Fiber - Optic Communication Systems and Techniques
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Lecture - 18
Modal analysis of step index optical fiber

Hello and welcome to NPTEL MOOC on Fibre Optic Communication Systems and Techniques. In this module we begin study of our optical fibre communication channel. So, we are going to look at the modes of an optical fiber following the systematic procedure; that we have already discussed at length in the previous modules ok.

Now optical fibers come in various types as we have seen in the earlier module what we want to do is to keep the analysis simple so, that we get the essential idea of, how to obtain the most of the optical fiber and the simplification is not so much has not to be useful I mean useless, because it is actually useful most fibers do have a profile which can be approximated by the profile that we are going to consider in this module ok. The index profile that we are going to consider is what is called as a step index profile. Step index profile is very similar to the slab waveguide index profile; where in the index of the film was constant n 1 and outside n 1 it was n 2.

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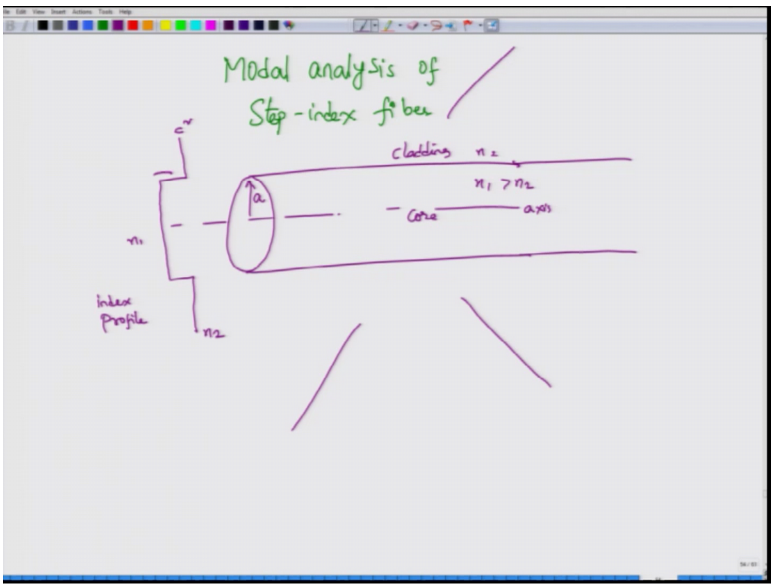

In the same manner we have an optical fibres sketched here please note that I have only sketch the core, because I am going to assume that this cladding essentially extends all the way to infinity.

So, you have this fiber which is in the form of a dielectric cylindrical rod ok, it has a radius a and the core is field with the material who is refractive index is n 1, but when you go to the cladding region that is outside the core then you suddenly go and then the profile actually changed suddenly from n 1 to n 2 ok. This difference between; n 1 and n 2 actually determines the amount of modes or the number of modes that can be guided and when the modes are actually confined within the optical fibre core itself.

One important point of difference between the slab waveguide that we have considered the so called planer slab waveguide to the optical fiber is that, optical fiber has a different geometry, it has a cylindrical geometry. And in practice n 1 and n 2 are not really very largely separately at least for the standard single mode fibers or fibers that are used for long distance communication, the value of n 1 and n 2 are very close to each other ok, but there is not really the case for an integrated optical waveguide such as this slab waveguide.

Of course, that is the superficial change you might in some sense think of, but the point of having a circular geometry makes our life very difficult when dealing with optical fibers, because except for a very special case, I cannot partition my modes into transverse electric and transverse magnetic ok.

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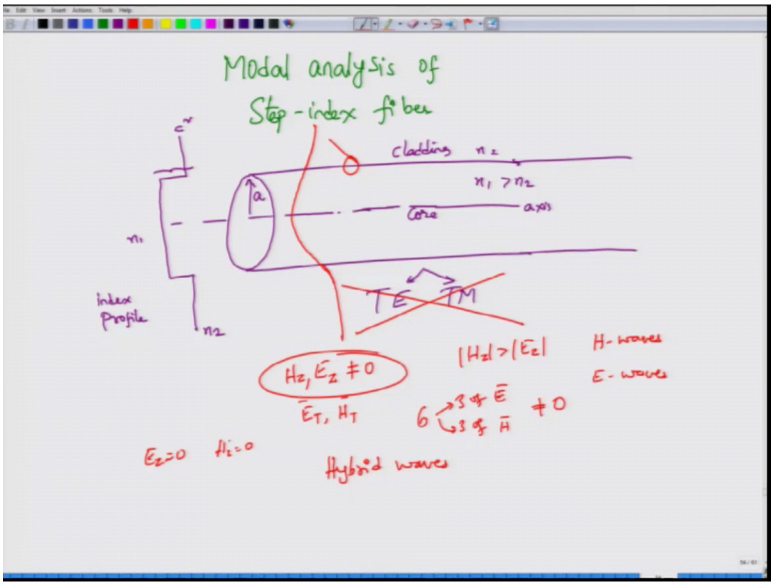

