Fiber- Optic Communication Systems and Techniques
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Lecture - 03
Uniform plane waves (UWPs) in free-space

Hello and welcome to NPTEL MOOC on Fiber Optic Communications Systems and Techniques course. In this module, we derive wave equation and discuss a few properties of wave equation, you may recall from the previous module that we introduced Maxwell's equations and said that Maxwell's equations describe all almost all known electromagnetic phenomena from frequencies as low as 0 to as high as optical frequencies that is from 0 hertz to terahertz a few tens of terahertz a few 100s of terahertz and these were the Maxwell's equations that we had. So, you had this Faradays law, then there is Amperes law which was modified by Maxwell so, this is Ampere Maxwell law.

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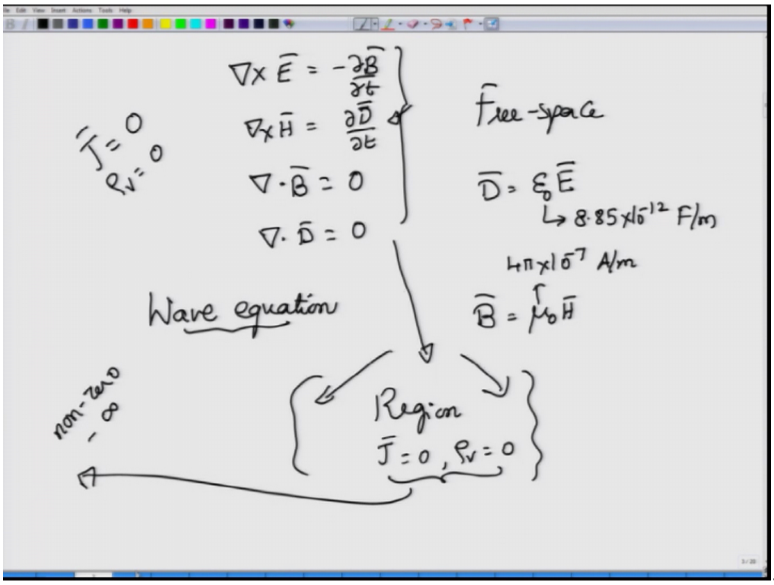

And the last 2 lost del dot B equal to 0 and del dot D equal to 0 are the Gausses law for electro magnetics right. So, one was for magnetic field and the other one was for the electric field case. What we do now is to consider a situation or what we have already done. So, is to consider a situation where the current density J is equal to 0 this is the conduction current density that is equal to 0 and the volume charge density the free charges are also considered to be 0. So, these 2 source quantities are considered to be 0 not everywhere, but at some distance far away from the region where we are considering the equation. So, the sources are considered to be 0 in the region that we are considering for example, let us say this is the region where we are considering this wave equation or the set of equations that we have.

So, in this region we assume J is equal to 0 and rho v is equal to 0, but these 2 quantities are nonzero at some point far away from this region and that point far away we will put this as minus infinity, but in the region that we are considering both J is equal to 0 and rho v is equal to 0 and consequently the terms del dot D equal to rho v becomes del dot D equal to 0 and J term in Ampere Maxwell equation disappears because we have assumed J equal to 0 right.

However, we do assume that they are nonzero, but they are nonzero not in the region where we are considering, but very very far away from that and in that region that is at minus infinity or plus infinity whatever the distance you want. So, we will keep minus infinity as a simple reference point. So, at this point very very far away these quantities J and rho v are possibly dependent on time. In fact, they should depend on time and then we ask this question, what happens to the electric and magnetic fields that is E field and the H field as I told you yesterday we are going to talk about E as the electric field and H as the magnetic field.

And what happens to these fields right of course, because this is free space E is related to D and H is related to B this relation is via the permittivity which in vacuum or in free space that we are considering as a numerical value of roughly 8.85 into 10 to the power minus 12 farad per meter and magnetic permeability of the medium that is permeability of the medium is by definition equal to 4 pi into 10 to the power minus 7 ampere per meter.

So, we have this region there are no J fields or the free charges and these are the equations which I mean these are the Maxwell's equations which have been applied for this case ok. So, these equations are valid in this region and you can see that we have removed the source terms or set the source terms equal to 0, does it automatically follow that because the source term is 0 in these Maxwell's equations in the region that we are considering does the field quantities E H B and D themselves become 0.

