Basic Electric Circuits
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Module 5: First Order and Second Order Circuits
Lecture 25: Second Order Response

Namaskar! So, in today's session we will discuss about the second order response of the circuit means those circuits which contain inductor, capacitor and resistor; all 3 components are there or we have 2 inductors or 2 capacitors which cannot be clubbed to become a equivalent inductor or capacitor. So, in those cases how we will solve the circuit, we will discuss in today's lecture.

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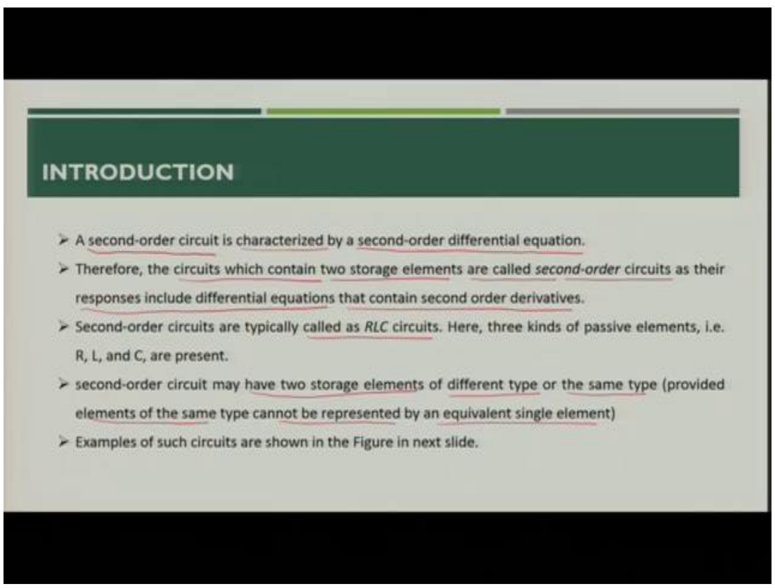

So, let us start the discussion about the second order response. Let us first understand what we mean by second order circuit. Second order circuit is characterized by the second order differential equation. When we have second order differential equation which governs that particular circuit that means that circuit can be considered as second order circuit. Now, we can also say that the circuit which contains 2 storage elements particularly inductors and capacitors then those are called as a second order circuit because their responses include differential equations that contain second order derivatives.

Second order circuits can typically series RLC circuit or parallel RLC circuits or maybe the 2 storage elements of the different type or same type; same type means that the elements can be either 2 inductors or 2 capacitors but these elements cannot be represented by an equivalent single element. So, in those case you need to create the governing equations which are second order in nature. Now, let us see the example of such types of circuits.

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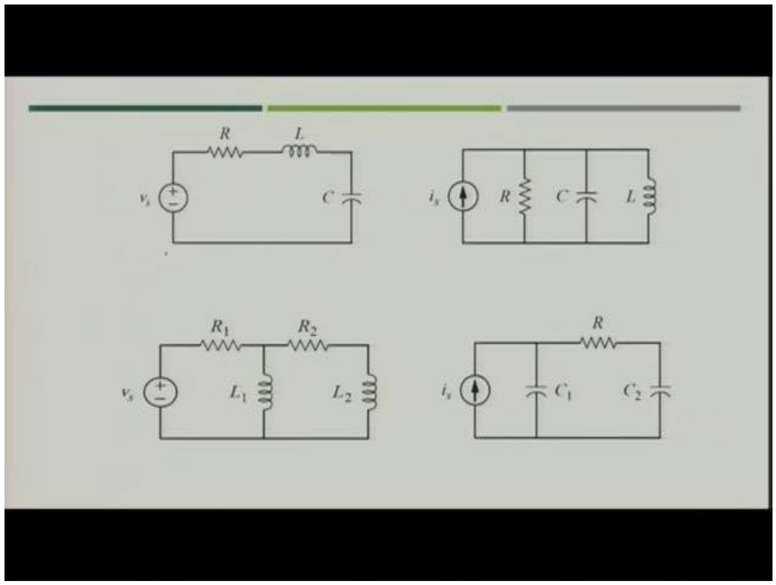

This is a series RLC circuit and is a second order circuit. Also, the parallel RLC circuit is also a second order circuit. Now, if you have 2 inductors which cannot be represented as a single inductor then also it can be considered as a second order circuit. Similarly, in this case also if we have 2 capacitors but cannot be clubbed to represent as a single equivalent capacitor of the circuit, the circuit would be considered as second order circuit.

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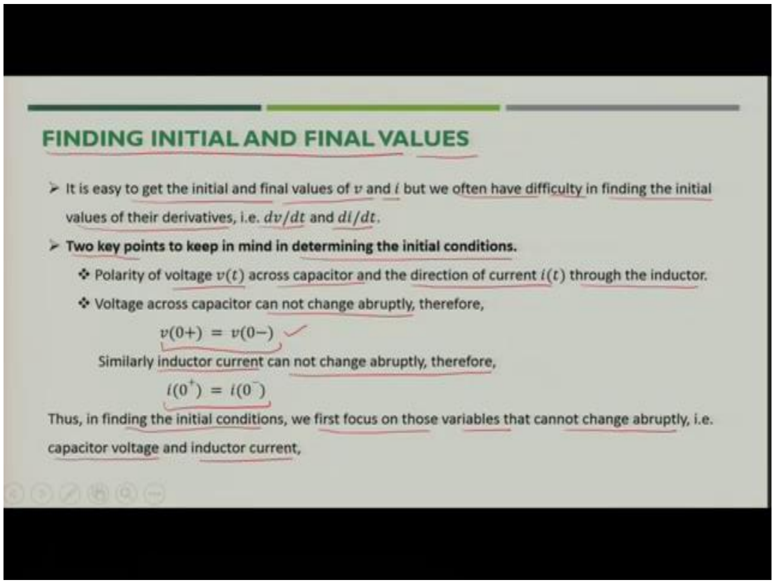