So this is not possible, because the geometry is not planer a, b the boundary condition that we are going to impose at the core and the cladding interface actually means that, the field does not stop and becomes 0 at the core cladding interface, but actually continuous along ok. So, from physical pictures of course, this is how the mode field would look like at least for the fundamental mode, where this is decaying outside the core right and then it is being guided in the cladding.

This is what you would actually expect to obtain and this is what is true for the fundamental mode except that the fundamental mode will in addition to having the components, it will also have the component E z and H z being non zero. Of course, sometimes H z will be larger than in magnitude corresponding to E z ok.

And in this case, you can call them as H waves and in the other case you call them as E waves these are older designations coming from microwave theory ok. So, the designations do not really worry about that, but what I was telling you is that in addition to the transverse components of the electric field and the magnetic field you will have usually the non zero horizontal components as well.

So, in general what would happen is all 6 components, that is 3 components of electric field and 3 components of magnetic field will be non zero for this waves. So, that is why they are sometimes called as hybrid waves, there is no easy way of separating them into a case, where E z alone is 0 and H z alone is 0 right.

So, we do not really have this kind of a suppression possible, because as I have told you for the two reasons ok. How to the go about analysing such a structure? Well we do know our 1st step is to choose a coordinate system.

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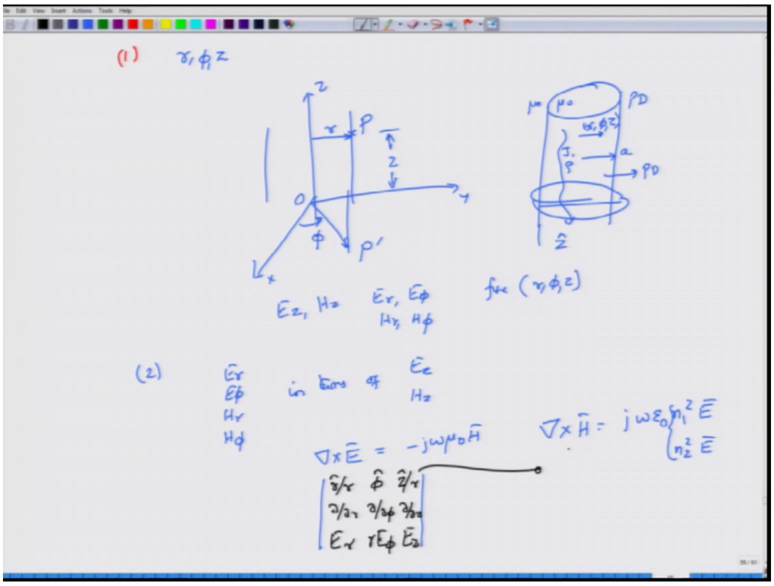

And in this case we are going to choose a coordinate system, which is circular cylindrical coordinates system in which any point ok, can be described by giving the radius at which the point occurs as well as the angle with which this particular point would make on the x and y axis ok.

So, this is the angle phi, this is a cylinder of radius r, this is the angle phi at which this point, when you projected onto the x and y axis, this is the z axis, when you projected onto the x and y axis this would be the angle made by the line that connects to the point where you have projected; the corresponding point P on to the x y plane, let say this is P prime the angle this line of P prime makes with respect to the x axis is the azimuthal angle phi ok. And of course, to indicate the elevation of the height of the point of the point above the x y plane, you will have to specify the z component as well.

