Fiber - Optic Communication Systems and Techniques
Prof. Pradeep Kumar K
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture -15
Systematic analysis of parallel plate metallic waveguide

Welcome to NPTEL MOOC on Fiber Optic Communication Systems and Techniques. In the previous module because of the time constraint, we had to stop our 6 step procedure of systematically analyzing a waveguide structure to just about step 1 where we wrote down the Maxwell's equations and beyond that we did not do anything.

So, in this module what I want to do is to go through the remaining steps and show that these steps lead to the same kind of expression that we have already seen for the rectangular or for a parallel plate waveguide, which is made out of perfect electric conductors. The reason why we are doing this I have already told you is because just a simple we know Ray picture kind of an approach to analyzing waveguides does not give us full information about that; so having said that we have covered, first step which is basically Maxwell's equation,.

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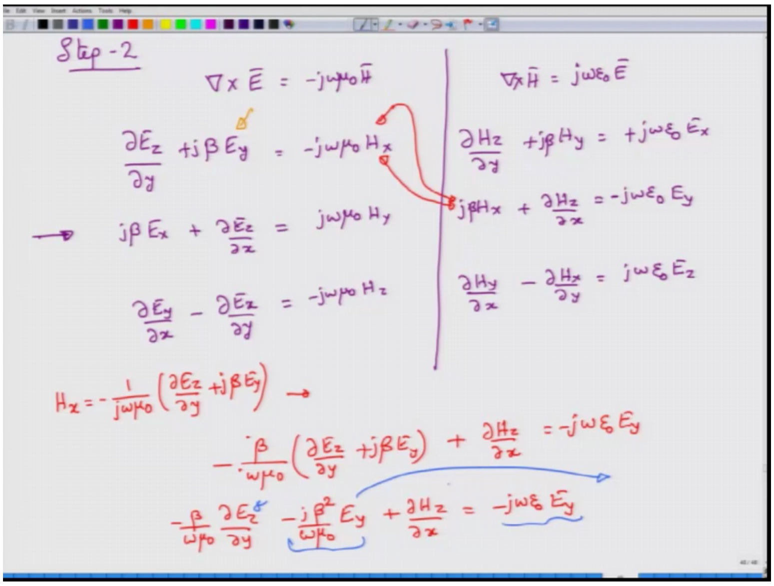

So now we are going to write down those Maxwell's equation in the coordinate system, the first was to choose the coordinate systems which choose as the rectangular coordinate system. Now we are going to write down Maxwell's equation in rectangular coordinates; so, this step should be familiar to you.

So, you have 2 curl equations of Maxwell's that are important in this context, So, I am writing the Maxwell's equations in the region between the waveguides that is the region where we have the waves propagating or the modes propagating and outside the region it is not really of our concern at this point,

But you can actually show that once you have a perfect electric conducting walls then those fields outside would not really be present ok; at least we cannot worry about whatever the radiation fields that might exist outside because we do assume that those radiation fields actually quickly go to 0, Anyway these are propagating structures they are not meant for radiating structures. We have the first equation curl of electric field being equal to minus j omega mu naught H and curl of H; equals j omega epsilon naught E,

Now, I am going to write down component vice the equations; you know the best way to write down these set of equations is to express this curl operation in terms of a determinant which I have shown you in the previous module, So, pulling those equations from the x component will be something like this. So, please note that each of those components themselves can be a function of x, y and z ok. So, E z can be a function of x y and z E y can be a function of x y and z,

So, this could be equal to say minus j omega mu naught H x. Similarly you have del E x by del z minus del E z by del x this is the y component; obviously, that I am writing, So, this minus j omega mu naught H y; similarly you have del E y by del x y and x component. So, del E y by del x minus del E x by del y,

So, this should be equal to minus j omega mu naught H z, So, I have this set of equations as I have obtained from the first curl equation. Now I have another curl equation which will allow me to write down the spatial derivatives of H in terms of the components E x E y and E z or rather equate the special derivatives of H with the electric field components on the right hand side,.

