function. So, what you can write? *f(t)* can be clubbed with the unit step function.

So, you can simply write *f(t) u(t)* or you can say *f(t)* for time t greater than equal to 0. Now, what we saw in this equation, the integration starts from 0^- to infinity. So that means that it is one sided that is unilateral Laplace transform. Now if you want both sides that is bilateral transform means you must integrate *F(s)* from minus infinity to plus infinity. In that case,

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt .$$

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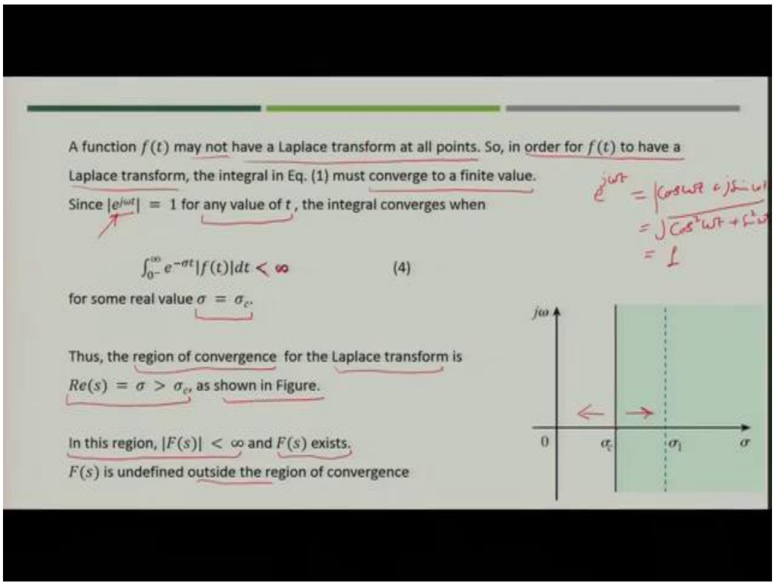

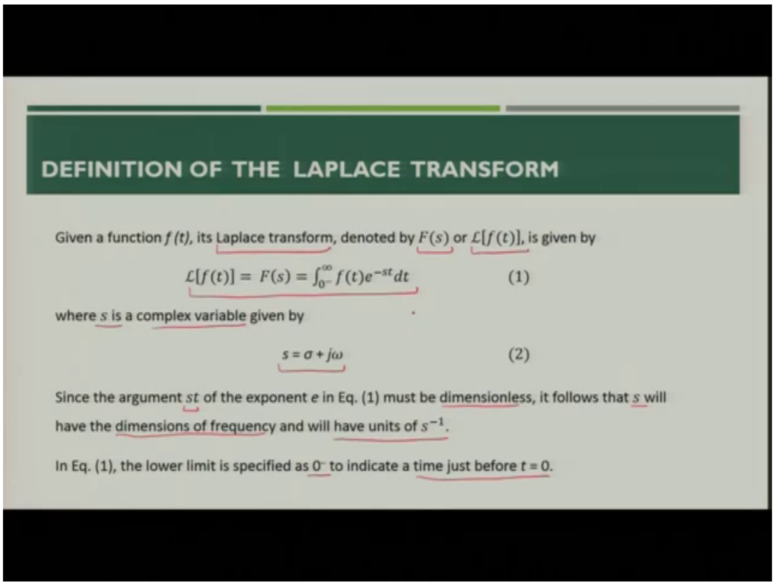

Now, the function *f(t)* may not have a Laplace transform at all points. So, for *f(t)* to have a Laplace transform, the integration, the integral what we saw here should be having a finite value or it will converge. Now, if you see the expression $F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$, if you put the value of *s* in this, so this will $F(s) = \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt$. So, now if you see the value of $\left|e^{-j\omega t}\right| = 1$. How? Because this is Euler's identity. So, you can simply write $e^{-j\omega t} = \cos\omega t - j\sin\omega t$. So, if you take the mod of this, this will become $\sqrt{\cos^2\omega t + \sin^2\omega t} = 1$. So, what you can say?

$\left|e^{-j\omega t}\right| = 1$ for any value of t. It means that anywhere it is a finite value. So, you can take it out.

