Applied Optimization for Wireless, Machine Learning, Big Data.
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Lecture - 09
Eigenvalue Decomposition of Hermitian Matrices and Properties

Hello, welcome to another module in this massive open online course. So, you are looking at examples to understand the framework or the mathematical framework or preliminaries required for developing the various optimization problems. Let us continue our discussion in this module, let us start by looking at the Eigenvalues of a Hermitian Symmetric Matrix, ok.

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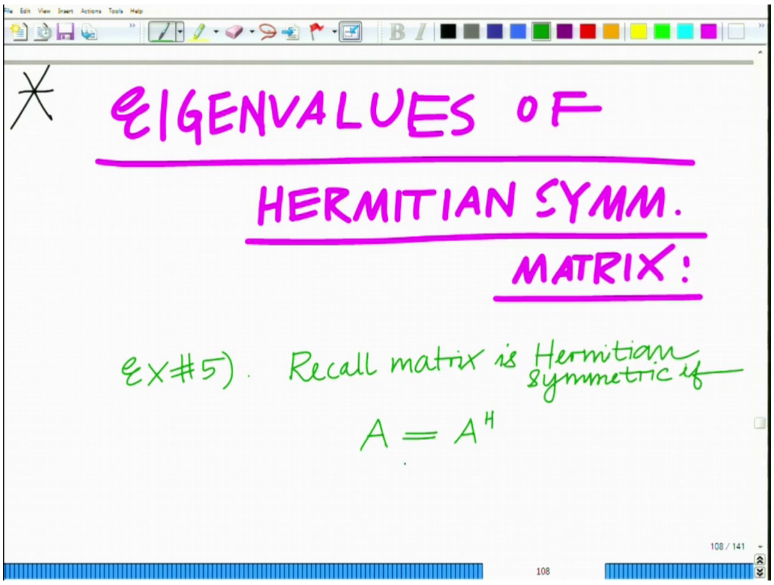

So, we want to look at the Eigenvalue Decomposition or Eigenvalues and Eigenvalue decomposition of a Hermitian matrix. Hermitian Symmetric that is if a matrix is Hermitian Symmetric we know. So, this is our example number 5. So, recall matrix is Hermitian symmetric if A equals A Hermitian ok.

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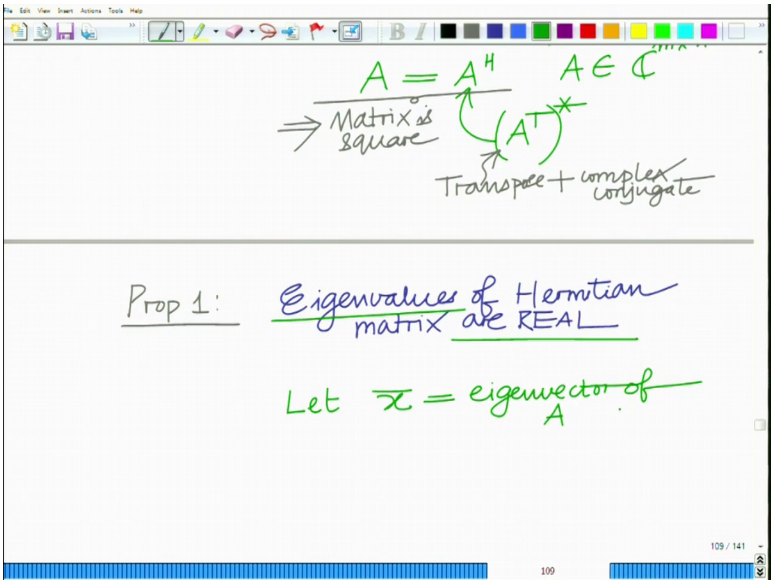

So, m cross, so for any m cross n. So, if A remember this is for complex m cross n matrix and remember Hermitian is nothing, but you first take the transpose of A and then take the complex conjugate. So, Hermitian involves two steps, Transpose plus Conjugate.

So, first you take the transpose of the matrix m cross n matrix becomes an n cross m matrix and then you take the complex conjugate of each element and if then you because we have a matrix which is A equals that is where is the property that A equals A Hermitian this is known as a Hermitian symmetric matrix or simply sometimes the Hermitian matrix and naturally for a Hermitian matrix for matrix to be Hermitian symmetric it must be the case that it is a square matrix correct. So, that an n cross n matrix remains an n cross n matrix when you take its transpose.

So, this implies for a Hermitian matrix, implies essentially also that matrix is square, makes sense only for a square matrix. Now we want to prove some of the properties in this example, we want to prove some of the properties of this Hermitian symmetric. Hermitian symmetric matrices, that is Eigenvalues of a Hermitian Matrix are real. The first property is Eigenvalues of Hermitian matrix are REAL that is if you look at I consider the Eigenvalues of Hermitian matrix these are Real ok. So, let to prove this let us consider x bar be the Eigenvector of A, Lambda equals the corresponding Eigenvalue, ok.