So, in the left hand side you had curl of E and the right hand said you have curl of H; you simply have to replace E by H to obtain the corresponding expressions and I am going to do that one now, So, I am writing this in parallel because I want to show you we know how to combine this equations and express E x E y H x and H y in terms of E z and H z and their derivatives ok. So, I have del H z by del y minus del H y by del z is equal to plus j omega epsilon naught E x.

Similarly, I have del H x by del z minus del H z by del x equal to j omega epsilon naught E y. Then you have del H y by del x minus del H x by del y equals j omega epsilon naught E z ok. So, I have these 2 equations; now what I am going to do is, so let me just go back to the equation.

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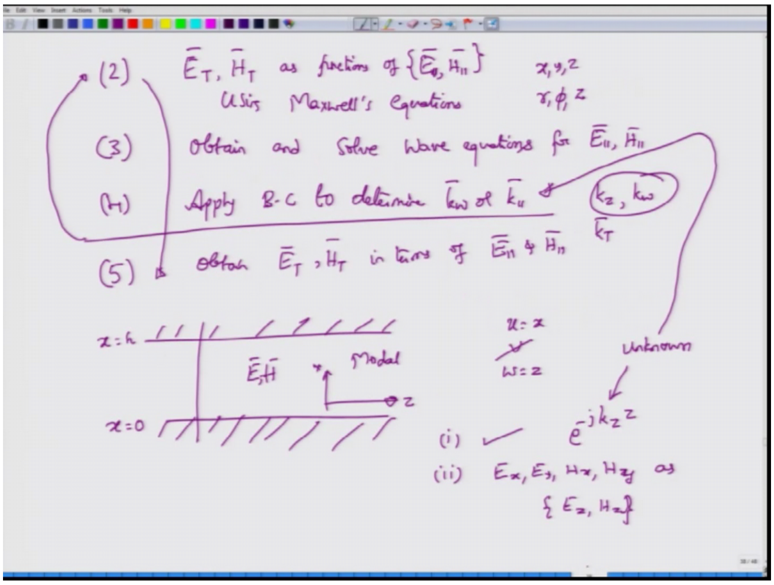

We have still in the step 2; where we had this transverse components that is E x E y H x and H y as and we need to express them as functions of E z and H z ok.

So, we are still in the second procedure itself or second step itself and for which we actually have these set of equations written already ok. So, in the same step right what we need to now do is to combine this equation or before doing that I already know that no matter what electric field component; I am going to consider whether I am going to consider E x, E y or E z and similarly H x, H y or H z all these components in terms of z dependence would show a dependence in the form of E power minus j beta times z,

That is because they are actually going to be modes which are propagating along z axis ok. So, these are the waves which would be propagating along the z axis. So, clearly if I take you know if I differentiate any of these components on the left with respect to z; what I would have is you know we already know that the way in which we will have our modes will be of the form F of x y E power minus j beta z of course, this F will be different for different components, So, if you actually differentiate this mode with respect to or the mode function, the total mode function with respect to z; clearly what you are going to get is this F of x y is independent of z; therefore, it will not be differentiated it could be treated as constant.

So, it will come out of the differential and then we have E power minus j beta z; when you differentiate it you are going to pull minus j beta. And because it is an exponential function this will be E power minus j beta as it is times F of x y. So, this was the original field and this is minus j beta. So, the operation del by del z can actually be replaced by multiplying that corresponding component by minus j beta; this is very important ok.,

Now put in all these equation that we have written this minus j beta I am going to substitute. For example, I have a minus del by del z in the first equation, So, I am going to substitute with minus j beta; so, that the first part can be rewritten as plus j beta ok. The reason is plus j beta because there is a minus sign out there, Similarly I will go ahead and make all the other substitutions; I request you to also do this substitutions in your note in case you are looking at them and we know you would actually be able to follow what we have been writing over here ok.

So, I have del by del z being replaced by minus j beta. So, I have a minus j beta at this point; the third equation has no del by del z component. So, will just leave it as it is where as on this side on the right hand side that I have again I am going to make the same substitution. So, here I have del by del z being substituted by plus j beta,.