Now, the first thing which we must understand that how we will find the initial and final values for these types of circuits. It is easier to find the initial and final values for voltage and current because these are some things which can be considered or you can derive it while just doing the inspection of the circuit for time t is equal to 0 or time t tends to infinity. But it is little bit difficult in finding the initial value of the derivatives of the passive elements. So, we have 2 major points to keep in mind while we determine the initial conditions. First is the polarity of voltage across capacitor and direction of current through the inductor. Second is the voltage across the capacitor cannot change abruptly. Therefore, we say that voltage $v(0^-)$ is equal to voltage $v(0^+)$ in case of capacitor. Similarly, in case of inductor the inductor current cannot change abruptly. Therefore, $i(0^-)$ would be equal to $i(0^+)$. Thus, in finding the initial conditions we first focus on those variables that cannot change abruptly. This means we will concentrate on the voltage of the capacitor and inductor current flowing through the inductor.

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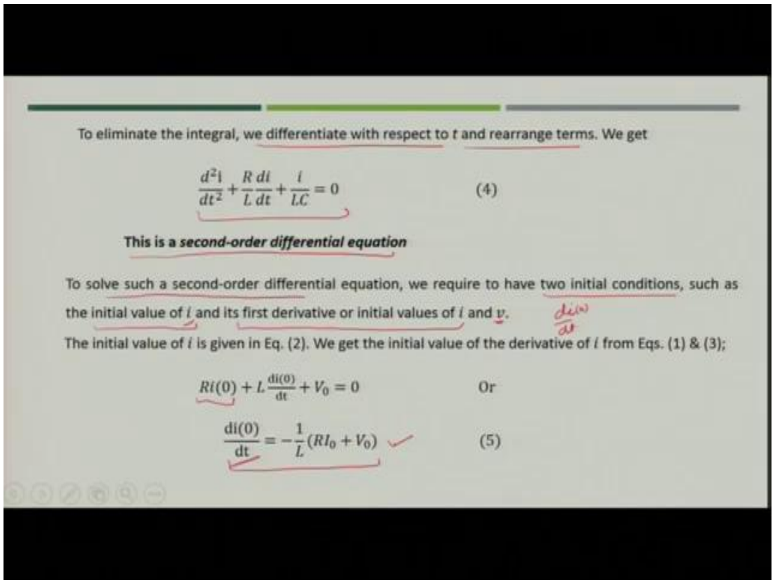

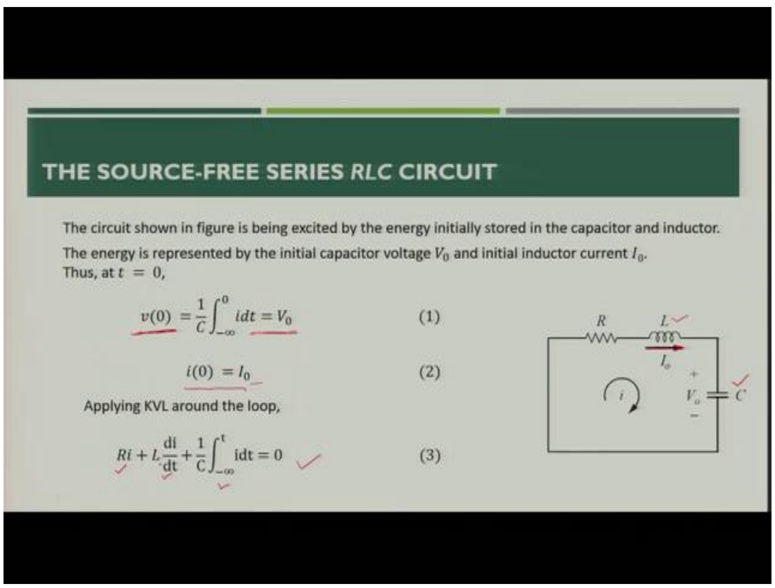

Now, let us first take the case of source free series RLC circuit. We will try to find the solution of those circuits which are series combination of R, L and C without any source. So, we assume that there is some initial current flowing through the inductor and the value is I_0 and capacitor is initially charged with some voltage V_0. So, these 2 would be considered as a source which provide current to the circuit. So, now what we can write at time t is equal to 0, *v(0)* can be written as integral of current as inequation (1) and that will be equal to V_0 because this is our initial condition.

Similarly, i(0), that is, the current at time t is equal to 0 will be I_0. Now, if you write KVL around this particular loop, where we are assuming that current is flowing in this particular loop. You can write the KVL equation for this loop as shown in equation (3). From this equation the integration part can be eliminated by differentiating it with respect to time t.

On differentiating you get equation (4). So, if you rearrange the second derivative of the equation you will get the equation (4) as shown in the slide. You can see that the equation has a second order derivative which is why it is called as a second order differential equation.