So, this is the coordinate systems is perfectly suited for the optical fibers, because optical fibers are essentially cylindrical in nature; where the radius of the core is a any point can be written by 3 coordinates can be specified by 3 coordinates r phi and z. The longitudinal coordinate in this case is again the z directed fields so you will have E z and

H z and the transverse component which could be in this particular plane, will be the E r E phi H r and H phi components.

All these components can be functions of r phi and z themselves. The second step as you know is to actually go ahead and obtain E r E phi H r and H phi in terms of E z and H z to do so you again use the curl equation. So you have del cross E equals minus j omega mu naught H clearly because, there is no j and no charges inside the optical fiber, I have not specifically kept any current in the optical fiber. I am going to assume that dielectric medium is perfect dielectric inside as well as perfect dielectric outside. So, the curl equations will be curl of E equals minus j omega mu naught H and curl of H equal j omega epsilon 0.

Now, n 1 square times E in the core and n 2 square times E in the clad correct. I am assuming the magnetic permeability to be the same in both cases, so, therefore, I do not need to worry about that, but I have this j omega epsilon 0 n 1 square E and n 2 square E being the right hand side corresponding to core and cladding separately, because in the core the refractive index is n 1 in the cladding the refractive index is n 2.

So, how do we write this curl of electric field expression or curl of magnetic field expressions in the theoretical coordinate system? Well we go to the determinant idea again, in the determinant idea you can look up any text book of which deals with vector analysis: these are the 3 unit vectors along r phi and z and this is the determinant form of the equation. So, you have; del by del r, del by del phi and del by del z and then you have E r r E phi and then E z. So, this is your curl of electrics field as we have written in this span this determinant out you are going to get the expression for curl of E similarly, the curl of H expression would involve H r H phi and H z.

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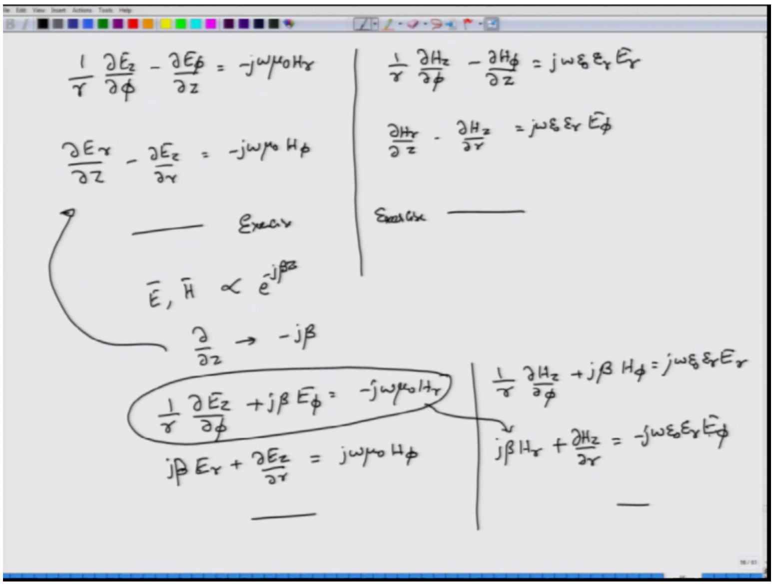

So, what I am going to now do is to write this equation side by side as I did for the case of the slab waveguides ok. So, this might take little while to write them down so in case, you are trying it or when you are revving this video and if you have a pen and paper with you I suggest that you stop the video here and then actually right down this equation expand the determinants and write the equation side by side and then you can continue to look at the equations on the video ok.

So, that will give a check; that you have done this carefully correctly we start by writing the r component; which would be 1 by r del E z by del phi. We have del E phi by del z being equal to minus j omega mu naught H r; the corresponding component for H will be this one where you replace E by H ok.

And then you have j omega epsilon 0 l at this point simply write this as epsilon r, I am not going to write down whether epsilon r is epsilon r 1 or epsilon r 2, because you can do the substitution later on ok. So, I am just writing this in a general media having a refractive index epsilon r to specialise this equation to core you substitute epsilon r by n 1 square and epsilon r by n 2 square for the cladding ok.

So, the phi component will be del E r del z minus del E z by del r equals minus j omega mu naught H phi. The corresponding equation here is del H r by del z minus del H z by del r equals j omega epsilon 0 epsilon r E phi. There is another equation which I leave as an exercise for you to write this down ok; this equation of course, connects the z components appropriately.