So, I am removing this under this thing; so, this is H y and then I have minus del by del z on this side. So, I am going to replace that one by minus j beta; now I look at this equations ok. So, look at these equations there is a minus throughout. So, you can remove all the minus signs and then convert everything into plus. In this case you remove the plus I mean minus signs here and then put a minus sign on to the right hand side ok, this is just 2 kind of you know simplify the equations little bit right.

So, wherever possible I will try to simplify the equations in this manner and these are the equations that we have simplified. So, I hope that you all have understood how we have come up to step 2, So, we started off by choosing the coordinate system and second step was to write down Maxwell's equations individually; the curl equations essentially what we are writing them down; the divergence equations do not really help us in this particular case. In the curl expressions you can see that wherever del by del z term appears because we have assumed a propagation along z axis. So, you can replace del by del z with minus j beta ok. So, anyway after doing all that this is the set of equations that we obtained,

Now, we get the interesting part what we do is I look at this equation and I pair up with this second equation that I have. What is the meaning of pairing up what I want to do is I look at this expression you know on the left hand side and express H x in terms of the left hand side expression that is I am going to invert this left hand and the right hand convention. And when I do that I can write H x as one by or minus j omega mu naught times del E z by del y plus j beta E y correct.

So, this is all I have done is to simply rearrange the first equation and now what I will do is I will after rearrange that equation I will put that when into the equation on the right hand side which is the second equation on to the right hand side. Equally I could have treated this y component express E y in terms of the others and then substituted into the second expression I could have done that one.

And in fact, I would like you to do that one because doing that will allow you to express E y in terms of the other components or rather H x in terms of the other components, So, now let us go forward I have written down H x in terms of del E z del y and E y. Now I am going to put that one into the second equation here to obtain an equation.

Now I am going to erase this part, I hope you have understood this one the reason I am doing it on this slide itself is because I want you to keep looking at the equations, So, where I am starting and where if the equation at which I am substituting, So, put H x in this place; so what you get for the second equation on the right hand side is that you have j beta there is a minus j omega mu naught del E z by del y plus j beta E y which covers the first part; plus del H z by del x equals minus j omega epsilon naught E y ok.

So, if you want you can if you wish you can cancel this j from denominator and numerator and then you take this term and multiply individually to each term in the bracket here, So, you get del E z by del y or rather that would be minus beta divided by omega mu naught del E z by del y. And then you have minus j beta divided by omega mu naught sorry that would be beta square.

Because there is already a beta here then there is one more beta it may remove the minus j here ok; so that now you are alright, So, I have written this minus j beta square divided by omega mu naught this would be E y plus del H z by del x equals minus j omega epsilon naught E y correct.

So, now what you have? You have one E y component here another E y component here, you can move this one on to this side and then interchange the equations, And then write down what would be E y in terms of E z or rather the derivative of E z and derivative of H z.

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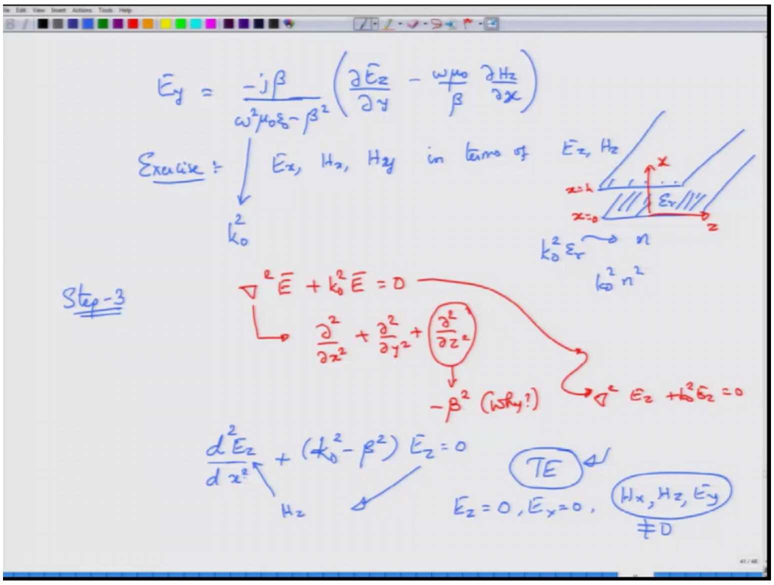