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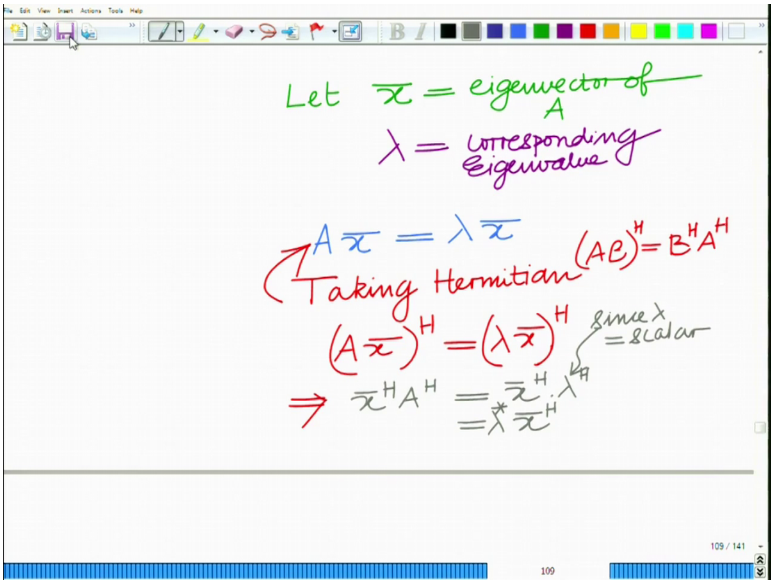

Taking the Hermitian, now what we are going to do is we are going to take the Hermitian on the left and right side. So, we have now, Ax bar Hermitian equals lambda x bar Hermitian and you will use the property that A B Hermitian product of two Hermitian. So, A B Hermitian that is the product of two matrices are Hermitian A B Hermitian is B Hermitian times A Hermitian.

So, this implies basically that x bar Hermitian that is Hermitian vector, Hermitian of vector x bar times A Hermitian equals x bar Hermitian times lambda Hermitian, but lambda is a scalar. So, this, I can simply write as and of course, this is just a number so, I can simply write as x bar Hermitian into lambda conjugate. Since lambda equals a scalar quantity that is lambda is simply A, is lambda is simply a number. So, for a simple number the Hermitian of the quantity is simply taking the complex conjugate and now we have this property x bar Hermitian A Hermitian equals lambda conjugate x bar Hermitian.

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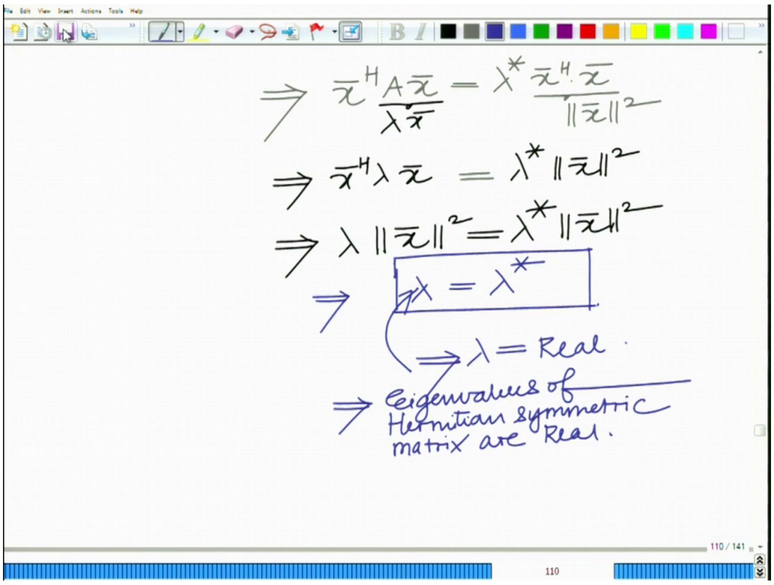

Now we multiply on left and right by x bar. So, this implies x bar Hermitian A Hermitian x bar equals lambda conjugate, x bar Hermitian x bar, but look at this x bar Hermitian x bar this is norm x bar square. So, this is equal to norm x bar square lambda conjugate times norm x bar square and further Ax bar equals lambda times x bar. So, this implies x bar Hermitian lambda x bar equals lambda conjugate norm x bar square and this implies now lambda is a number.

So, this is lambda x bar Hermitian x bar equals norm x bar square equals lambda conjugate norm x bar square and from this from the left hand side and right hand side since x bar square, norm x bar square is not equal to 0, this implies lambda equals lambda conjugate, which basically leads us to the conclusion that lambda is a Eigenvalue of a Hermitian symmetric matrix this implies that lambda equals a real quantity ok.