So, I am not writing that one and leave these as an exercise. Now what is our objective? Well we know that all these components whether I am looking at the electric field, the transverse or the longitudinal component or I am looking at the magnetic field longitudinal or z components. These components go with respect to z as e power minus j beta, z that is these are the most that are actually propagating along the z axis and therefore, because of their propagation along the z axis, they should have the face factor E to the power minus j beta z.

At this point please go back to slab waveguide in case, you do not really understand, why this e power minus j beta z comes up ok, but if you do understand you notice the similarity between the analysis of the slab waveguide on one hand with the optical fibers on the other hand, the difference is in the geometry and much more complicated expressions for characteristics equation as you will soon see either in this module or in the next module.

So, you have electric field and magnetic fields going as e power minus j beta z with respect to z axis. So, therefore, their del by del z can be replace by minus j beta. I am going to make though substitutions into these equations and obtain 1 by r del E z by del phi plus j beta E phi equals minus j omega mu naught H r. Again this is something that you have to do it to fully appreciate that these equations are correct ok.

So, you have j beta H phi equals j omega epsilon 0 epsilon r this is E r component, the second equation become j beta and because there will be a minus j minus on all sides you can remove that minus sin ok, just have we did for the slab waveguide some removing this minus sin everywhere. So, you have j beta E r plus del E z by del r equals j omega mu naught H phi. On the right hand side again we will have the same kind of a thing on the left hand side you will have a minus sin because del by del z will be minus j beta I transpose the minus sin on to the right hand side.

So, I will have j beta H r plus del H z by del r equals minus j omega epsilon 0 epsilon r E phi ok. The 3rd equation anyway does not really have any del by del z turns out to simplify that one is really no point to do that one. So, what you need to do is to actually look at this equation and then express H r in terms of E phi and E z and then use that equation into this second here the cross coupled equation that I am showing you substitute for H r and simplify this equation to obtain an expression for E phi ok.

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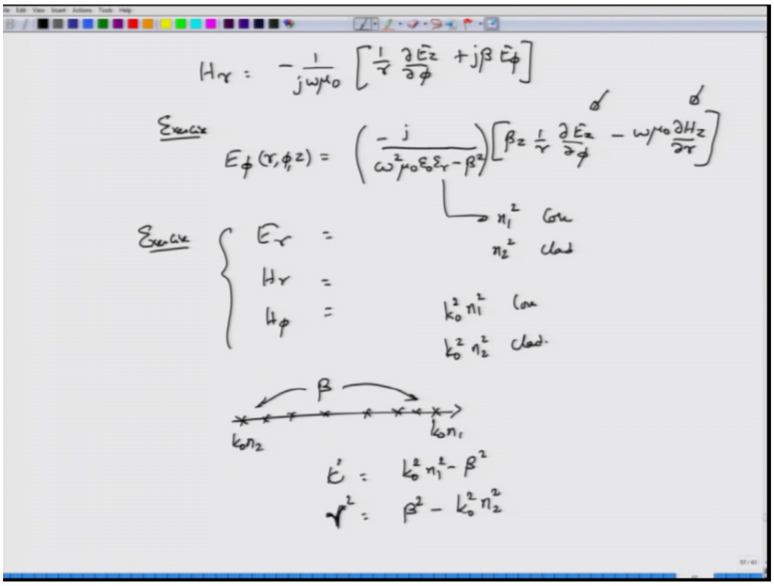

So, the expression for H r from the above equation would be minus 1 by j omega mu naught. So, I have 1 by r del E z by del phi ok, plus j beta E phi no problem at this point and when you substitute and then simplify I leave these as an exercise to you to show that the phi component which is the function of all 3 components r phi and z is given by minus j divided by omega square mu naught epsilon 0 epsilon r minus beta square ok.

So, there is no surprise here in this expression this is kind of the same thing that we have seen earlier as well times beta z 1 by r del E z by del phi minus omega mu naught del H z by del r. Pause here a little bit to understand what is going on. What we have done is to express E phi in terms of E z and H z as or rather there derivative with respect to phi and r and of course, you can do the same thing for E r you will have to do the same thing for H r and H phi.