Now, if you are asked to solve such a second order differential equation you require 2 initial conditions. Now, we know that initial value of current is equal to I_0. Now, next you need to find the first derivative of initial value of current. So, you must calculate the $\frac{di}{dt}$ at time t equal to 0. We also know the initial value of the voltage across the capacitor we know. We can utilize that value as given in equation (1).

So, what you can write for this equation? You can simply write equations as given in (5) and (6). So, now you have the initial value of I and initial value of di by dt in your hand which you can utilize to solve the equation.

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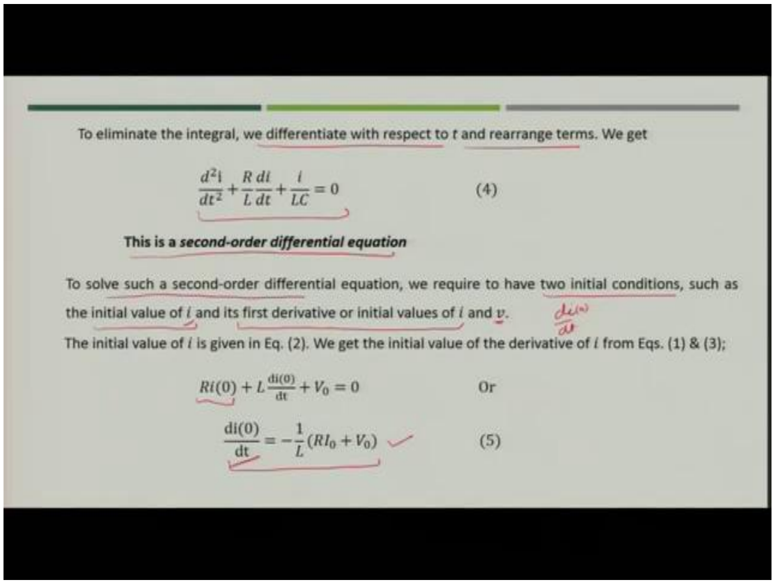

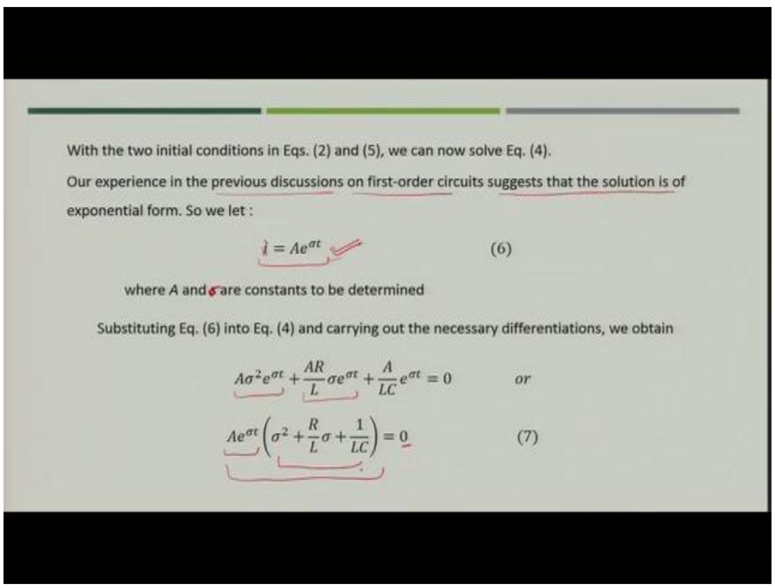

Now, if you see that the previous discussion what we did when we discussed about the source free response of RL circuit and RC circuit we came to know that typically the solution of these kind of circuits is exponential. So, let us assume that the solution for this case is as given in equation (6). So, A and σ are constants which we must determine. Now, what you will do? You will put the value of current assumed in equation (6) in equation (4). This means that you will differentiate it twice so, you will get the first component as in equation (7). Similarly the other terms of equation (7) can be obtained. This equation can be rewritten to obtain its final form. Now, the solution of these type of equations, if you see in that case the factor that is $Ae^{\sigma t}$ cannot be equal to 0 because this is our initial assumption for the solution. So, this cannot be 0. We are left with the second term which would be equal to 0.

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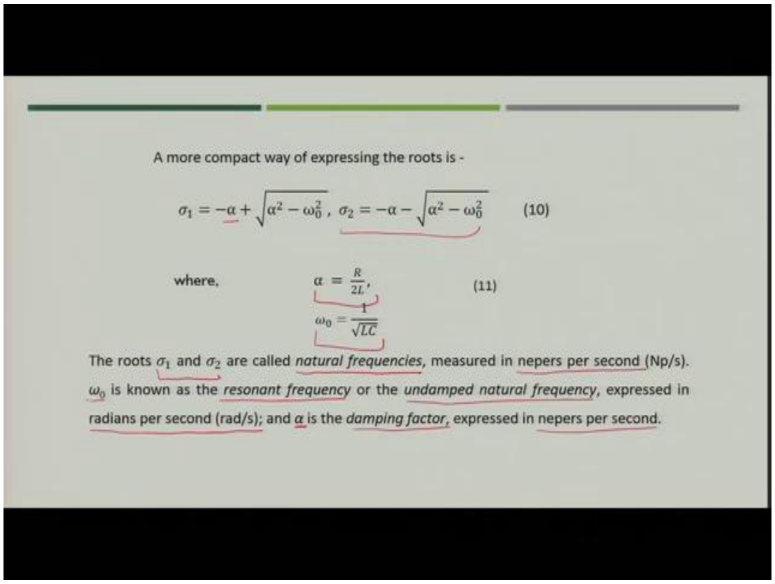

