Applied Optimization for Wireless, Machine Learning, Big Data

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Lecture - 08
Null Space and Trace of Matrices

Hello welcome to another module in this massive open online course. So, we are looking at the example problems to better understand the preliminaries required for optimization all right. In this module let us start looking at another important concept that is the Null Space of a Matrix.

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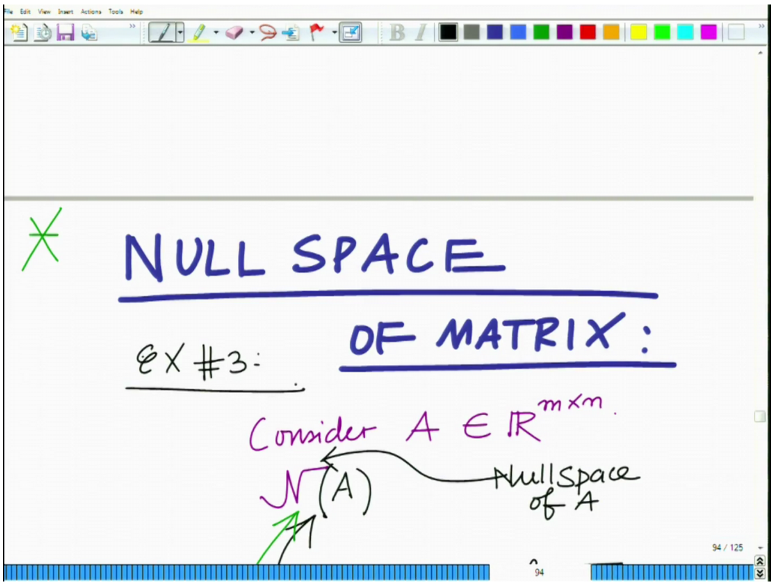

So, what we want to understand as part of our examples for the mathematical preliminaries is this concept of what is known as the null space of a matrix.

Now, what do we mean by the null space of a matrix; now consider a matrix A. So, that is an m cross n matrix, then the null space of A denoted by the symbol N of A this denotes the null space.

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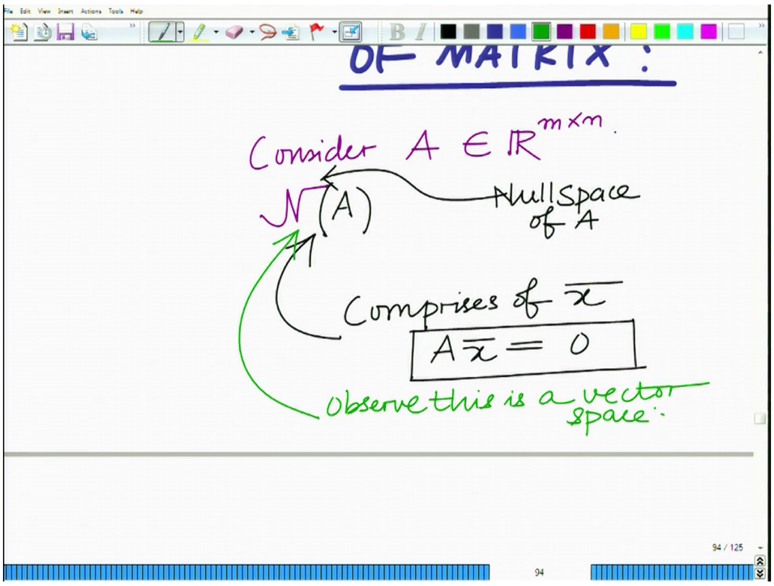

The null space of A; this comprises of all vectors x bar the null space of A comprises of x bar such that A x bar equal to 0 that is all set of all vectors A x bar x bar such that A x bar that is mean multiplied by A; A x bar equal 0 all right.

So, that that is basically that set all right in fact, it is the space that space of all vectors x bar is called the null space of the matrix A. In fact, this is the space vector space you can see this as follows.