So, when you rearrange and write it I will not write the intermediate step I will leave that as an exercise; for you can just look at this one. And then see that you are going to get the same expressions that I am getting over here. Of course, if I have made a mistake here you would also be able to figure that out, but most importantly what you have is that you have kind of expressed everything in terms of you know E z and H z right or their it derivatives,

Now, the reason why we are doing this is because remember we have to express all are components E x, E y, H x and H y in terms of E z and H z; so, that when we solve a wave equation for E z or H z depending on which mode we are considering. Once we have solve that equation for E z or H z then it is very easy to go back and find out the mode functions or how the modes I mean other components of the mode function behave because I have all the information in terms of their expressions of E x E y H x and H y in terms of E z and H z.

Again it is not necessary that you choose E z and H z as the coordinates in which way I mean components in which you express other components ok. You could I equally chosen E y and H y, but this is a very systematic procedure which in almost all cases is going to give you solutions which you are looking for in a very least possible suffering from your side if I were to say that ok.

So, you have E y expressed and please take this exercise very seriously; once in lifetime you are you can do this one. What you have to do is to express other components that is E x, H x and H y in terms of derivatives of E z and H z, I have shown you one way to do this, when if you go back to this set of expressions that we have.

So, you had this expression where we you know we substituted for H x and then we express the component E y. If you substitute for E y then you can obtain H x from these 2 equations and the remaining 2 equations also can be similarly manipulated to obtain H y I know H y and E x components accordingly.

So, please do that and once you also look at what we have on E y expression; the denominator that we have is omega square mu naught epsilon naught which you recognize from your earlier classes that this is equal to free space wave number; that is square of free space wave number which I am going to write this as k 0 square.

If the waveguide were to be filled with a certain material of epsilon r; then instead of k 0 square you would have obtained k 0 square epsilon r and since epsilon r is the refractive index; so, you could have written this as k 0 square n square where n is the refractive index of the material that fills up this perfect electric conducting parallel plate waveguide ok. So, I hope that is kind of understood from your side.

So, you can simplify this expressions by removing omega square mu naught epsilon naught and then writing this as k 0 square, Now the step 3 that we are looking for is that we need a wave equation for even, And wave equation we have already derived are there in one of the introductory classes, where we have seen that curl of curl of electric field E will be equal to or rather let me just write down the final expression which we are already familiar with,

So, I have derived this equation with says del square E plus k 0 square E equals 0 right this is the so called Helmholtz equation that we wrote down, And here this del square is the scalar Laplacian which is individually given by del square by del x square plus del square by del y square plus del square by del z square, Luckily in the coordinate system that we have chosen this equation can be separately applied to E x, separately applied to E y, separately applied to E z,

So, in all that cases what you get is this is the wave equation that you are going to get right, And since we have expressed all field components in terms of E z; this is the equation that I am actually looking for, I do not want to look at these equations which correspond to E x and E y wave equations; I just want to know how E z wave equation will be looking like and this is exactly the equation that I am going to get,

And in this equation I can further make the substitution of del square by del z square by minus beta square, Why is this so? Go back to the start of this module and you will understand what why we have written this which works out this way. So now quickly; so I have del square by del x square plus del square by del y square ok.

And since my parallel plate waveguide is assumed with a coordinate system such that this is x and this is the plate x equal to 0 this is the plane x equal to H. And this direction is the z axis and along y direction we have assumed it to be infinity ok. So, you have the axis direction in which the parallel plate waveguide actually extends to infinity.

So, all of those del by del y terms can be made equal to 0 that is the modes are not changing as a function of y ok. So, the only thing that is they are changing as the function of x and now there is no need to write a partial derivative, we simply have a total derivative of the component that you are considering. So, you have d square E z by d x square plus d square E z by d x square plus basically k 0 square minus beta square times E z equals 0.