The left part of the expression is $F(s) = \int_{-\infty}^{\infty} f(t)e^{-\sigma t}dt$ and the integration should be finite. So that means the value of this expression should be less than infinity and this would be applicable for a real value sigma.

Let us assume that, value of sigma is sigma c. So, with this you can find out the value of the region of convergence for Laplace transform. You can say that the real component of s that is $\sigma > \sigma_c$, for solution to converge. So, if you see the particular condition in the complex plane so σ is varying in the at the x axis and ω at the y axis. So, σ_c is your condition, which is the boundary line.

So, your solution of this expression to converge your c should be more than the critical value that is σ_c. So, right side of σ_c if you have the solution then you will say that your Laplace transform will exist. For left side condition the solution will not converge. So, you will say that your Laplace transform will not exist. So, what we can say?

That mod of *F(s)* if it is less than infinity it means that the solution exists and for rest of the section the outside the region the solution will not converge and therefore the Laplace transform will not exist in that particular region.

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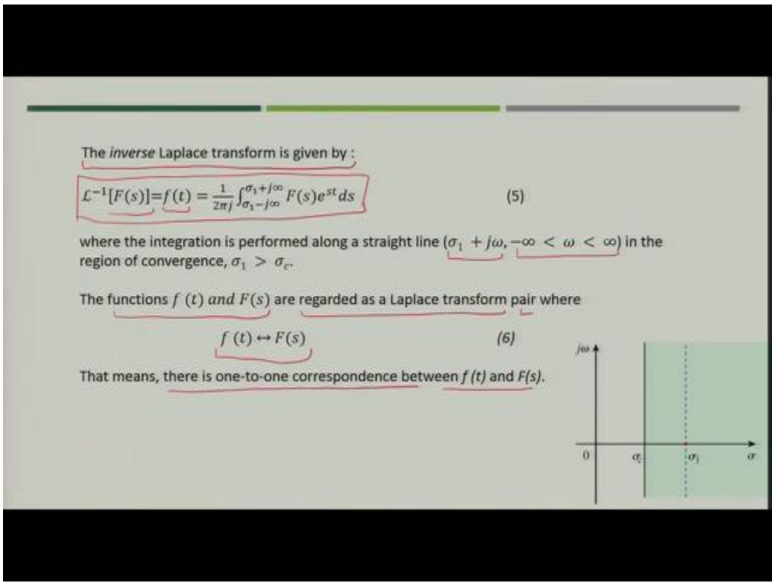

Next you need to find out the inverse Laplace transform and inverse Laplace transform is given by

$$\mathcal{L}^{-1}[F(s)]=f(t)=\frac{1}{2\pi j}\int_{\sigma_1-j\infty}^{\sigma_1+j\infty}F(s)e^{st}ds$$

So, this would be the expression for finding out the inverse Laplace transform where integration is performed along a straight line.

So, that is $\sigma_1 + j\omega$, $-\infty < \omega < \infty$. So, you will see that again the integration for any σ_1 will be carried out along this particular line. So, what we can say the function *f (t)* and *F(s)* are regarded as the Laplace transform pair where *f (t)* and *F(s)* are the Laplace transform pairs. So, that means that there is one to one correspondence between *f (t)* and *F(s)*.

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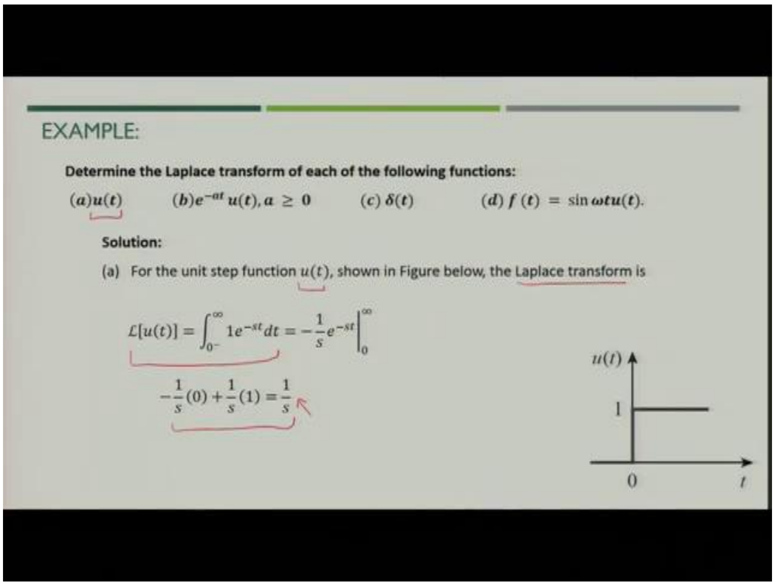