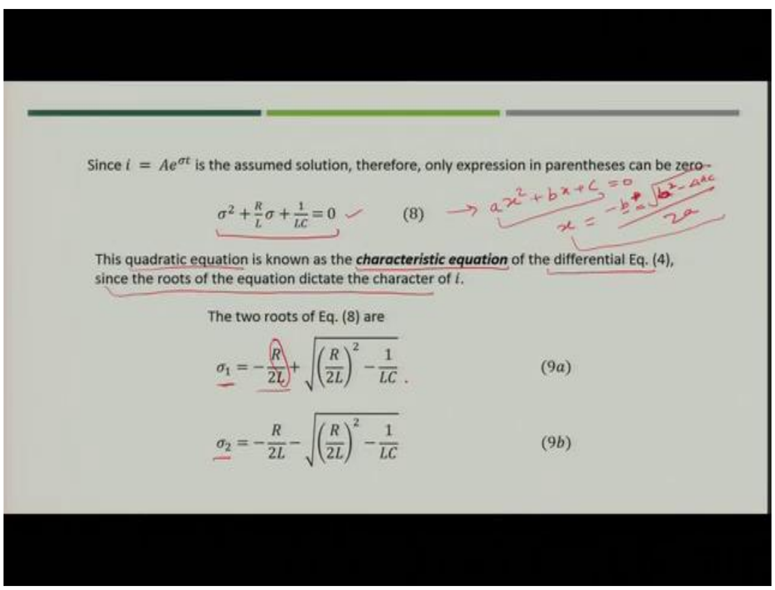

So, we will say that $\sigma^2 + \frac{R}{L}\sigma + \frac{1}{LC} = 0$ would be the governing equation for this. This is a quadratic equation and we say this type of equations as characteristic equation for the differential equation which we just saw. So, this will be considered as a characteristic equation of the differential equation. Now, if you see this is second order equation solve you will say there will be 2 roots of this equation.

So, let us say these roots are σ_1 and σ_2. You can find out the value of these roots as given in equations (9a) and (9b).

We must solve this quadratic equation. Solving this you get σ_1 and σ_2. So, now you have these 2 roots in your hand then you can represent it in a more compact manner. How will you represent? Let α represent $R/2L$ and $\omega_0 = 1/\sqrt{LC}$

So, if you assume these 2 factors and put the value of σ_1 and σ_2 in this particular form then you can simplify this as as given in equation (10). Now, these 2 roots that is σ_1 and σ_2 are called natural frequencies and are measured in nepers per second. Well, ω_0 is known as the resonant frequency or undamped natural frequency of the system which is expressed in radians per second. α is the damping factor which is again expressed in nepers per second.

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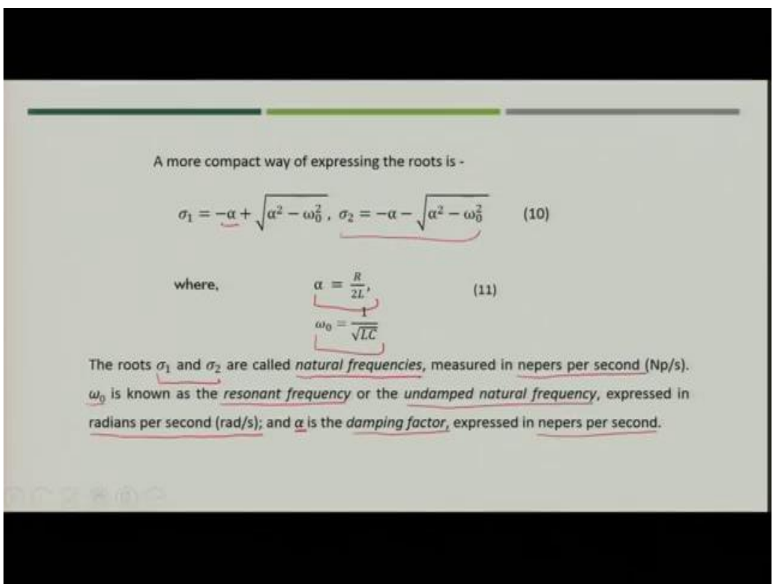

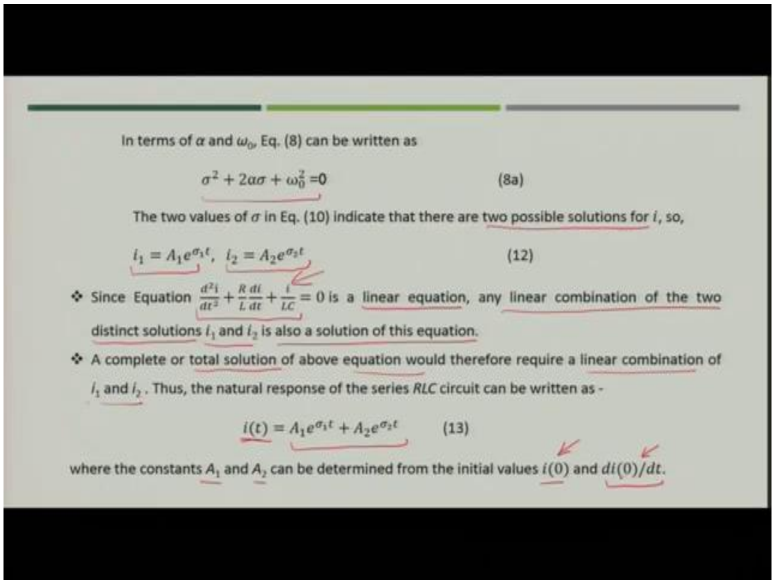

The values of α and ω obtained is inserted in equation (8) to get equation (8a). Now, this indicates that there are 2 possible solutions. So, what we can say? There can be 2 solutions like as in equation (12).