Now, this is the equation that we wrote for E z, but now I mean you can write a similar equation for H z as well. So, replace E z by H z in this equation and then you will be able to obtain an equation for H z. Now let us ask are we go on to talk about transverse electric modes or are we going to talk about transverse magnetic modes right?,

The essential difference between transverse electric and transverse magnetic modes we have seen is that transverse electric modes will have a component of electric field along y direction; they do not have any component along z direction. Whereas transverse magnetic waves will have component of the magnetic field along y direction and 2 in plane components E x and E z in the x and z plane,.

Since we find it little bit easier to talk about transverse electric modes just for the mathematical convenience; we are going to go with transverse electric mode in literature sometimes you will you will see that this is written as T E z, where this z is just to indicate that this is the mode which is propagating along z axis ok. So, I will not write this z every time at in the super script that will confuses as a little bit,

So, I am going to consider transverse electric modes which mean I cannot have E z at all. So in fact, for transverse electric mode E z is 0; so is E x the only thing that is non zero is H x right H z and E y, So, these are the non zero components for transverse electric mode, And now the wave equation I should not be writing in terms of E z, but I should be writing in terms of H z, So, this is the equation that I would like to now write down for H z,

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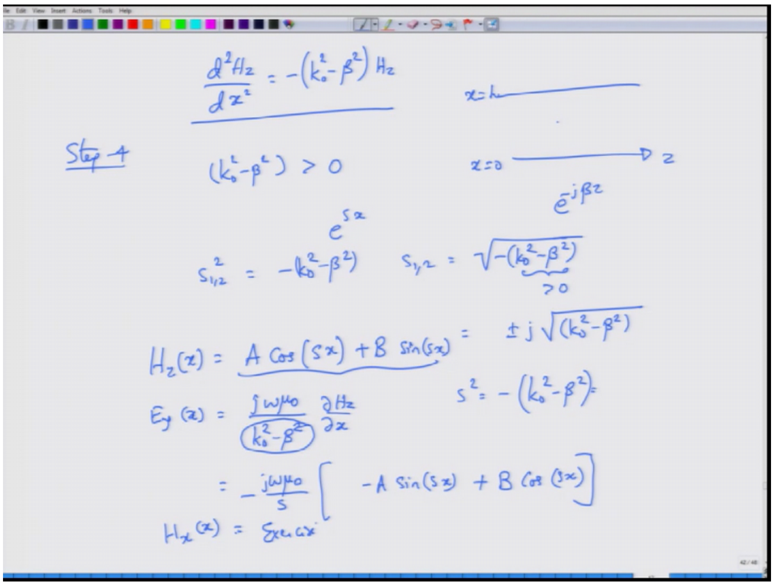

And that simply comes from the Helmholtz equation; so, I have del d square H z by d x square plus k 0 square minus or rather let me write it like this I will take this term to the right hand side I will have k 0 square minus beta square types H z, So, I hope this equation written in this fashion should be alright,

So, now I have this equation now what are the solutions of this equation? Well this is where we go to step 4 where our solutions are to be considered within the waveguide ok. So, I have x equal to 0 and x equal to H here as usual and z right. So, along z I already know what is this E power minus j beta z; so, I do not need to worry about the z dependence,

But on the x dependence the solution; now depends on the sign of the argument in the brackets right. If k 0 square minus beta square is positive, then I get solutions right. So, since in that case I will get a solution which would be the characteristic equation of this one will be what let us first write down the characteristic equation remember the solutions are assumed of the form E power S x,

So, I will have S 1 and 2 square being equal to minus k 0 square minus beta square is the characteristic equation, And therefore, S 1 2 which is the square root of the characteristic equation will be; square root of minus k 0 square minus beta square,.