Now, let us take few of the examples so that you can understand the concept. So, first is let us say unit step function so that is $u(t)$. Now, what we have to do we have to find out the value of the Laplace transform of unit step function. So if you see this, this is the unit step function. The value of unit step function is 1 at time t is t greater than 0 and it is 0 for time t less than 0. So, when you when you are asked to find the Laplace transform you will use the expression for conversion of Laplace transform

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$

Now, this value is 1 for 0 plus. So, you can simply change the limit.

So, it will become 0 to infinity now and you will only have the component to integrate is e^{-st}. When you integrate you will get $-\frac{1}{s}e^{-st}\Big|_0^{\infty}$. So, when you put the limit you will get the value of Laplace transform equal to $\frac{1}{s}$. So, for the unit step function the value of Laplace transform is given by $\frac{1}{s}$.

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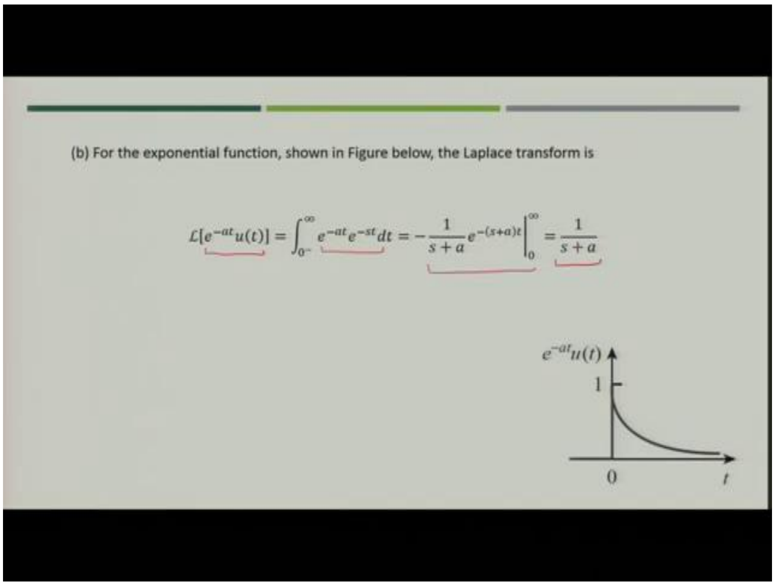

Now, for the exponential function $e^{-at}u(t)$for a greater than 0, you must put the value of function. When we put the value of function in the integration and we solve we get the output of this integration as $\frac{1}{s+a}$. Here also we will integrate between 0 to infinity because the unit step function is 1 only from 0 to infinity. So, you will simply get the Laplace transform of $e^{-at}u(t)$ is nothing but $\frac{1}{s+a}$.

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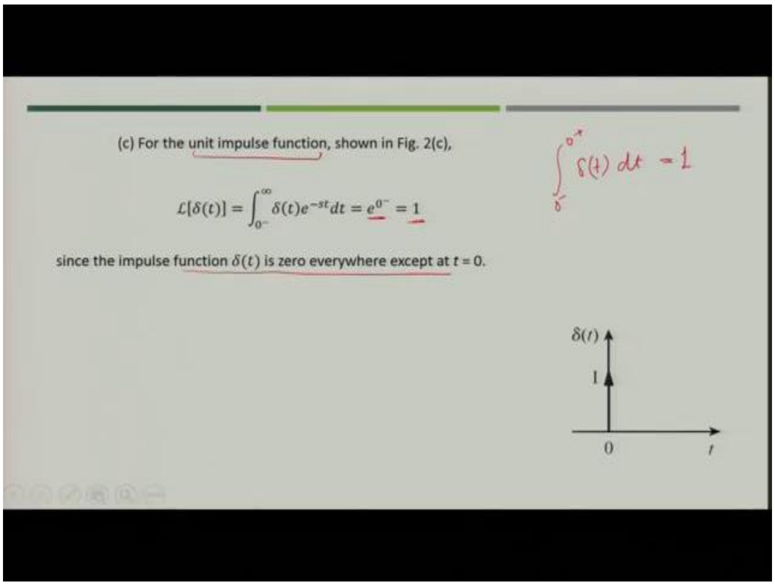