So, Eigenvalues of, so this implies Eigenvalues Hermitian Symmetric Matrix are real, Eigenvalues of Hermitian symmetric matrix are real. Now how about eigenvectors of a Hermitian symmetric matrix.

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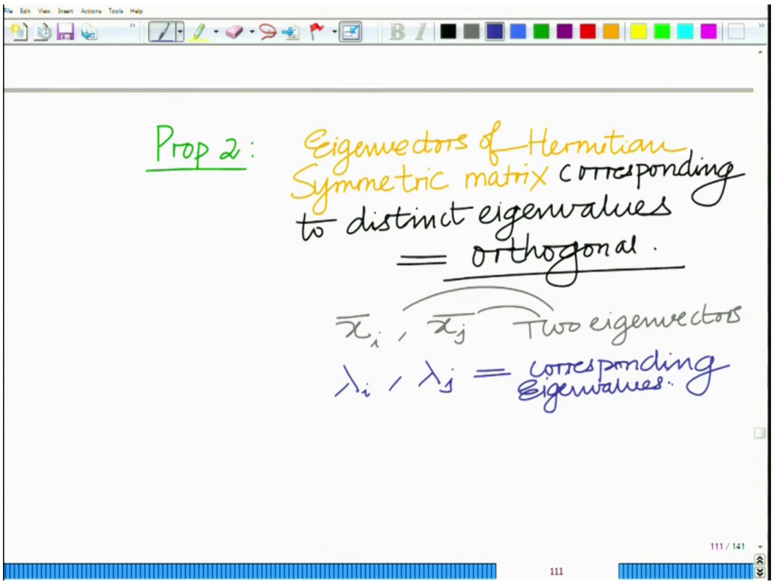

Now eigenvectors of Hermitian symmetric matrix satisfy an interesting property that is eigenvectors of a Hermitian symmetric matrix corresponding to distinct Eigenvalues are orthogonal that is their inner product is zero, we will demonstrate this fact. So, the property number 2 and another very interesting property, both these properties of Hermitian symmetric matrices very interesting and have immense utility.

The second property is that Eigenvectors, Eigenvectors of, Eigenvectors of Hermitian symmetric matrix corresponding to distinct Eigenvalues, corresponding to distinct Eigenvalues,these are, these are orthogonal and this is a very important and interesting property ok.

So, let us consider two Eigenvectors x i bar comma x j bar, these are two Eigenvectors and remember these correspond to distinct Eigenvalues, ok. These are the corresponding Eigenvalues and remember these are distinct, lambda i not equal to lambda j that is these are distinct.

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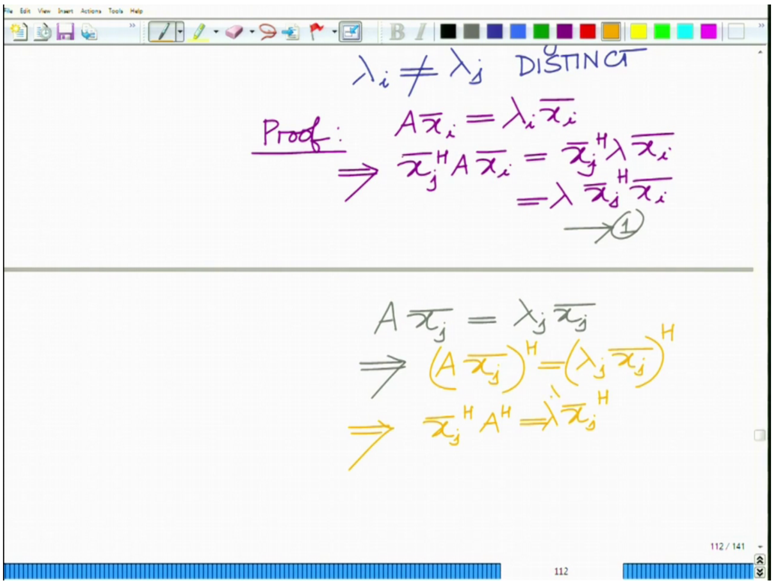

And now what we want to show that the inner product x i bar Hermitian x j bar equals zero and the proof can proceed as follows that is if you look at a x i bar equals lambda i x i bar and this implies now if you take multiply by xj bar Hermitian we have x j bar hermitian A x i bar equals lambda i or x j bar Hermitian lambda x i bar which is basically lambda x j bar Hermitian x i bar lambda x j bar Hermitian x j bar.

So, let us call this as the first observation or the first result x j bar Hermitian Ax i bar equals x j or Hermitian lambda x i bar its lambda x j bar Hermitian x i bar. Now we also have x j bar is an eigenvector corresponding to the Eigenvalue lambda j which implies A x j bar equals lambda j x j bar this implies that x j bar Hermitian taking the Hermitian x j bar or a x j bar, let us just write that one step A x j bar Hermitian equals lambda j, x j bar Hermitian this implies x j bar Hermitian A hermitian equals x j bar once again, x j bar Hermitian, we have already seen lambda j conjugate because lambda is once again it is an Eigenvalue its simply a scalar quantity.