Now I want k 0 square minus beta square to be positive because I want a sinusoidally oscillating solutions in the region between the parallel plate waveguide ok, I do not want a decaying solutions in between the waveguide, So, this quantity will be greater than 0 accordingly you will have plus or minus j square root of k 0 square minus beta square or instead of writing this plus or minus j kind of a thing; you can write down the solution for H z of x in terms of the linear combinations of this complex exponentials which are nothing, but sin and cosine function.

So, you will have A cos; now I need to find out certain, you know interesting letter for this ones. So, let me just write down this as S itself, So, I have cosine of S x plus B sin of S x ok. So, I have this solution now I know what is E y; right E y as a function of x we have already I mean derived it.

This would be if you go back and substitute or if you go back and look at the equation, you will have j omega mu not k 0 square minus beta square in the denominator; I have del H z by del x ok. So, you can see the previous equation out there and of course, what I have now is that k 0 square minus beta square is actually a positive S quantity right. So, I mean I can just write down as it is I do not want to worry about that one,

So, now write down what is del H z by del x here. So, writing del H z by del x by substituting from the top equation what you get is; j omega mu naught by k 0 square minus beta square is as it is, So, the solution that we have assumed is that this is going to be minus S square anyway, So, del H z by del x will give you differentiating H z will pull S out from the first equation, then you have minus A sin S of x plus B,.

So, S is the common thing; so I am going to remove this one from the entire thing. So, I will have S here; so I have B sin of a S x derivative will give you cosine of S x and this is the solution for electric field that we are going to obtain, And what is this k 0 square minus beta square? Well k 0 square minus beta square is basically; so, squaring up on both sides will give you S square equals minus k 0 square minus beta square.

So, I can instead of writing k 0 square minus beta square in the denominator; replace that one by minus S square and one S is in the numerator one S is in the denominator. So, I can remove that S from the numerator and in the denominator I will have a minus S here ok. So, this is the expression that I have; now I have obtained E y,

Since I have did not derive the expression for H x; you please derive the expression for H x and then you can obtain the corresponding solution ok. So, this I will leave it as a exercise for you ok; now we are not completely finished.

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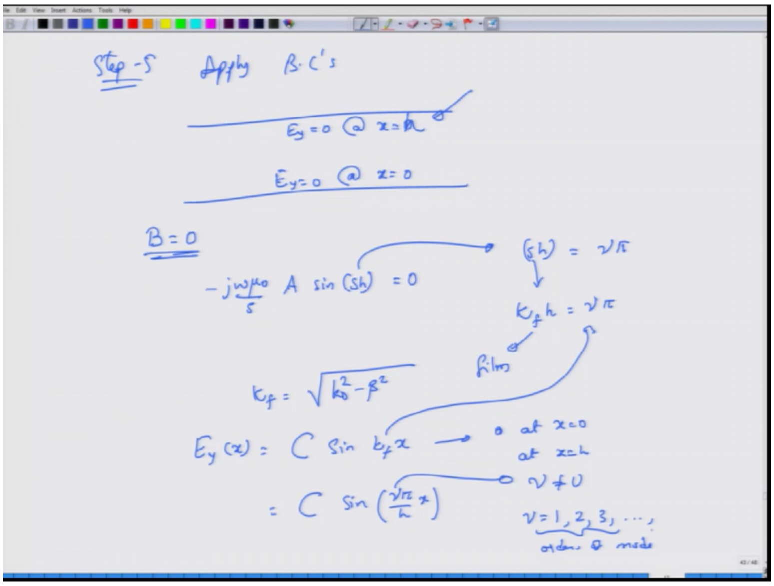

Let us now go to step 5 where we are going to apply boundary conditions,

So, what is the boundary condition? I know that on this surface right my E y which is essentially tangential component must be equal to 0 at x equal to 0 or rather this is x equal to H. And similarly E y being a tangential component will be again equal to 0 at x equal to 0 as well ok.