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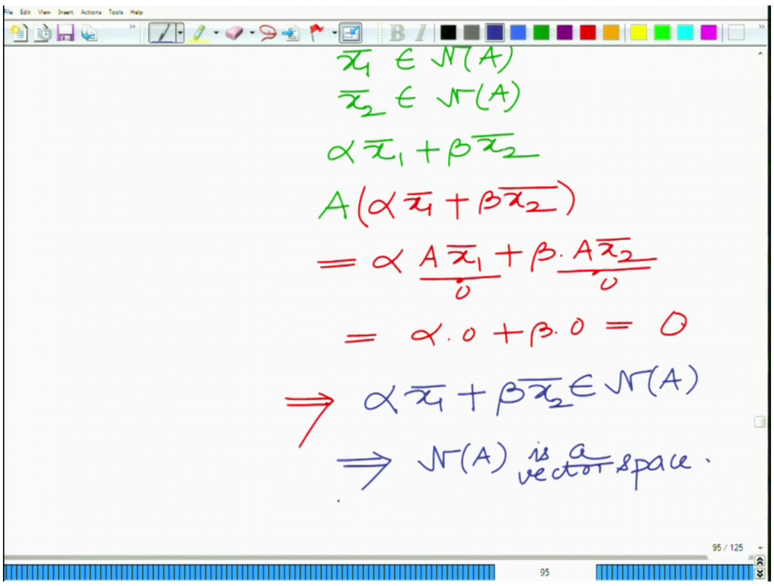

If x bar; so, observe that this is a vector space, this can be seen as follows if x 1 bar belongs to the null space of A x 2 bar belongs to the null space of A then if you perform any linear combination alpha x 1 bar beta x 2 bar then A times alpha x 1 bar plus beta x 2 bar equals, alpha times A x 1 bar plus beta times A x 2 bar, but remember x 1 bar and x 2 bar are both elements of the null space.

So, A x 1 bar a 2 bar is 0 which means this is alpha time 0 plus beta time 0, which is 0 and this implies this implies that alpha x 1 bar plus beta x 2 bar also belongs to the null space of A, which means that the null space of A indeed is a this is a vector space. That is because if x 1 bar belongs to what is the definition of vector space if x 1 bar belongs to the vector space x 2 bar belongs to the vector space, then any linear combination of these vectors was also belong to that set that is known as a space or a subspace a vector space or a vector subspace all right.

And therefore, the null set that is the set of all vectors x bar such that A x bar equal 0 is also a vector space. Because if you take any 2 vectors belonging to this set their linear combination also belongs to the set, also if this is known as the null space this vector subspace is known as the null space ok.

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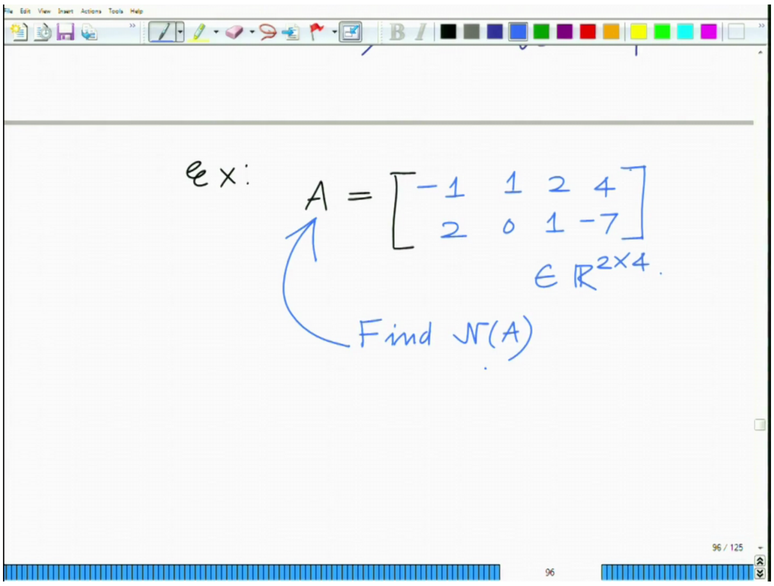

Now, example let us point for instance, let us point let us consider an example within this example; this is example number this null space this is the example number 3 that we are looking at example number 3 ok. And for instance let us consider a matrix A that is equal to minus 1 1 2 4 2 0 minus 1 7 and what we want to do. So, this is a 2 cross 4 matrix belongs to set of 2 cross for real matrices, and what we want to do is we want to find the null space of this matrix find the null space of this matrix A and this can be found as follows.