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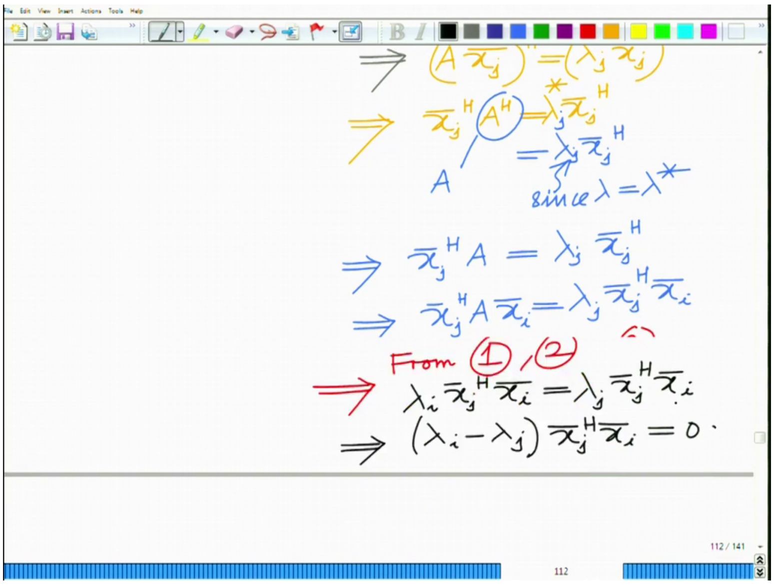

So, this is lambda conjugate, but remember the Eigenvalues of Hermitian matrix are real which implies that lambda j conjugate equals lambda j ok. So, we will use that property here, this is equal to lambda times x j bar Hermitian since lambda equals lambda conjugate and this implies that well again realize that this is the Hermitian symmetric matrix A Hermitian is simply A ok.

So, we have x j x j bar Hermitian into a equals lambda conjugate into x j bar Hermitian. This implies that now if you multiply by x i, I am sorry this is lambda now if you multiply by x i bar we have x j bar Hermitian A x i bar I am sorry this is lambda j this is lambda j this is lambda j this is equal to lambda j times x j bar Hermitian lambda j x j bar Hermitian x i bar ok.

So, x j bar Hermitian x i bar equals lambda j x j bar Hermitian x i bar and this we can, this we can denote as result 2 and now if you see from result 1 and result to x j bar Hermitian a x bar x i x i bar equals lambda or lambda. In fact, lambda i, lambda i x j bar Hermitian x i bar. Similarly if you look at result 2 x j bar Hermitian A x i bar equals lambda j x j bar Hermitian x i bar. So, this implies from 1 comma 2 from results, 1 comma 2 what we have, this implies well we, this implies lambda i x j bar Hermitian x i bar equals lambda j x j bar Hermitian x i bar which implies now lambda i minus lambda j into x j bar Hermitian x i bar equals 0.

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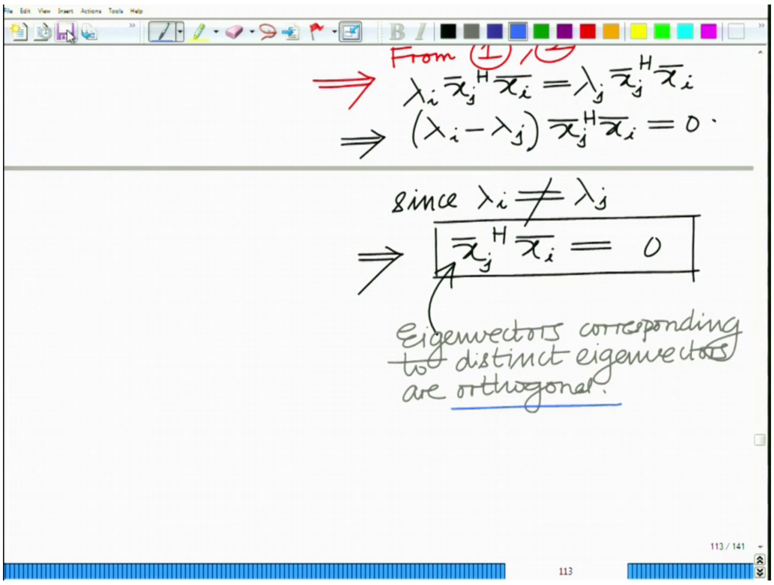

And now, since lambda i not remember that is a key point lambda i is not equal to lambda j otherwise lambda i minus lambda j can be zero. So, this implies x j bar Hermitian x i bar this equal to zero.