So, when you substitute these equations; essentially what you get is B equal to 0 ok. So, from the expressions that you substitute you will see that B will be equal to 0 and for the second boundary condition that had x equal to H, when you apply you will get a very interesting solution; you will get minus j omega mu naught by S; A sin of S H equal to 0,

Now, when can this sin function go to 0? When this term right S times H is equal to some integer multiple of pi ok. Now S seems to be pretty bad way of representing the solution. So, let me actually rewrite this one in terms of what is called as kappa f; f stands for the waveguide or the film or the slab if you wish that would be, So, what we have the condition is kappa f h equals nu pi where kappa f is given by square root of k 0 square minus beta square,

So, what we have seen is a very interesting expression out there that you will get electric field component E y of x being proportional to I will put all the proportionality constant into some C and then you have sin of kappa F times x. Clearly this function will be equal to 0 at x equal to 0 as well as at H equals to H because we have written kappa f in terms of some integer nu pi right,.

So, this would be C sin nu pi by H times x, So, this is the expression and nu of course, can take on different values; nu equal to 0 makes no sense because then the field itself will be equal to 0, So, nu is not equal to 0 , but the first term that you are going to get will be nu equals 1, 2, 3 and so on and these are all the different orders of mode ok; so, these are all the different modes,

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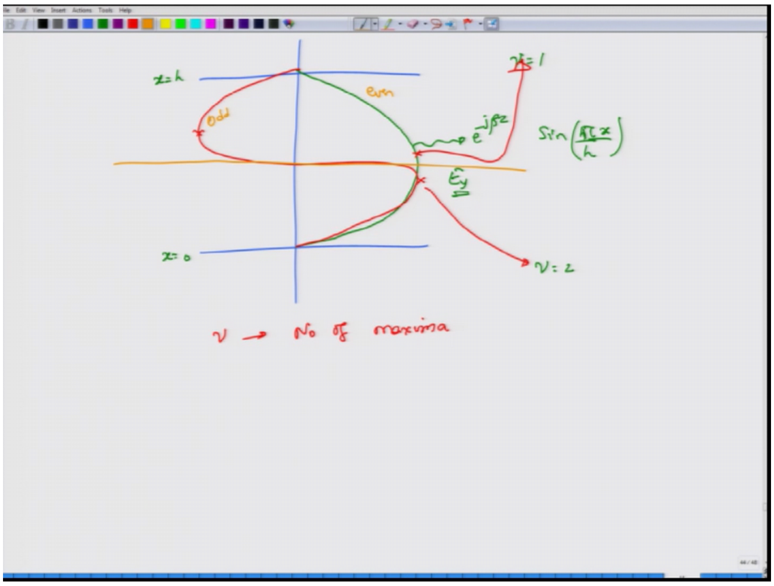

And now let me just sketch for you how would the functions look like; for let say nu equal to 1, So, nu equal to 1 will have sin of pi x by h which of course, must start at 0, this is the x equal to 0 plane this is the x equal to h plane; it must start at 0,

And then at h equal to I mean x equal to h, it must go 0 again. So, this is E y which I am plotting; so, you can of course, plot h z and h y separately for you I mean if you wish and this is the mode that would actually be propagating along the z direction with the E power minus j beta z function, When you take nu equal to 2 you will actually see that again the function will start at 0,

But now, it will actually go something like this ok. So, it will actually exhibit 1 maxima; 2 maxima; so, it will actually exhibit 2 maxima which is equal to nu. In fact, the first one actually exhibited one maxima which was equal to the value of nu. So, in fact nu tells you the number of maxima that your wave is going to go or the mode that is going to present between the 2 parallel plates,

So, I have left out couple of; in fact, let me just point out finally, that this mode. So, if you want to draw a line here at the centre you would find that this mode is an even function the green one. So, this is called as the even mode and this mode is called as the odd mode because it is basically in a odd function,.

The even and odd modes actually depend on the symmetry of the line that you would draw. So, because of that this become even and odd; so, this is not really a standard convention out there, So, you have seen that all the steps that we have written except for the last step, where you will be writing E x, E y, H x and H y in terms of the other components ok.

So, you can derive all the other component mode functions and this is how you analyze systematically any given waveguide structures; so, starting from choosing the coordinate system to writing the final solutions ok.

Thank you very much.