So, we will have 2 solutions for the current that is i_1 and i_2. Now the original equation $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} = 0$ is a linear equation. So, any linear combination of the 2 distinct solutions that is i_1 and i_2 is also a solution of this equation. So, for the complete solution of the above equation we would require therefore a linear combination of i_1 and i_2. This can be written as given in equation (13). So, this would be the total solution of the equation that is the second order equation related to our series RLC circuit. Now, here A_1 and A_2 are some constants which you can determine with the help of i at time t is equal to 0 and $\frac{di}{dt}$ at time t is equal to 0 and these values we already have in our hand.

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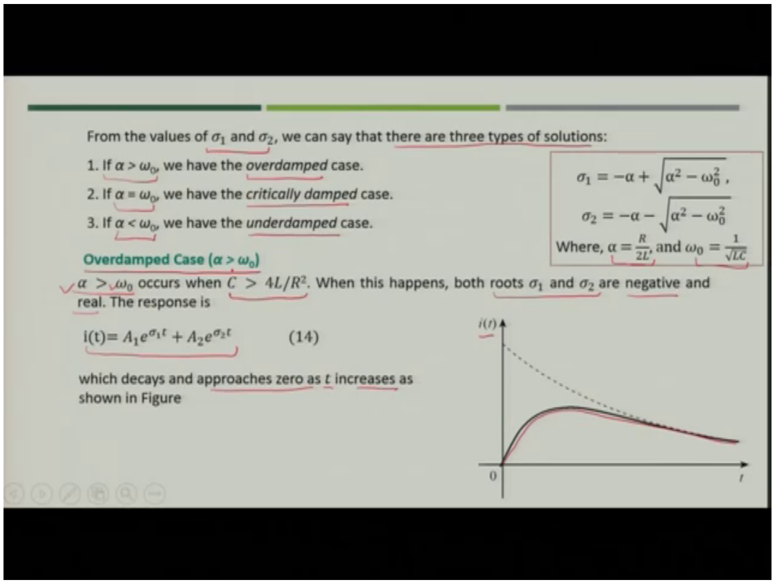

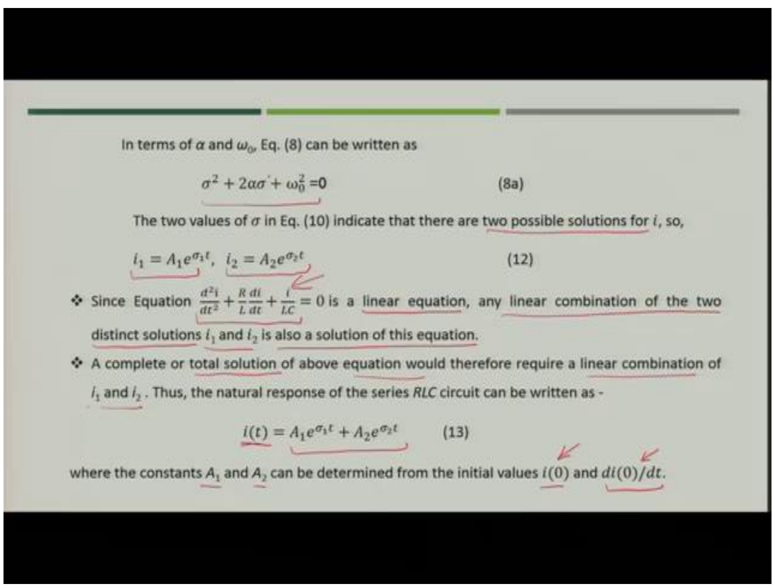

So what we can say that when you compare these equations then you can say that σ_1 and σ_2 which are the roots of the characteristic equations we have 3 conditions or we say that there are 3 possible types of solutions related to the roots of σ_1 and σ_2. What are those solutions?

First is when $\alpha > \omega_0$ the system is overdamped, $\alpha = \omega_0$ we say this is a critically damped case and if $\alpha < \omega_0$ we have underdamped case. Now, let us see the first case when the system is overdamped means $\alpha > \omega_0$. $\alpha > \omega_0$ means the value of $C > \dfrac{4L}{R^2}$. We get this by putting the value of α and ω_0

whose values we already know. In this case, both roots σ_1 and σ_2 of the characteristic equation are negative and real. So, the response would be as in equation (14) and the figure the curve which you plot the value of current it with respect to time t, it will be like this as shown in the figure which decays and approaches to 0 as time t increases. So, this would be the type of response for the case when $\alpha > \omega_0$ that means the system is overdamped.