So, this finally, verifies the fact that, Eigenvalues correspond Eigenvectors corresponding to distinct Eigenvalues are basically real. So, the Eigenvalues corresponding to, the distinct Eigenvalues are basically real. Ok let me just mention this or eigenvectors corresponding to distinct Eigenvalues are orthogonal that is what I meant to say. Eigenvalues corresponding to distinct Eigenvalues are eigenvectors corresponding to distinct Eigenvalues are orthogonal or a Hermitian symmetric matrix and these are two important and interesting properties of Hermitian symmetric matrices that one uses frequently during the development of various, various techniques for optimization, all right.

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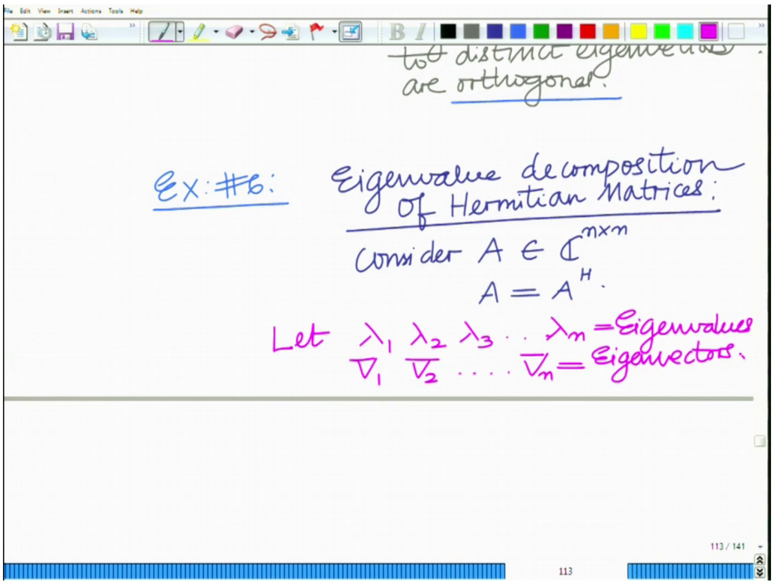

Let us look at another example,example number Eigenvalue decomposition of Hermitian matrices, Eigenvalue decomposition of Hermitian matrices for a Hermitian matrix that is consider that is n cross n complex Hermitian matrix equal to A Hermitian that is Hermitian symmetric, let the Eigenvalues be equals or the Eigenvalues and V 1 bar V 2 bar V n bar be the corresponding eigenvectors ok.

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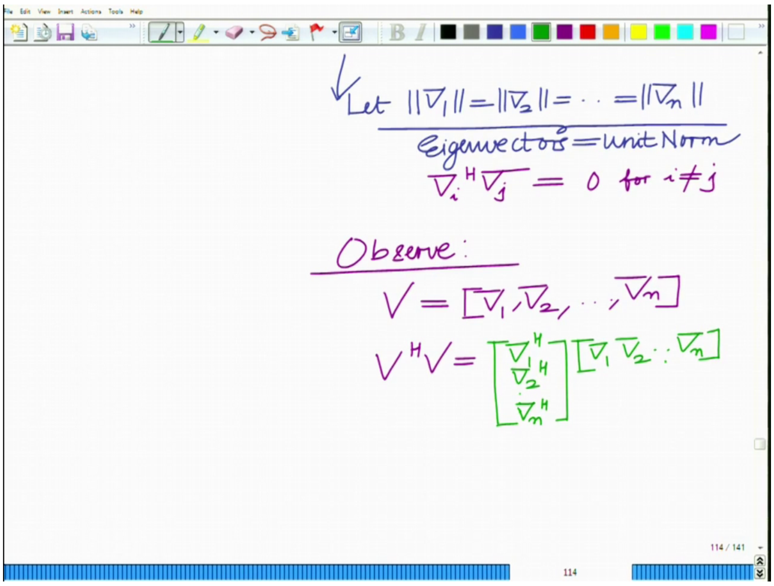

Now observe that let these eigenvectors be unit norm, we can simply normalize remember an eigenvector if you normalize it, it still remains A eigenvector because you are simply dividing it by a by it is norm or by a constant that is you are simply scaling an eigenvector ok. So, let us consider the Eigenvectors to be unit norm, all right.

So, let norm V 1 bar equals norm V 2 bar equals norm V n bar, this implies eigenvector is equal to unit norm and further from the property previously let us assume that the Eigenvalues are distinct which implies V i bar Hermitian V j bar equal to zero for i not equal to j that is if the Eigenvalues are distinct then the eigenvector satisfies the property V i bar Hermitian V j bar equals zero ok.

Now, notice that if you consider this, observe that if you consider this matrix V 1 bar, V 2 bar, V n bar, the matrix of eigenvectors if you now perform V Hermitian V then what you are going to have is you are going to have V 1 bar Hermitian V 2 bar Hermitian V n bar Hermitian times V 1 bar V 2 bar V n bar which is equal to.