It turns out that they do not become 0 and in fact, because E can sustain, H field and H field can in turn sustain E field, what happens is in the region that we considered even though there are no source terms it is possible for us to have wave phenomena and electric and magnetic fields are part of this particular wave phenomena. And the equation that we are going to derive now describes how electric field and magnetic fields behave in the region even though the source quantities are themselves 0 in that given region of course, please remember that these source quantities are nonzero somewhere at the origin of the universe which we have taken to be around minus infinity ok.

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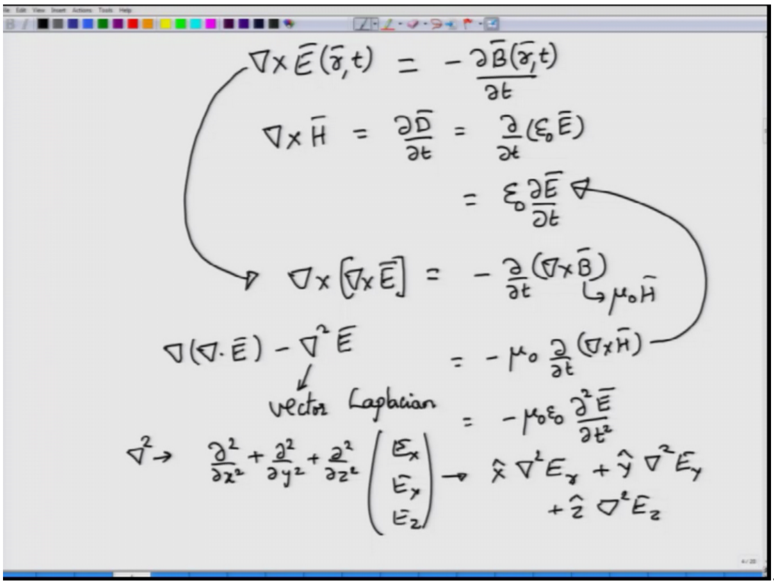

So, we start with an equation that we want to describe how the electric field varies with respect to both space and time and remember that space and time variation we covered it by writing electric field as a function of r and t, r was the position vector t of course, is your time variable and this equation according to Faradays law should be equal to minus del B, again B is a function of both r and t and this is the differential form of Faradays law that we wrote down right.

Now we also have an equation which says del cross H is equal to del D by del t I have dropped r and t from this equation for notational simplicity. Now in free space I know how D is related to E. So, I will actually use that expression or use that relationship and write in place of D epsilon 0 times E. Now epsilon 0 in free space is a constant is a scalar quantity it is a constant. So, this constant can be taken out of the differential so, you can move this epsilon 0 out and then write down this as del E by del t.

So, we do not have to simplify this one further we will wait for the other simplification to occur. Now what we do is a very interesting thing, I take this equation which was describing the curl of electric field and I take the curl of that equation again. So, basically what I have done is to take the curl of faradays law itself. So, naturally if I operate curl on the left hand side I should operate the curl on the right hand side as well and what I will do is I will interchange the curl and the differentiation operators right.

So, what I get is minus del by del t del cross B it may seem that we have no further way of proceeding with this equation, but we are wrong because we do have a way B is related to H by a simple magnetic permeability of the free space medium that we have already seen and B is basically equal to mu naught h and mu naught being a constant can again be pushed out of this differentiation as well as the curl expressions.

So, what I have on the right hand side of this expression is minus mu naught, mu naught has been taken out and then I still had del by del t del cross H. Now from the second equation I know that del cross H is actually equal to epsilon 0 del E by del t right. So, I will put that one into this expression take epsilon 0 outside of the integral and there is already a del by del t in this expression and there will be one more del by del t therefore, this becomes a second order partial derivative with respect to time.

So, what you have is minus mu naught epsilon naught del square E by del t square. So, on the right hand side we have an expression where the electric field the second time derivative of the electric field is considered and on the left hand side let us see how we can simplify. This we can simplify the left hand side by following a certain I mean certain identity of vectors ok, that is curl of curl of a vector quantity vector field quantity. In fact, is given by gradient of del dot E, del dot E if you remember is the divergence of the electric field and then minus del square E.