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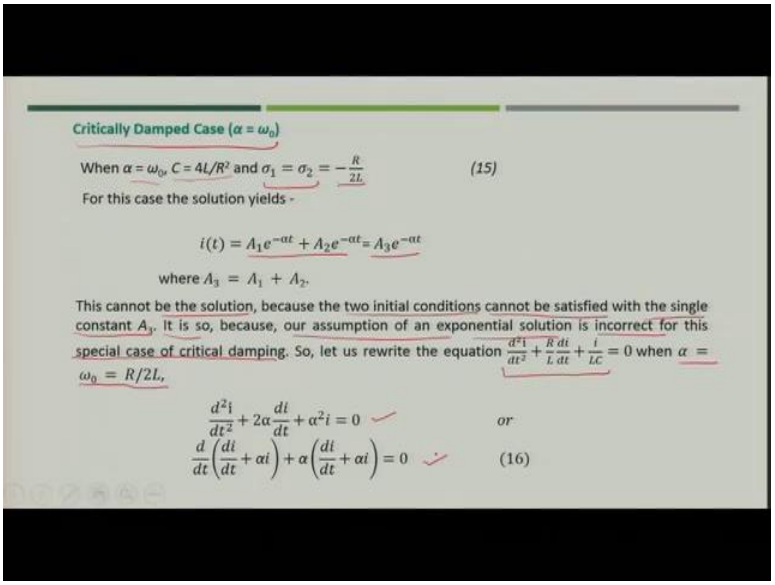

Now, second case is the case when the system is critically damped means $\alpha = \omega_0$. So, what will happen in that case? $\alpha = \omega_0$ means $C = \frac{4L}{R^2}$ or we can say $\sigma_1 = \sigma_2 = -\frac{R}{2L} = \alpha$. The solution for this is $A_1e^{-\alpha t} + A_2e^{-\alpha t} = A_3e^{-\alpha t}$.

Now, this cannot be a solution because the 2 initial conditions which we saw, that is *i(0)* at time t is equal to 0 and $\frac{di(0)}{dt}$ cannot be satisfied with single constant A_3. So, our initial assumption of exponential solution is incorrect for this particular case of critical damping. So, now what we will do? Let us rewrite the equation as given in equation (16).

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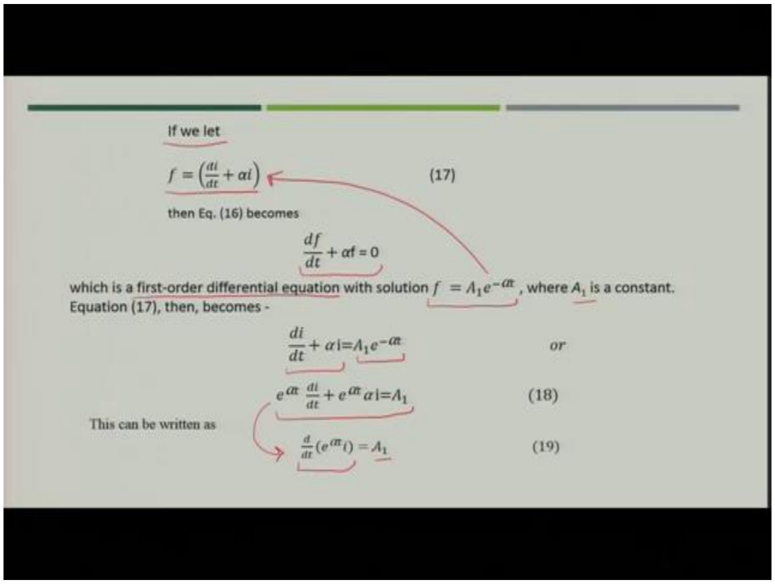

Now, let us assume that the term in bracket in (16) is some function *f*. So, what you can write? So, when you put the value of *f* in (16), this equation will become $\frac{df}{dt} + \alpha f = 0$. Now, the solution of this because this is a first order differential equation and we have seen these kinds of equations when we discussed the series RC or RL circuits in case of source free response. So, you can simply write that the solution of this equation would be $f = A_1 e^{-\alpha t}$ where A_1 is a constant. Now, if you use (17) in the above solution $\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$ .This equation can be rewritten as (18) to obtain (19).

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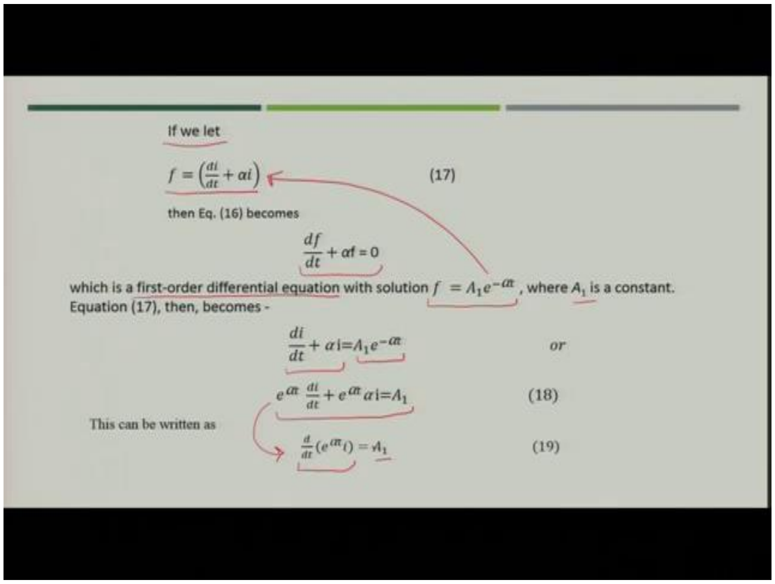