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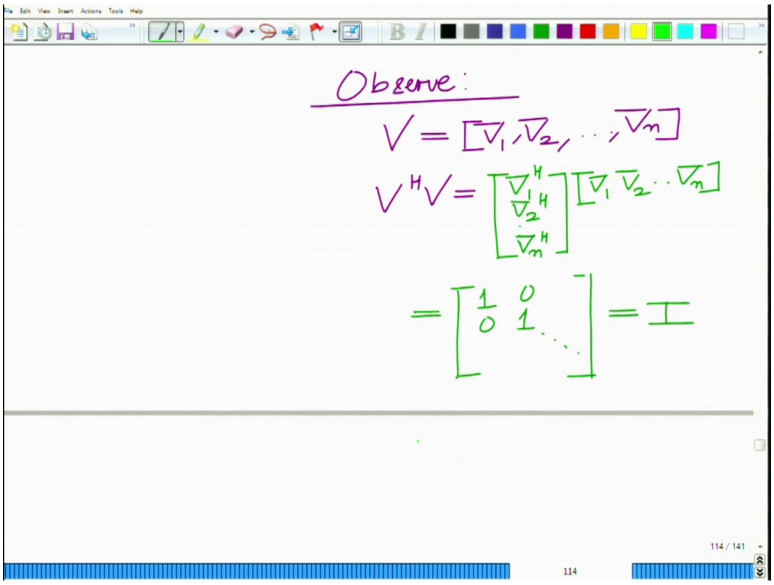

Well if you look at V 1 bar Hermitian, V 1 that is 1, V 1 bar Hermitian,V 2 is 0 V 2 bar Hermitian V 1 is 0 V 2 bar Hermitian V 1 is 1. So, this you can see is simply the identity matrix which implies V is the inverse of V Hermitian and V Hermitian is the inverse of V. So, we have, well we have something interesting, what we have is, we have V Hermitian V.

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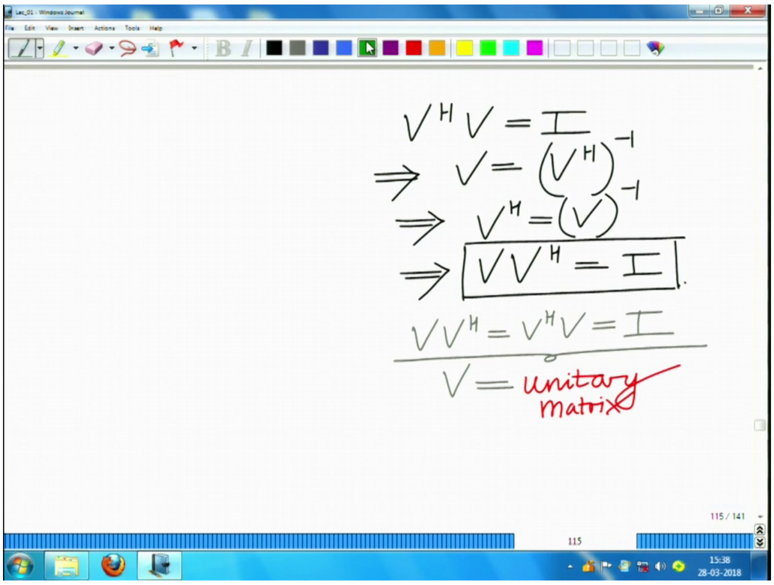

So, we have V Hermitian V equals identity implies V equals V Hermitian inverse and since the inverse of A square matrix is unique and this also implies V Hermitian equals V inverse and this also implies that since if A is B inverse A B is identity be A is also identity this also implies that V, V Hermitian equals identity since if A B is identity then B A is also identity for square matrices.

So, we have and such a matrix V which satisfies V V Hermitian equals V Hermitian V equals identity such a matrix is termed as a unitary matrix. So, V is termed as this satisfies this interesting property that is termed as a unitary matrix that is V matrix V, square matrix which satisfies this property V V Hermitian equals V Hermitian V equals identity is said termed as a unitary matrix, unitary matrix ok.