This del square is called as the vector laplacian operator. So, this is called as vector laplacian operator in the Cartesian coordinate system that we are considered this del square actually becomes del square by del x square plus del square by del y square plus del square by del z square and this del square by del x square del square by del y square del square by del z square has to operate independently on E x E y and E z.

So, the result of this del square E operation turns out to be a vector such that it will be x hat this is a scalar laplacian operating on E x plus y hat the scalar laplacian operating on E y plus z hat the scalar laplacian operating on E z. Of course, I should have differentiated the scalar laplacian and the vector laplacian maybe by some other notation, but I hope that you have already seen this equation. So, there is not that much of confusion amongst what constitutes a scalar laplacian and what constitutes a vector laplacian.

So, we now have on the left hand side of this equation we have some space derivatives going on and on the right hand side we have time derivatives or rather second order time derivatives going on ok. What is the quantity del of del dot E? Is it possible for us to further simplify this quantity? Yes it is possible for us to further simplify this quantity.

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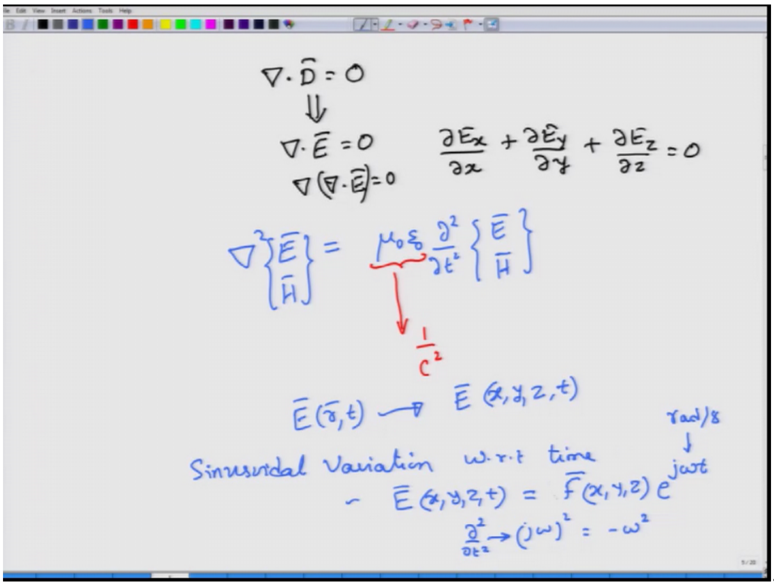

Because we know that del dot D is equal to 0 in the region that we are considering and using D is equal to epsilon times E and recognizing that epsilon is epsilon naught is just a constant you actually end up having an equation which tells you that del dot E is equal to 0. So, the divergence of electric field is equal to 0 right.

Let me write this down in Cartesian coordinate system so, that we all are aware of what is really happening here. So, you have del dot E equal to 0 in Cartesian coordinate system meaning that E x that is partial derivative of E x with respect to x, partial derivative of E y with respect to y and partial derivative of E z with respect to z is equal to 0 please keep this expression in mind we will come back to it after some time ok.

Now that we have made del dot E equal to 0 gradient of del dot E naturally will also be equal to 0, now we have an equation right. So, now, we go back to this expression since we know that del of del dot E is equal to 0 let me substitute a 0 here. So, what I have is this one. So, minus del square E is equal to minus mu naught epsilon naught times del square E by del t square and what is this del square, del square is second partial derivative with respect to space coordinates.

We have considered for simplicity Cartesian rectangular coordinate system, but the expressions for other quadrate systems are slightly complicated, but they are all available in the literature. So, if you look at Google or look at Wikipedia you will be able to find expressions for del square in any other coordinate system. The most commonly used coordinate systems are circular cylindrical coordinate systems spherical coordinate systems and Cartesian coordinate systems as you would know.

Now focus on this equation which I have rounded off with this blue color right, I have a minus sign on the left hand side I have a minus sign on the right hand side. So, I will put you know cancel that out. So, what I have remaining is on the left hand side a space variation or the second time I mean second derivative with respect to space coordinates of the electric field and on the right hand side what I have is apart from the scalar multiple mu naught epsilon naught the second derivative second partial derivative with respect to time of the same field quantity and if you go back to your high school or maybe you go back to your first year you know physics courses you know that this is how a wave equation would look like right.