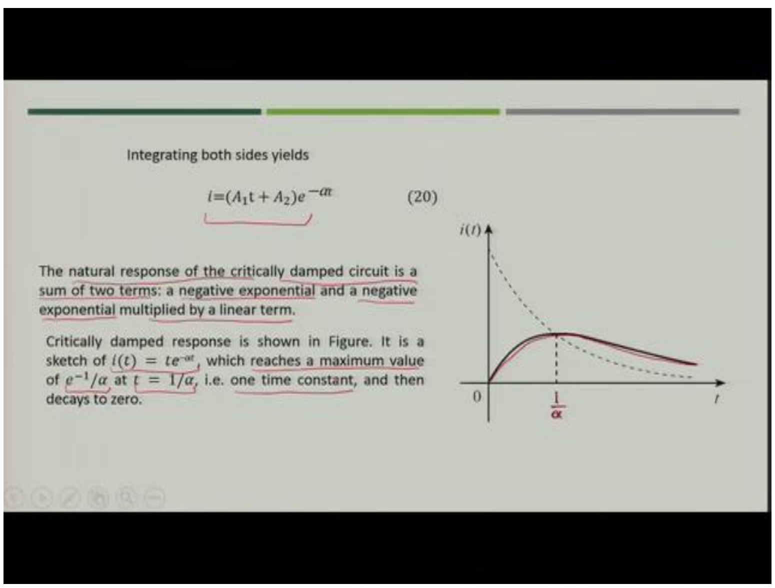

You will get the value of *i* as given in equation (20). You have to just simply integrate it and you will get another constant *A2* Now, the natural response of the critically damped circuit is sum of 2 terms the negative exponential and negative exponential multiplied by a linear term.

So, in that case what will be the response? If you plot the variation of current *i* with respect to time t the response would be like shown in the figure. Now, if you see at time t is equal to 1/α what will happen? So, the value will be maximum at t is equal to 1/α and the value of maximum is e^{-1}/α. So, that means after 1 time constant the current will reach to its peak value and then it will slowly decay to 0. So, this kind of response you will see when the system is critically damped.

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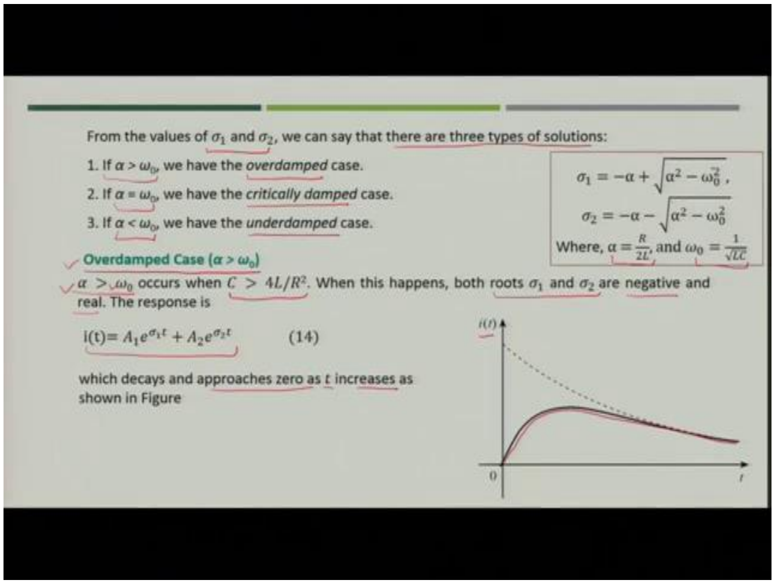

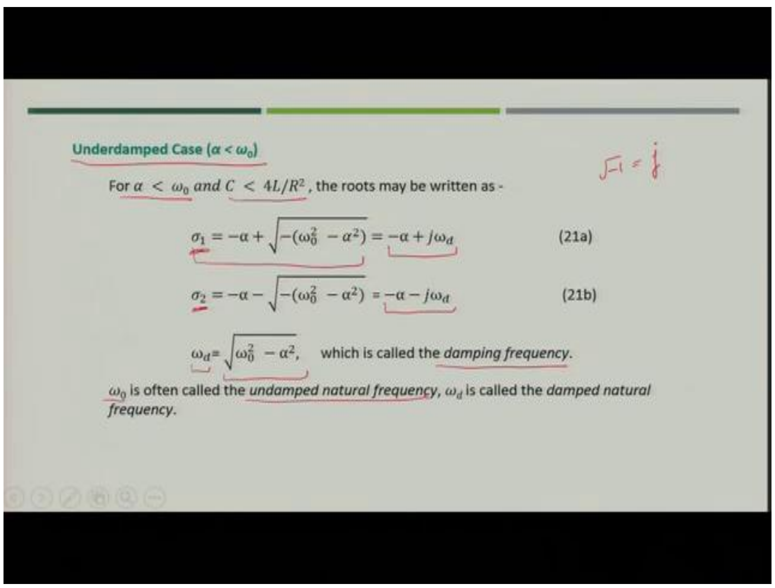

Now, the third scenario is that the value when $\alpha < \omega_0$. When $\alpha < \omega_0$ means $C < \frac{4L}{R^2}$. So, the roots which you have seen that is σ_1 and σ_2 are is given in (21a) and (21b). The root of -1 can be replaced with the complex variable j. So, you can say that $\sigma_1 = -\alpha + j\omega_d$ and $\sigma_2 = -\alpha + j\omega_d$, where $\omega_d = \sqrt{\omega_0^{\,2} - \alpha^2}$. Now, if $\alpha < \omega_0$ this factor will always be positive. So, that is why we have taken -1 out. So, this ω_d is nothing but damping frequency and ω_0 is the undamped natural frequency.