And this is again you know an exercise, I will give the answers in the assignment for you to verify ok, but you will have to go through this E r H r and H phi and this term would essentially remain the same. And please remember that these terms are to be separately written for core and cladding right, because in the core epsilon r will be n 1 square and it would be n 2 square in the cladding. We can of course, simplify this omega square mu naught epsilon naught n 1 square by writing this as k 0 square n 1 square in the core and then you have k 0 square n 2 square in the clad.

So, you can simplify this omega square mu naught epsilon naught epsilon r by the writing them as, k 0 square n 1 square and k 0 square n 2 square. Just like in the slab waveguide the allowed values of beta will range from k 0 n 2 on the lower side, to K 0 n 1 on the higher side. And of course, not all values of beta r permitted only certain values of beta which will satisfy the characteristic equation will be permitted. And that beta has to be within this range k 0 n 2 to k 0 n 1 ok. Just as we did for the slab waveguide; where we defined kappa f square as k 0 square n 1 square minus beta square, we are going to the same thing here, but instead of calling this as kappa f I am going to simply call this is kappa ok.

I could have of course, called it is as kappa c to indicate that this is the core, but then I would also have the same later cladding so, I do not want to you know put any subscript that we confuses. And outside the core we know that the fields are going to decay without decay constant let say gamma or dependent on gamma for which; we define in a manner very similar to the slab waveguide as well so, gamma square will be beta square minus k 0 square n 2 square.

So, please do these exercises, these are very important for you to understand the next steps or other at the boundary condition where you are going to derive the characteristic equation ok. Now what you have here is completed on the step 2 step 3 corresponds to our wave equation.

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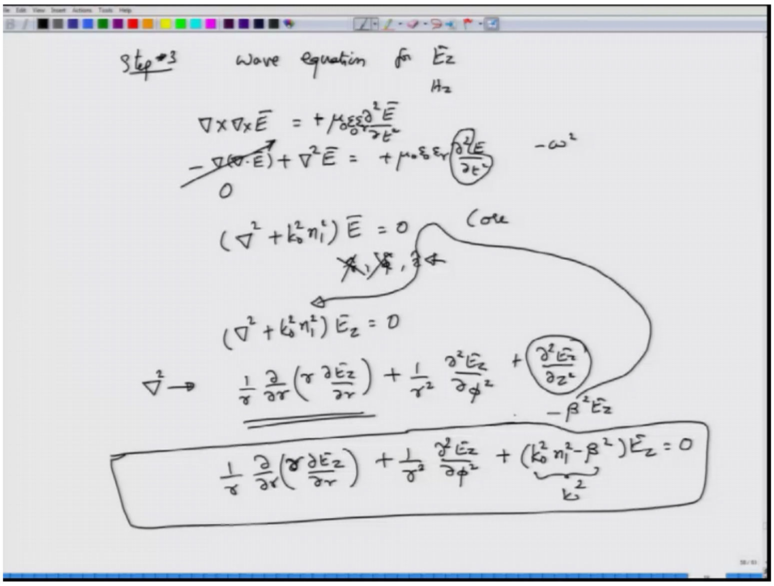

In this case the wave equation has to be for E z and the same equation will hold for H z as with the form of the equations are essentially the same. And what is the wave equation for E z or H z? Where it turns out that you are now looking at this del cross del cross E write. So, you started off with del cross E with this one so, this del cross del cross E was equal to something like mu epsilon del square E by del t square if I am correct ok.

So you can go back and then check so, of course, there is mu 0 epsilon 0 epsilon r. Again E z wave equation will have n 1 square in the core and n 2 square in the cladding. So, this del cross del cross E can be written as minus del del dot E plus del square E are believe. So, this could be minus mu naught epsilon naught epsilon r del square E by del t square.

So, I think there is the minus sign here not a plus sign so, please verify this for my; for us. So, I am not I do not remember this sins of here. I will set these to 0 I am going to talk more about; how I can set this to 0 or whether can I set this to 0 and this new for the optical fiber, it turns out in general I cannot set this to 0, but only in this special case I can do so ok. And the wave equation therefore, becomes del square and since we are assuming all harmonic type of solutions del square by del del t square becomes minus omega square.

So, then there is a plus sin here, because then minus sin comes from this; when you pull this omega square mu naught epsilon 0 epsilon r to the left hand side so, you will have plus k 0 square and 1 square in the core ok. In the cladding of course, you will have k 0 square n 2 square ok. So, this is just the core wave equation for electric field. Again this electrics field will have r component, phi component and z component; we do not want any wave equation for r and phi because, we want a wave equation for z component ok.