So, if you take a string and then you know you shake the string with some applied force the displacement of the string from it is equilibrium position would follow this kind of an equation there you would have a one dimensional equation. So, you would just have maybe say if y is the displacement then you would just have del square y by del x square where x is the motion I mean x is the direction of the wave propagation.

Here you in general have all the 3 partial derivatives in them right. So, you have x square, you have y square and you have z square ok. This equation that we have written or that we have derived can be derived for magnetic field as well. So, the equation would look very similar in fact, the only change that you need to do is to replace E by H and the equation that you would have would continue to have on the right hand side a time derivative with I mean second order partial time derivative second order derivative and you have E and H.

The way I have written the equations is slightly standard in optical fiber textbooks what it means is that, this del square applies both to electric field and to magnetic field, but separately ok. So, the one way to read this equation is del square E equals mu naught epsilon naught del square by del t square times E del square H is equal to mu naught epsilon naught del square by del t square times H ok.

What about this mu naught epsilon naught is there any significance to this one, it turns out that if you put the values of mu naught and epsilon naught which I showed you at the beginning of the lecture you will see that this would actually be equal to 1 by C square. In fact, this kind of a identification was crucial for Maxwell to actually think of light as an electromagnetic wave. So, he postulated by calculating the values of mu naught and epsilon naught not in the S I coordinate system, but in the coordinate system that he used and he found out that the product of these 2 turns out to be exactly equal to 1 by C square.

So, the history of that is not something that we are interested in, but this product mu naught epsilon naught equaling 1 by C square is an extremely important step in formulating electromagnetic theory right. So, this is our wave equation that we have and you need to understand that in this wave equation we have electric fields which are varying with respect to r that is position vector as well as time since it is the cartesian coordinate system we can be slightly explicit about that.

So, we replace the position vector by the 3 coordinate values x y z and time naturally magnetic field will also be varying with respect to x y z and time and so, do these D and B fields, but since it is free space we do not really need to calculate D and B because they are very simply related to I mean electric fields E and H respectively ok. Is it possible for us to simplify this equation any further well there are couple of things that we need to consider ok.

So, first let us assume that instead of allowing for any arbitrary variation with respect to time we have a sinusoidal variation with respect to time ok. So, I have a sinusoidal variation with respect to time for all the field quantities which means that the electric field as a function of x y z and time can be written as some function of course, this function has to be a vector function. So, x y and z and with respect to time it goes as e power j omega t where omega is the angular frequency measured in radians per second.

And naturally when I am writing for e you also write the same equation for H do not use the same function F you can use another function called g maybe for example, right, now can I look at this equation and substitute this equation into wave equation that is possible for me to do.

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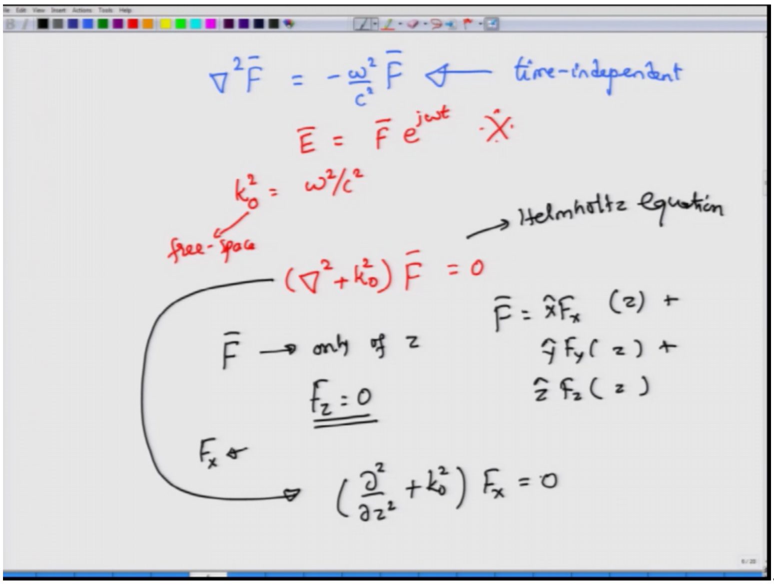

So, and when I do that what I actually end up on the left hand side is this del square F and please remember F is not a function of time E is function of time, but F is not function of time, but on the right hand side what I get is because there is second derivative with respect to time right.