Now, the third function is unit impulse function. So, you will see the unit impulse function here. So, this the if you remember the condition for the unit impulse function we discussed that the value that $\int_{0^-}^{\infty} \delta(t)dt = e^{0^-} = 1$. Because the area for this particular unit step function would be 1 and this infinity to plus infinity can be replaced by the limits as 0 minus to 0 plus because the rest of the reason the value of $\delta(t)$ would be 0. So, this we already know. So, if you apply this knowledge and you try to point out the value of the Laplace of $\delta(t)$ because the impulse function is 0 everywhere except at t is equal to 0, so you will finally get the value of integration at e to the power 0 minus and the value of this is 1. So, the Laplace of the unit impulse function is 1.

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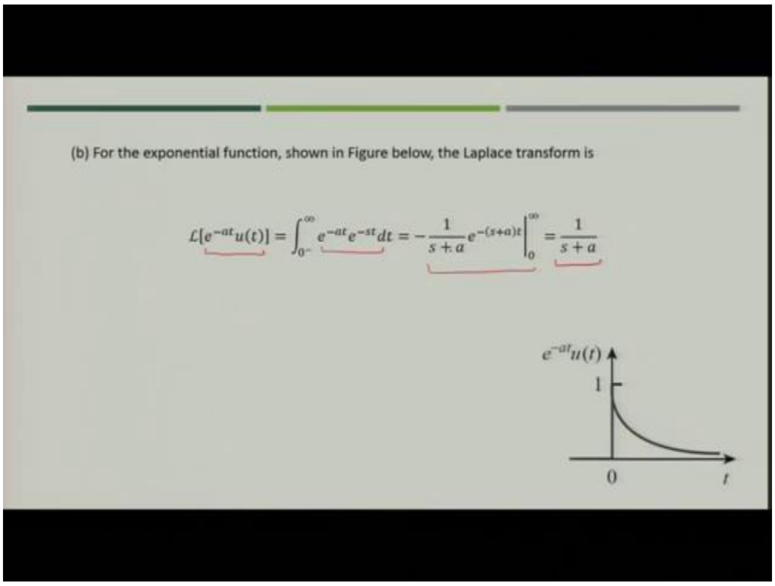

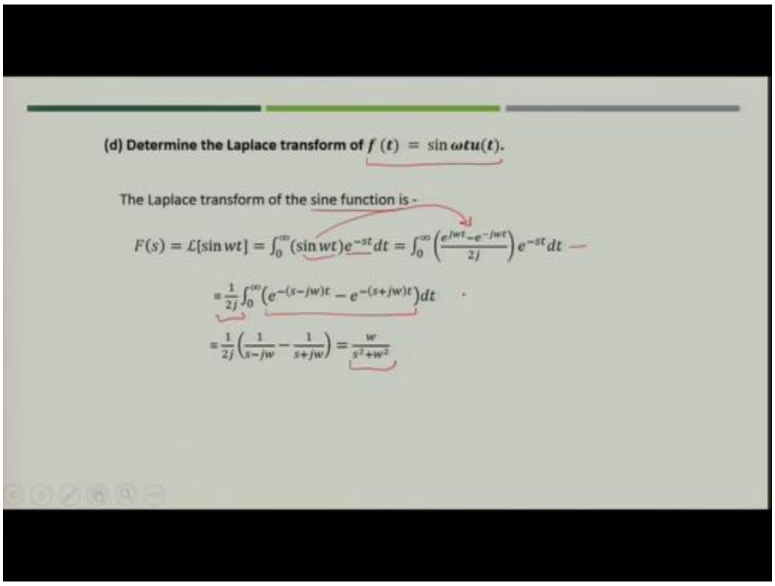

Now, next let's see the function of ***f (t)* = sin *ωtu(t)***. The Laplace function of the sin function can be given as, if you integrate between 0 minus to infinity, so since it is a ut that is a unit step function attached with sin omega t. It means that instead of having limits 0 minus you can put limit from 0 to infinity. Because for that period only the function will be having the value of sin ωt. Otherwise, the function value is 0. So what you can write?