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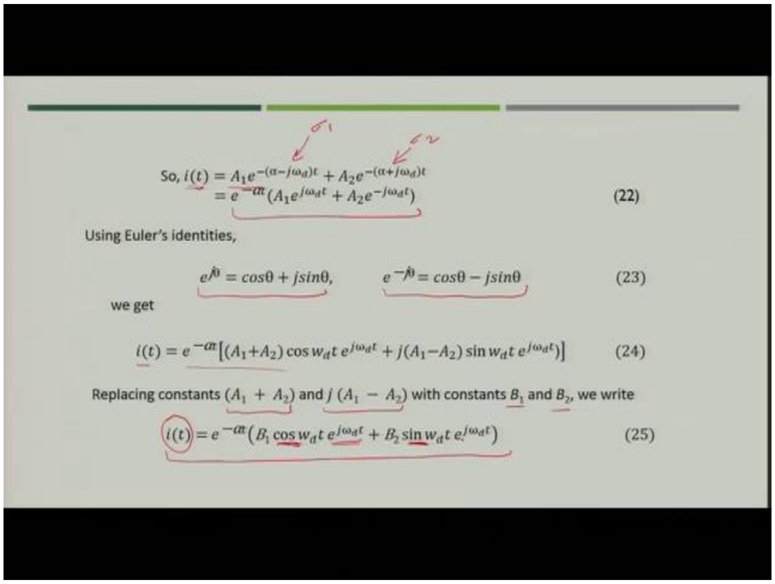

So, when you will put the value of σ_1 and σ_2 in the solution which we have got and rearranging it as in equation (22). So, you will get the response of current in terms of t. Then applying Euler's identity in equation (22) we get equation (24). So, this will be simplified for expression for current with respect to time t. Now, if you see this equation the expression for it you will say that the system will have cosine terms and sine terms that is sinusoidal terms you will see plus exponential terms.

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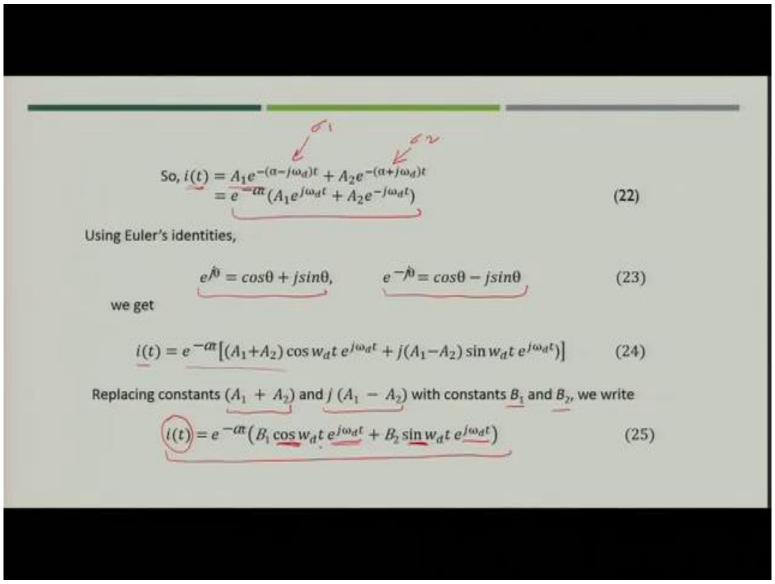

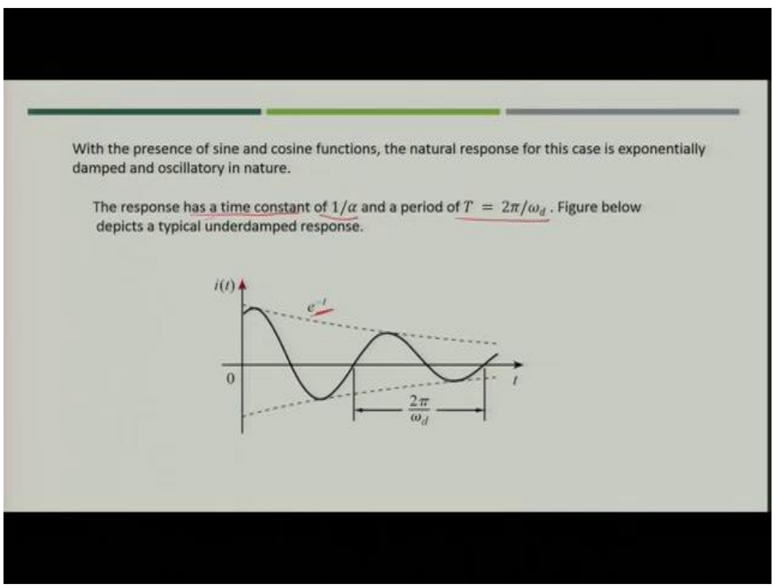

So, how the response would look like? In that case if you see the response for current it would be exponentially decaying because of the exponential factor of e to power minus factor which you have. So, you will simply see that it is exponentially decaying plus sinusoidally varying also because you have sine cos terms in the equation. So, the response at time constant of 1/α and the period will be t is equal to $2\Pi / \omega_d$. So, this you can see from the figure. So, with this we close our today's session. We will continue our discussion in the next lecture, and we will see when we apply step input to the RLC circuit how it will behave. Thank you!