So, first time when you differentiate this E field with respect to time you will pull out j omega. So, del by del t differentiation gives you j omega when you do it for the second time you get square of this and what is square of j omega, it is minus omega square of course, because this is an exponential function this operator actually results in minus omega square times whatever the function that was originally there right.

So, this is a simple exercise for you to calculate and show that this actually becomes minus omega square by C square times F itself. The power j omega t factor drops out from left hand side and the right hand side and what we have is now a time independent equation right. So, this in this equation F does not depend on time, but please remember the full electric field is actually a function of I mean is actually a product of the function F which is completely space dependent time derivative which is e power j omega t. So, please remember this I am going to mark this as very important ok.

So, now, this is the equation and we further define a quantity called k and in this case I will put a subscript 0, 0 indicates that this is free space that we are considering and in that free space case k 0 square is equal to omega square by C square and I can pull this right hand side equation on to the left hand side and essentially obtain even more simplified equation which would be del square plus k 0 square times F. So, F equal to 0 and in this equation I have assumed Cartesian coordinate system which means that del square will be expressed in Cartesian coordinate system and F is a function of x y and z coordinates ok.

So, far so, good is there any further simplification possible, well we have been given a choice slightly in the sense that I am going to assume that this function F is a function only of z coordinate system ok, which means that f which actually has 3 values right F x. So, it is actually x hat F x x y z plus y hat F y the same coordinates plus z hat F z the same coordinate dependence right. In this I am going to remove any dependence on x and y in fact, I can do this ok.

So, doing this does not contradict my Maxwell's equation. So, as long as I am consistent with Maxwell's equations this choice is perfectly legitimate. So, what I have is that all of these components F x F y and F z are functions of z. Now I am going to also say that I mean I am going to now assume another thing that F z is equal to 0 ok. Further for now let me just focus on F x. So, whatever I do on F x would be that would be the same thing that you can do it for F y with some sign changes you please keep that in mind. So, what I have done is to assume that there is no electric field component along the z direction ok.

And whatever the field components that I have will have will be either along x or along y or along the combinations of both because both are actually allowed ok. This k 0 that we have written will remain unchanged so, the equation that we have the time independent equation which is sometimes also called as Helmholtz equation ok. So, this equation is also called as Helmholtz equation what this equation becomes if you assume F to be having only the x component is that del square by del z square plus k 0 square times F x equal to 0.

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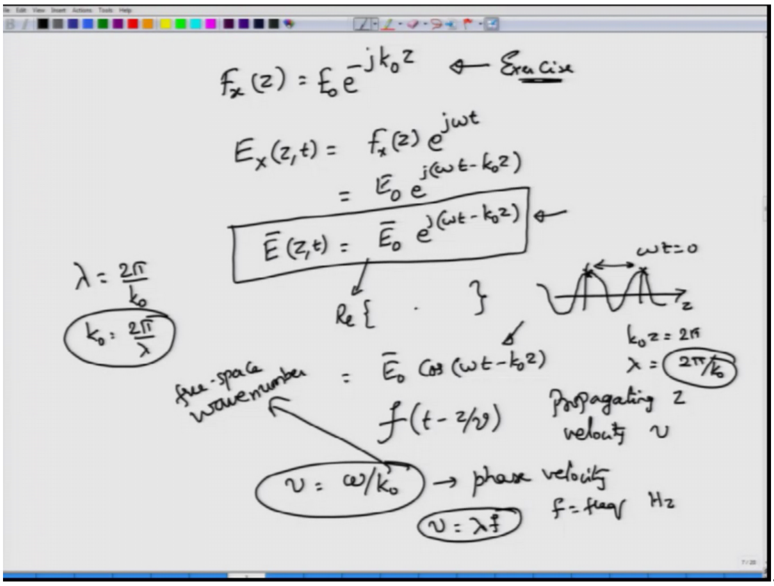

This equation actually has a solution which is given by F x of z being equal to e power minus j k 0 z I will leave this as an exercise for you to show that this is the case all you have to do is to use this expression go back to this equation and substitute of course, when you substitute you are going to realize that this equation is true and what you have done is that you have found out how the electric field actually varies when we assume that we have functional dependence of the electric field quantities and of course, the magnetic field quantities only on z and furthermore we have assumed that there is no z component out there.