$$F(s) = \mathcal{L}[\sin wt] = \int_0^\infty (\sin wt) e^{-st} dt$$

We can convert the sin ωt term in the form of exponential that is $\int_0^\infty \left(\frac{e^{jwt} - e^{-jwt}}{2j}\right) e^{-st} dt$. Now, $\frac{1}{2j}$ you can take it out because it is constant and when you club, you will get the expression in the form of exponent as

$$\frac{1}{2j} \int_0^\infty \left(e^{-(s-jw)t} - e^{-(s+jw)t}\right) dt.$$

Now, we just now calculated the value for expressions that is if you see,

$$e^{-at} e^{-st} dt = \frac{1}{s+a}$$

So, we will use the same. The Laplace of this will become

$$\frac{1}{2j}\left(\frac{1}{s - jw} - \frac{1}{s + jw}\right) = \frac{w}{s^2 + w^2}$$

Similarly, if you put cos ωt instead of sin ωt you will get expression as $\frac{s}{s^2+w^2}$. So, this you can verify offline by solving the integration and putting up the value of cos ωt, the expression will change in that case and you will solve.

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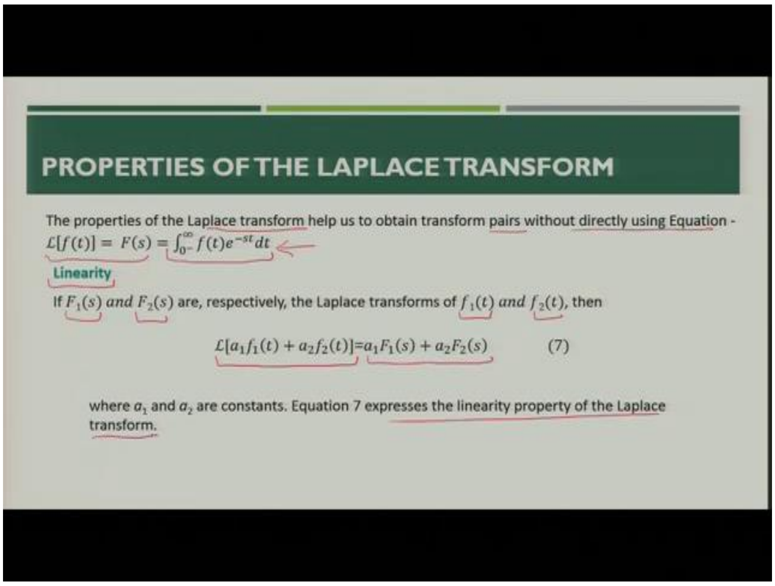

Now, let's see few of the properties of Laplace transform. The properties of this Laplace transform help us to obtain this pairs without directly using the integration method, so that is the initial expression which we started with was $\mathcal{L}[f(t)]=F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$. So, we will use various properties to directly find out the value of Laplace transform rather than going with the standard procedure of the integration of the function. So, let us see what are those properties.

First is linearity. So linearity means $F_1(s)$ and $F_2(s)$ are the Laplace transform of $f_1(t)$ and $f_2(t)$. So what we can say that Laplace of $\mathcal{L}[a_1f_1(t) + a_2f_2(t)]=a_1F_1(s) + a_2F_2(s)$, where a_1 and a_2 are the constant. So this will be showing the linearity property of the Laplace transform.