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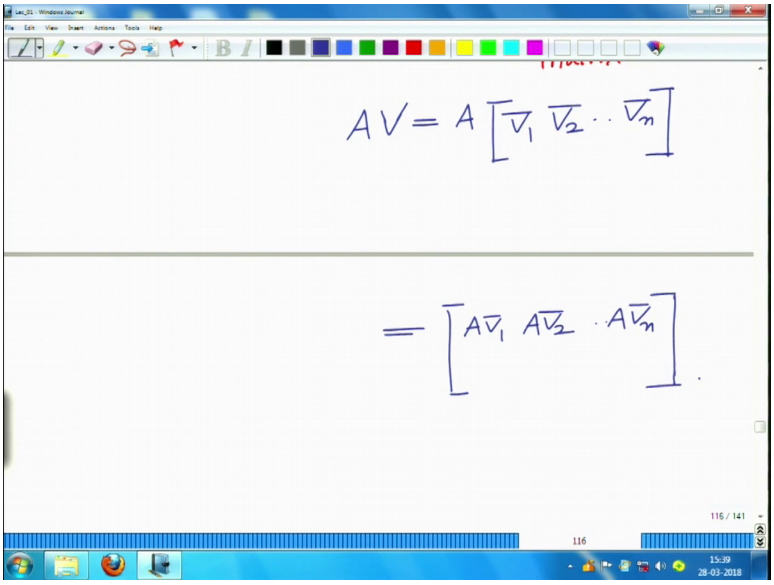

Now let us look at the product A V A times matrix V that is A into the eigenvectors V 1 bar, V 2 bar, V n bar, you can see this is nothing, but this equals well A V 1 bar, A V 2 bar, A V n bar which equals if you look at this, but if these are eigenvectors so, A V 1 bar is lambda 1 times V 1 bar, AV 2 bar is lambda times V 2 bar A V n bar is lambda n times V n bar, these are the various columns which you can now write also as remember we are looking at a times the matrix V.

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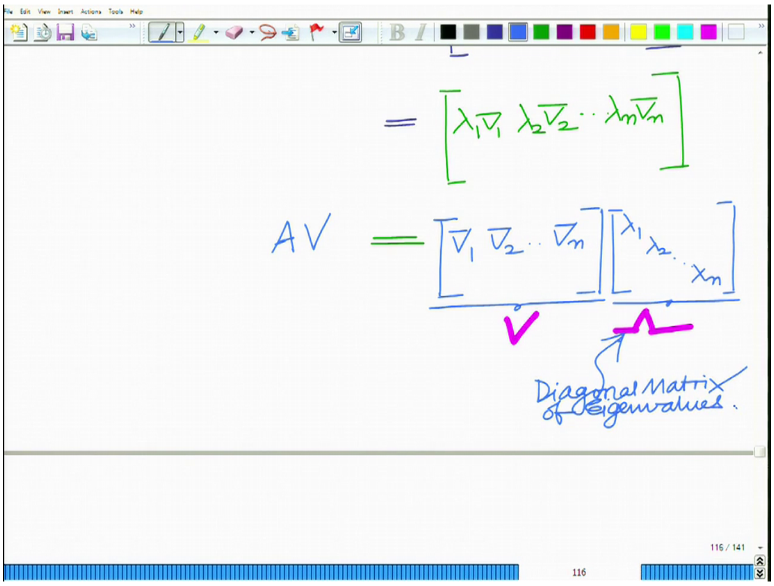

So, A times the matrix V equals now you can write this as V 1 bar, V 2 bar, V n bar times the diagonal matrix, lambda 1, lambda 2, lambda n this is a diagonal matrix.

So, this is nothing, but your matrix V and this we denote by the matrix capital lambda, which is the let me just write it with a little so, this we denote this by the V and this is basically your capital lambda and what is this, this is the diagonal matrix of Eigenvalues,

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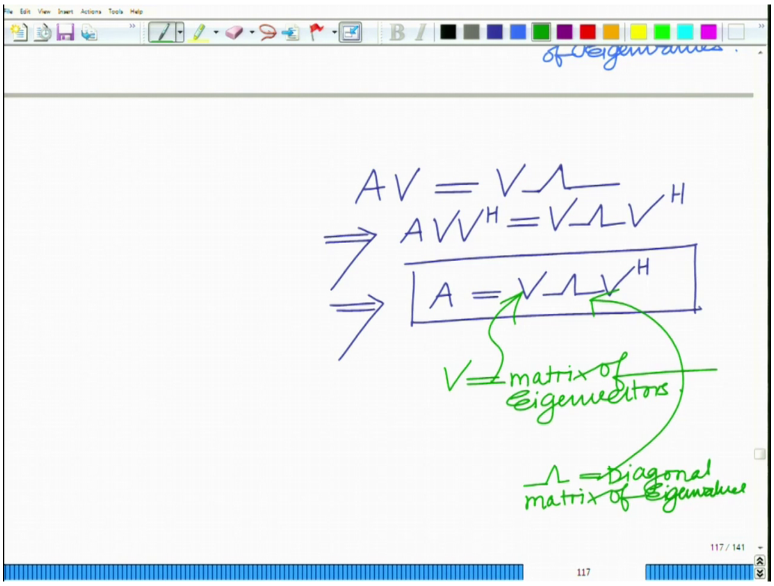

This is the diagonal matrix of Eigenvalues. So, we have A times V equals V times lambda multiplying on both sides by V Hermitian A times VV Hermitian equals V lambda V Hermitian, this implies the matrix A which is Hermitian symmetric can be expressed as V lambda V Hermitian where V is the matrix of Eigenvectors and lambda is the diagonal matrix of Eigenvalues. So, what is V, V equals the matrix of Eigenvectors lambda equals diagonal, equals diagonal matrix of lambda, equals diagonal matrix, lambda equals diagonal matrix of Eigenvalues and this is termed as the Eigenvalue Decomposition.