Now this is just F x of z that we have obtained what would be the electric field quantity or electric field component if you remember E x right would now be function of both z and time and that is obtained by putting in the space variation F x of z times e power j omega t and what is F x of z from the above equation we already know that is some e power minus j k 0 z of course, you can even have some sort of a initial or a constant multiple for this one.

So, and that constant multiple I will take it as e naught instead of F naught because that goes nicely with this expression. So, what I have is e 0 or e naught which is just a constant times omega t minus k 0 z if you want you can express this in terms of the vector notation. So, in that case E will be a function of z and t and it has a vector E naught which of course, in our case will be directed along the x axis you have e power j omega t minus k 0 z.

Is it possible for us to have electric fields which are complex of course, not the correct expression for to write for this one would be to take the real part of it, but we are going to use this complex notation ok, because it simplifies lot of our discussion. There is also some relation of these equations to Fourier transform which we will explore in the exercises that we are going to give. So, I am not going to comment on that at this point.

So, what I have now is, an expression for electric field which then shows that it is in complex form it is of the form e power j omega t minus k 0 z, but in the real form when you take the real part of it would actually be something like cosine of omega t minus k 0 z and this cosine of omega t minus k 0 z is some function F of the form t minus z by v and now do you remember where we had seen this function f of t minus z by v yes this is exactly the solution of a wave equation which is propagating or this is an expression for a wave which is propagating along z axis with a velocity or speed of v.

So, what is this velocity v in our case you can compare this expression with the cosine expression and then see that the velocity is basically given by omega by k 0 right. So, this is called as the phase velocity and this expression k 0 is called as the free space wave number ok. So, we call this as free space wave number right and you can see because this is a cosine wave if you just plot this one, but there are certain so, if you hold time to be constant right. So, I take omega t is equal to 0 and then plot this expression as a function of z.

Since this is the cosine k 0 z or because it is an even function what you see is that it follows like a cosine wave right. So, at z equal to 0 it will have it is maximum and again after passing through one cycle it will have another maxima when k 0 z will be equal to 2 pi right and the distance between these 2 which of course, is given by 2 pi by k 0 is what is called as the wave length of this wave.

So, you have wavelength lambda being given by 2 pi by k 0 or sometimes more popularly you write down k 0 as 2 pi by lambda. So, please remember that you have velocity which is given by omega by k 0 and then k 0 which is given by 2 pi by lambda you can even put these 2 equations and then essentially see that v is equal to lambda f where lambda is the medium wavelength, in this case it is free space F is the frequency v is the velocity in free space.

Of course in free space v is equal to c because k 0 omega the ratio if you take by substituting for the value of k 0 right you will actually see that the velocity v turns out to be c itself ok. So, this is how we have a wave which is propagating along z axis and then it is electric field component is actually along the x direction. So, you have a wave which is propagating along the z axis and then the wave is directed along the x axis.

Another solution that we could have done using all these mathematics is that a wave I mean with that a solution where the electric field would be pointing along the y axis, I have taken the direction towards me as the y axis and this is the direction along the wave that is propagating. So, please note that there is no component of the electric field along the z axis whatever the component that we have is the component either along y or along x and when they propagate or when this wave propagates along the z axis there are certain planes where the amplitude of the electric field would be some value, but the amplitude in that cross plane right. So, if I consider z equal to constant then you can see from this equation cos omega t minus k 0 z that the fields do not depend on what position x and y you have.

So, you can go here, you can go here, any direction that you or any point that you can take in the transverse plane the electric field value will always be the same similarly magnetic field will also be the same, but they will be in pairs for example, if the electric field is along the x axis, then the magnetic field will be along the y axis, and if the electric field will be along the y axis, magnetic field will be along the minus x axis ok.

There is a relationship between these 2 which we will see in the next module which tells you that the direction of electric field cross h should point in the direction of the wave propagation. So, the wave is propagating along z axis and at every so and so, values of I mean at every certain values of z or constant values of z you actually have a plane the transverse plane in the x and y and the fields are independent of any value x and y in this particular case.

So, these I mean these equations are describing what is called as a uniform plane wave. It is a plane because electric field and magnetic fields are crossed with respect to each other and they do not change with respect to x and y, the amplitude remains the same along x and y, but they do change along z axis. So, this is the wave equation and the solution of Maxwell's equation in free space.

Thank you very much.