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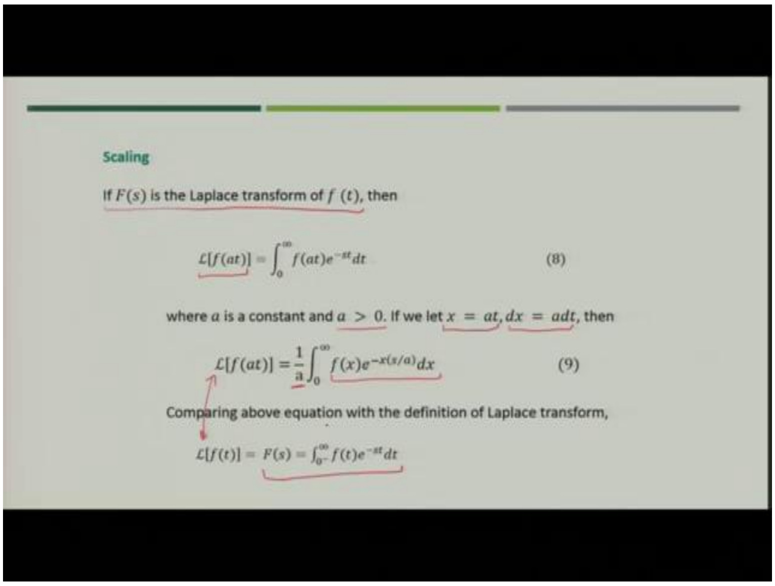

So, next is the scaling. So if *F(s)* is the Laplace transform of *f (t)*, then for Laplace of $f(at)$, we need to find out. So, let us solve it,

$$\mathcal{L}[f(at)] = \int_0^\infty f(at)e^{-st}dt$$

where *a* is a constant and *a* > 0. If we let *x* = *at*, *dx* = *a dt*, then $c = \int_0^\infty f(x)e^{-x(s/a)}dx$. If you replace the value of *at* by putting the value of x. So, $f(at)$ will become $f(x)$. Then, e^{-st} will become $e^{-x(s/a)}$ and *dx* = *a dt*.

So,

$$\mathcal{L}[f(at)] = \frac{1}{a}\int_0^\infty f(x)e^{-x(s/a)}dx$$

Now if you compare these two equations because this is the standard definition of the Laplace transform, so that means that you are simply replacing s by a. If you replace s by s by a both will be same.

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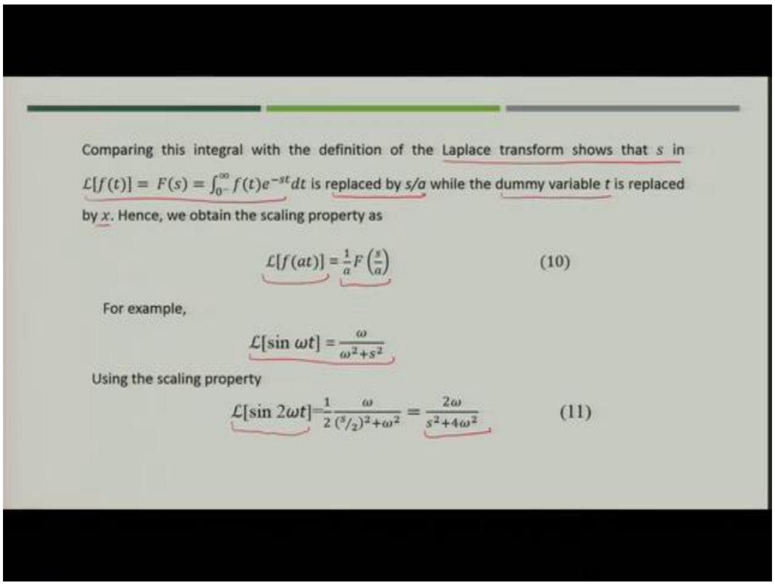