So, the equations for z component will be, del square plus k 0 square n 1 square in the core E z equal 0, this is the Helmholtz equation. Now base yourself for a little bit of a mathematical problem because del square does not have a simple form as it had for the Cartesian coordinate system. Del square in this case actually has a form; which is little complicated because, we are dealing with cylindrical coordinates ok. So, this is the del square equation that I am writing. So, 1 by r del by del r or del E z by del r ok, plus 1 by r square del square E z by del phi square.

So, in terms of phi and z you still have the same second derivative format, but this is the complicating fellow ok. Then you have plus del square E z by del z square, but I know the del square by del z square can be written as minus beta square E z. And of course, that is that minus beta square that I am going to substitute into the wave equation.

So, what I have is this simplified equation which depends only on r and phi. So, this is r actually, so, please keep this in mind; this are not gamma these are actually r. And then you have del square E z by del phi square plus k 0 square n 1 square minus beta square, which of course, as what we have describe can be rewritten as kappa square I will do the substitution later on.

So, but now it is K 0 square n 1 square minus beta square times E z which is equal to 0. This is the wave equation that we need to solve; if we want to understand how E z depends on the coordinates r and phi ok.

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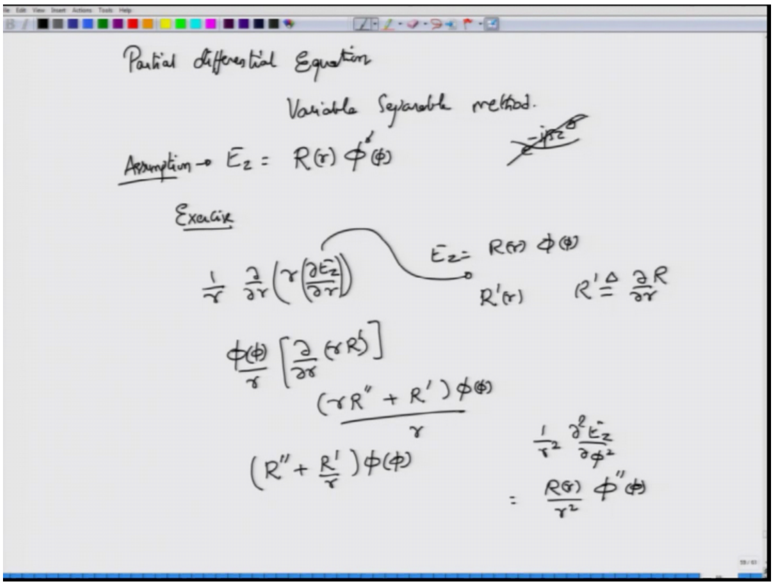

This equation is what is called as a partial differential equation ok. In the slab waveguide we converted with the partial differential equation into a ordinary differential equation, because of the one dimensional dependence on the mode fields. Here it is not possible for me to avoid the PDE because, this is exactly a PDE, because there are 2 variables and there are partial derivative sitting in this equation.

The standard way or at least one of the simplest way to begin to see whether this PDE has a solution or not, is to employ what is called as the method of variable separation or sometimes called as variable separable method ok. In the variable separable method what we try to do, is to assume that E z actually, can be written as a function only of r and a function only of phi ok.

So, R of r is a function of you know function, which is named R, which depends only on the radial component r and phi of phi is a function; which depends only on phi. Of course, along z that you already know e power minus j beta z, which you already substitute therefore, we do not need to worry about that but, now what we have is an E z which is function of small r and small phi, we are going to write this as a product. Now this is an assumption ok. It may turn out that this assumption is sufficient for us to find the solution it may turn out that is assumption is not sufficient for us to find the solution we will have to find this out.

Luckily, these equations have been solved by mathematicians and those equations are known to actually give you the correct solutions ok. So, without further do will actually substitute this R of r, phi of phi and as you could have guessed this turns out to be another exercise for you ok, because you know this is not very difficult is just a tedious steps that you need to follow.