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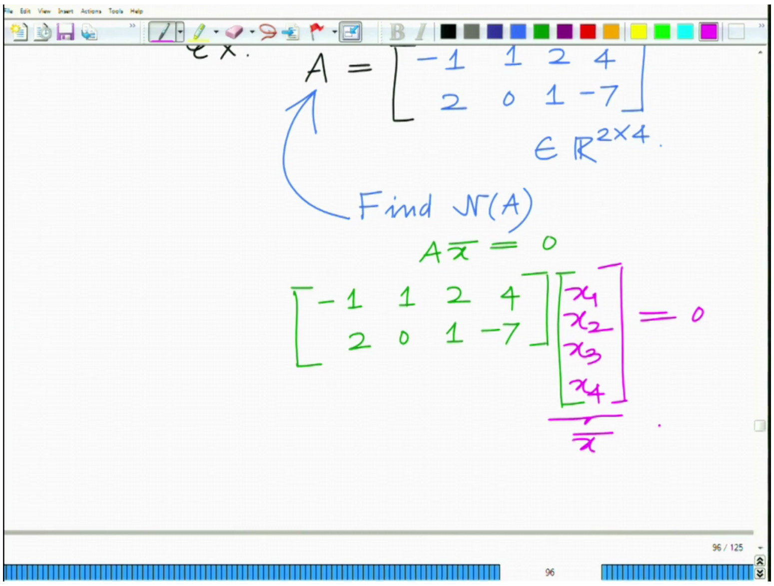

So, you want to find the set of all vectors remember minus 1 1 2. So, you want to find the set of all vectors such that A x bar equal 0 which is minus 1 1 2 4 into 2 0 1 minus 7 times well remember. So, 2 cross 4 matrix so, you have to multiply this by a 4 dimensional vector x 2 x 3 x 4 equal 0 this is your x bar.

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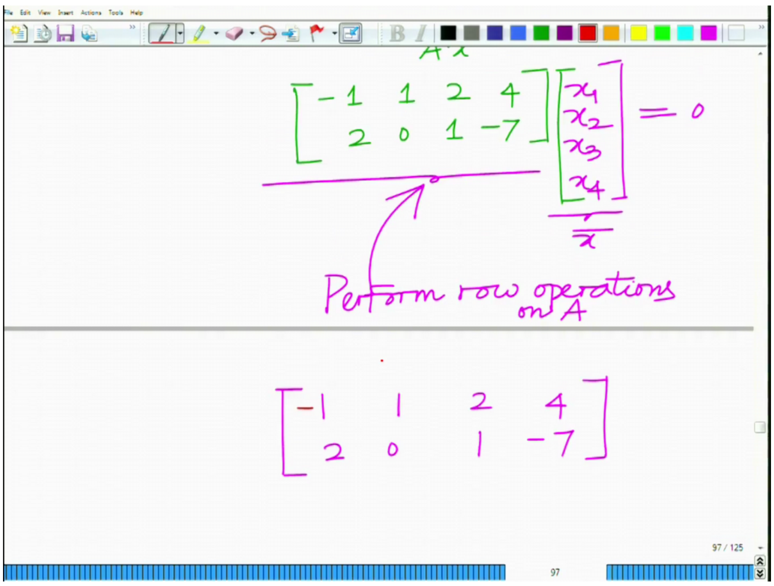

What we will do now is we will perform row operations on the matrix A; perform we will perform row operations on the matrix A.

So, we have 1 minus 1 minus 2 minus 4 2 0 minus 1 or 2 0 1 minus 7 first what we are going to do is we are going to perform reduce the pivot to 1 I am sorry. So, 1 I am sorry this is minus 1 1 2 4 minus 1 1 2 4 2 0 1 minus 1.