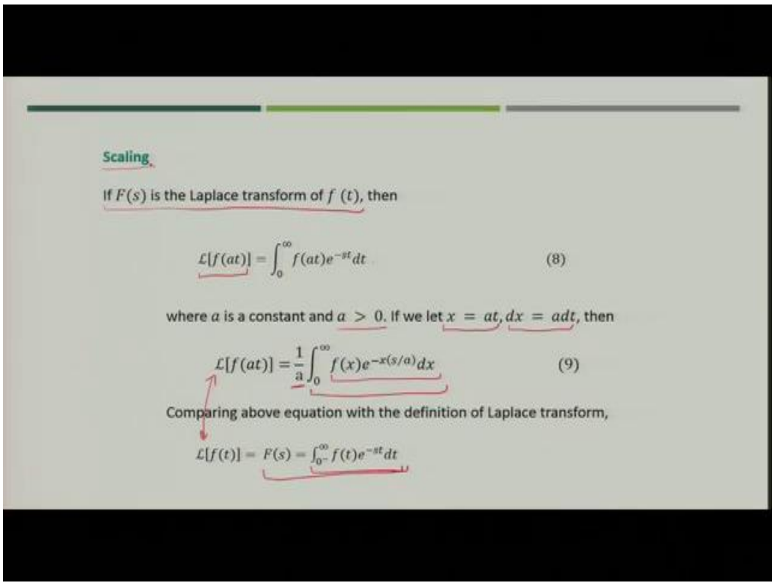

So you can say that when we compare the definition of Laplace transform shows that s is this, the standard definition and you simply replace s by s by a while you change the dummy variable t with x. So when you compare both of them, you can simply say that

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

So, this is what you got. 1 by a is anyway constant, so you can take it out. So now if you compare these two, the only difference is that you are simply replacing s by s by a. So, with this simple comparison by visualizing both of the equations, you can say that

$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$. So if you use this particular the condition, you can say that, since you know that

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{\omega^2+s^2}$$

then

$$\mathcal{L}[\sin 2\omega t]=\frac{1}{2}\frac{\omega}{\left(s/2\right)^2+\omega^2} = \frac{2\omega}{s^2+4\omega^2}$$

This is what we call as scaling.

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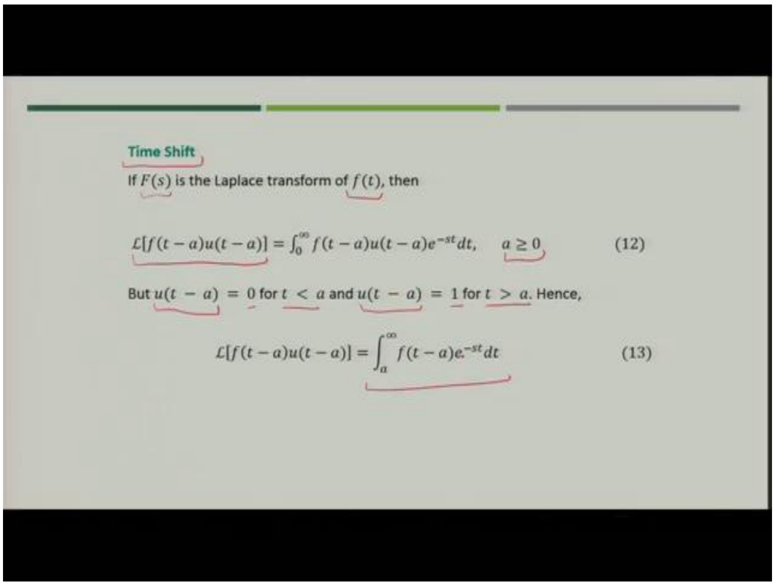

Now, let us see another property of the Laplace transform that is time shift. So if *F(s)* is the Laplace transform of *f (t)*, then $\mathcal{L}[f(t-a)u(t-a)]$ will be given by, you can simply use this standard definition of Laplace transform. So you get,

$$\mathcal{L}[f(t-a)u(t-a)] = \int_0^\infty f(t-a)u(t-a)e^{-st}dt, \qquad a \geq 0$$

Now, $u(t - a) = 0$ for $t < a$ and $u(t - a) = 1$ for $t > a$. Hence, you will use this property and then your integration will become,