So, and anyway I will show you at least, how to simplify one equation Ok, then the other equation is kind of very simple for you. So, what we have is 1 by r del by del r r del E z by del r correct. Now I substitute E z equals R of r phi of phi. And these partial derivatives are with respect to r therefore, phi of phi can be treated as a constant, when you are doing the differentiation with respect to r, so we can put this out.

So, there is a divided by r here because, that is 1 by r multiplying the whole term are there, then you have del by del r r multiplied with so, when you differentiate E z with respect to r, what do you get? You will get R prime of r, where R prime is by definition del by del r of capital R.

So, you have r R prime and the derivative of the product r R prime with respect to r is fairly simple. So, you have r R double prime here plus you have R prime because del by del R of r will be equal to one this needs to be multiplied with phi of phi divided by r. So, you will have R double prime plus R prime by r multiplied to phi of phi. The other term that you had one by r square del square E z by del phi square is fairly simple you can easily show that this is nothing, but R of r divided by r square, that is the constant outside the differentiation. And then the second derivative of E z with respect to phi means second derivative of phi with respect of phi, which would be phi double prime of phi.

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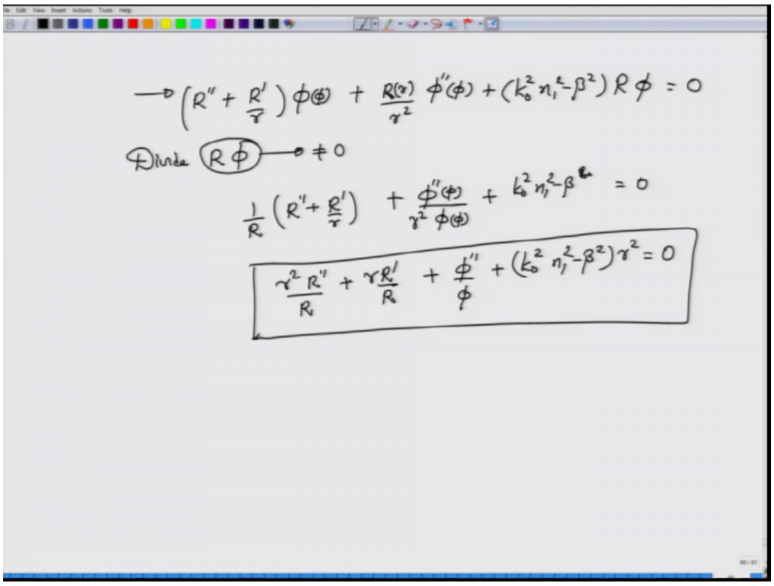

So, these 2 equations are what we were looking for now that we have them you can combine this equation. So, I will now have an equation which tells me that, R double prime plus R prime by r times phi of phi plus R of r divided by r square and then you had phi double prime of phi plus k 0 square n 1 square minus beta square times R phi equals 0 ok. Now what we do is we divide so, this as I have told you than exercise, so if you do not understand it please go back and then you know do this carefully, ensure that you understand how this equation was derived.

So, what we now do is we divide throughout by R phi of course, I am assuming that the solution does not vanish anywhere. So, the solution is not supposed to go to 0 otherwise this differentiation operation would not tell you work. So, when you do that you will have 1 by R R double prime plus R prime by r plus phi double prime of phi divided by phi there is an r square here as well so will keep that for the moment ok. So, this is phi of phi actually plus K 0 square n 1 square minus beta square equals 0. Now I multiply throughout by r square ok. So, when I do that when I get r square R double prime by R plus r R prime by R plus phi double prime by phi plus k 0 square n 1 square minus beta square times r square equals 0 ok.

So, this is the simplified equation that we now have which we you know from the method of variable separable. And what we now need to do is to solve this equation, it turns out that this equation can be solved relatively easily at least for the phi case, but inorder to solve this equation in terms of small r that is r dependence small r dependence; you need to know something called as Bessel functions ok.

So, before we go to the solution of this equations when you have some time brush up your Bessel functions at least look at how the Bessel functions of different kinds and different orders look like you have Bessel function of the first kind, j you know have Bessel functions of the second kind, when you are the linear combinations of those and you can actually look at those equation ok, we will be using only first kind with j and first kind of type two which is y ok.

So, these Bessel functions are necessary because that is how the fields are going to depend on r with respect to phi the equation is kind of simple. So, in the next module we are going to you know solve this equation.

Thank you very much.