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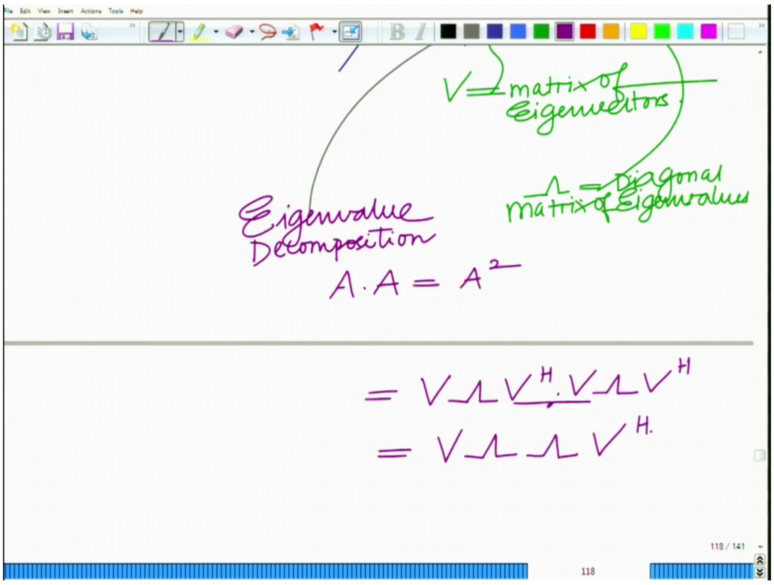

This is termed as the Eigenvalue Decomposition ok, Eigenvalue Decomposition of A and this has many interesting properties for instance if you want to compute the square A into A which is equal to A square, this will be equals you can write it in terms of V lambda V Hermitian into V lambda V Hermitian which is equal to now V V Hermitian is identity. So, which is equal to V lambda, lambda V Hermitian equals V lambda square V Hermitian.

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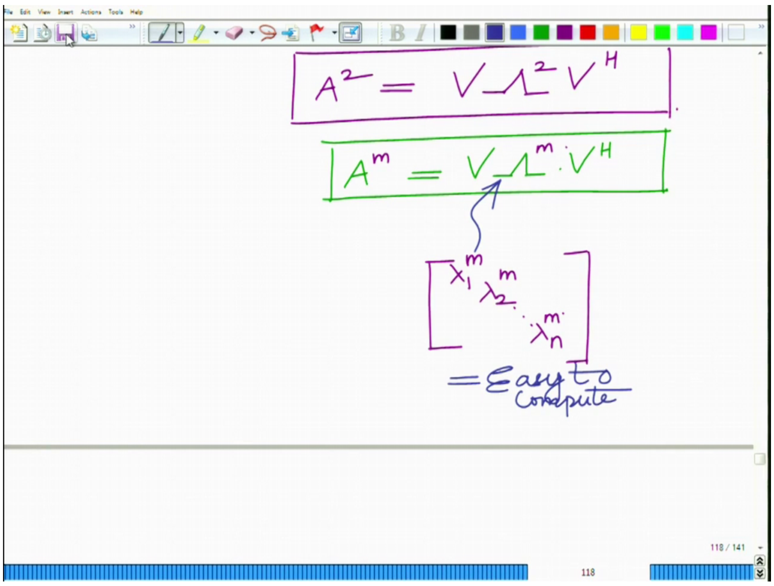

So this is A square. Similarly you have many other interesting properties for instance you can show, you can generalize this as Araise to n for this Hermitian symmetric matrix is V lambda n V Hermitian and lambda to the power of n is easy to compute because since it is a diagonal matrix. So, all it is it contains lambda i each lambda i raised to the power of n on the diagonal. So, lambda raised to the power of n this is very easy to compute, this is simply the matrix if you think about this, this is lambda 1 to the power of n, lambda 2 to the power of n. So, on lambda n, I am sorry I should have used a different integer here lambda m, lambda n raised to the power of m ok.

So, this is simply raised to the power of m, raised to the power of m, raised to the power of m. So, this is easy to compute and frequently you will see this is a very interesting property as well as this is a very handy tool to perform several matrix, matrix manipulations that is a Eigenvalue Decomposition of a matrix all right and it is also one of the fundamental decompositions of a matrix or one of you can also call it as one of the fundamental properties or one of the of a matrix, all right. So, we will stop here and continue with other aspects in the subsequent modules.

Thank you very much.