$$\mathcal{L}[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-st}dt$$

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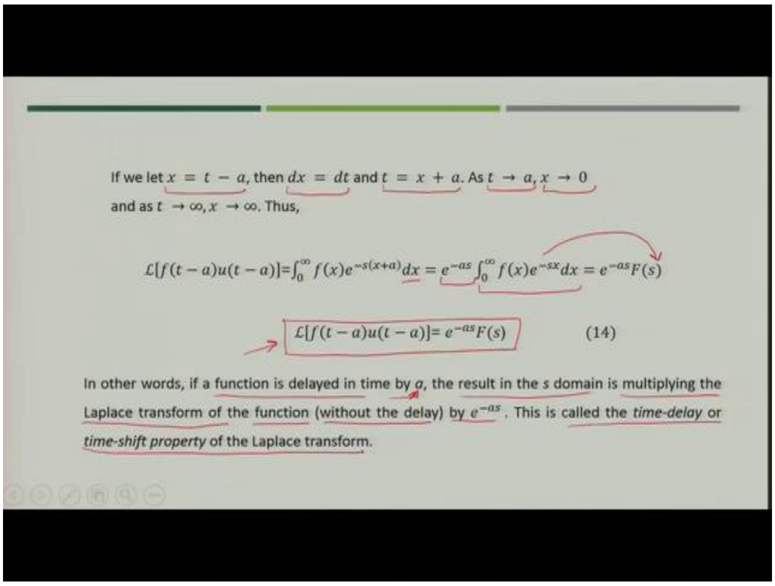

Now, let us assume x is another variable which is let $x = t - a$, then $dx = dt$ and $t = x + a$. As $t \to a$, $x \to 0$ and as $t \to\infty$, $x \to\infty$. Thus, by using these conditions you can simply say that the updated integration would be

$$\mathcal{L}[f(t-a)u(t-a)]=\int_0^\infty f(x)e^{-s(x+a)}dx$$

Now, if you see this expression e^{-as} is constant. So you can take this out. What we are left with is $\int_0^\infty f(x)e^{-sx}dx$ which is nothing but the Laplace transform of the function $f(x)$.

So if you compare with the standard definition you can simply say that this is the Laplace transform of $f(x)$. So you will simply say that the Laplace transform of this is s. So finally what you got is

$$\mathcal{L}[f(t-a)u(t-a)]=\int_0^\infty f(x)e^{-s(x+a)}dx = e^{-as}F(s)$$

This is called as a time shift means that if a function is delayed in time by a, the result in the s domain is multiplying the Laplace transform of the function without the delay by e^{-as}.

If you have a time delay in function, you will simply multiply the Laplace transform by e^{-as} and then you will get the Laplace transform of a function which is delayed in time by some constant value a. So this particular property is called time delay or time shift property of the Laplace transform.

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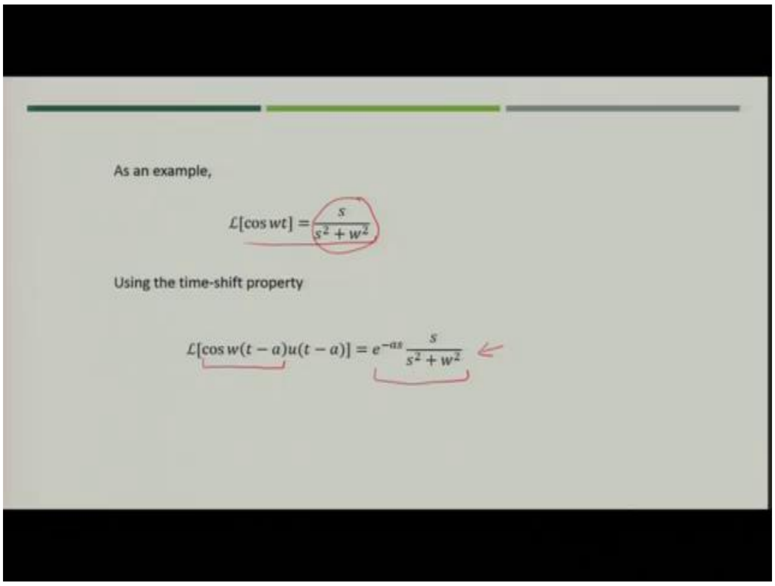

So with this let us take the example of

$$\mathcal{L}[\cos \omega t] = \frac{s}{\omega^2+s^2}$$

Now, if you are asked to find out the value of $\mathcal{L}[\cos \omega(t\text{-}a)]$, what you need to do?

You get,

$$\mathcal{L}[\cos w(t-a)u(t-a)] = e^{-as}\frac{s}{s^2 + w^2}$$

So now let us close our session at this point of time. We will continue our discussion in the next class where we will discuss few more properties of the Laplace transform. Thank you.