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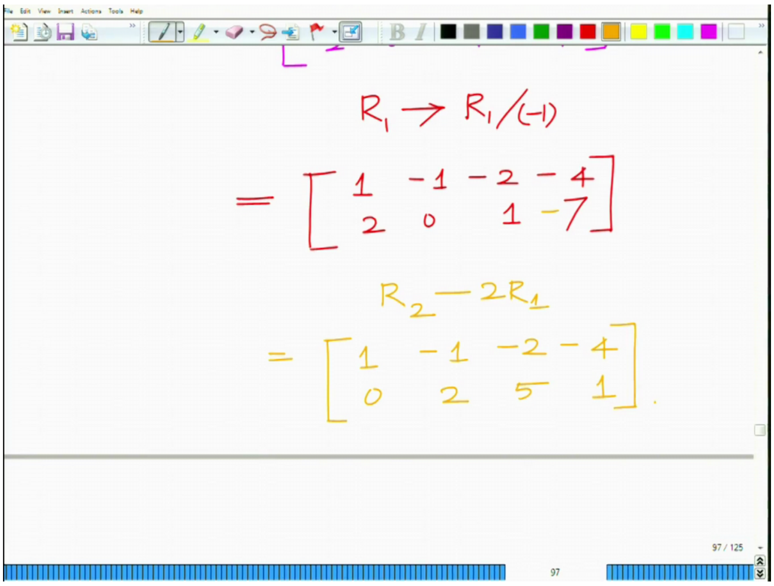

So, first we will divide R 1 goes to R 1 divided by minus 1. So, this will become equivalently this will equivalently become well when you divide by minus 1 this becomes 1 minus 1 minus 2 minus 4 2 0 minus 1 7. Now what we are going to do is we are going to perform R 2 minus twice R 1 the row operation, this will become the matrix this for first we will remain as it is minus 1 minus 2 minus 4 2 minus 2 0 this will be 2 this will be 5 minus 1 in this will be minus 1 minus 4 this will be basically you can see that this will be 2 R 2 minus 2 R 1 I am sorry this is 2 0 1 minus 7. So, 2 0 1 minus also 1 plus 4 this will be 5 and minus 7 plus minus 7 plus 8 this will be 1. So, this is what you get.

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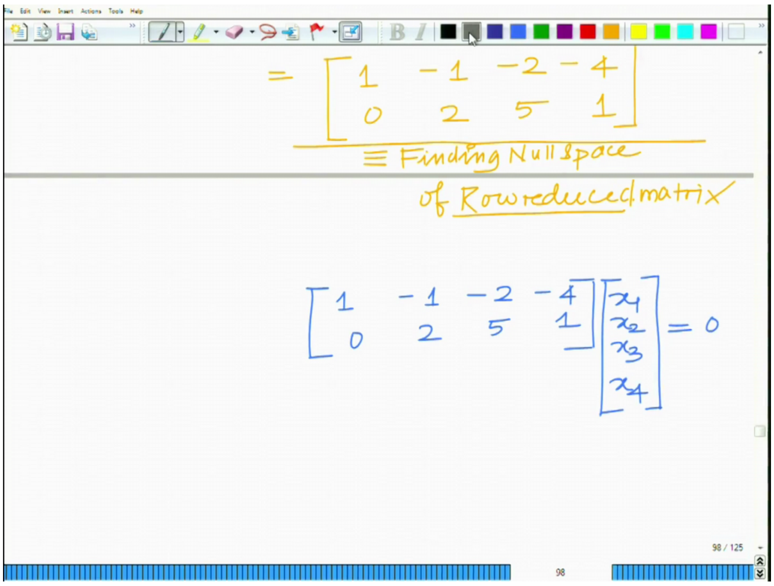

And now what you can show this is a equivalent matrix after row operations, and what you can show is that finding the null space of this matrix is equivalent to finding the null space of this row reduced matrix. This is equivalent to finding null space of the row reduced; y is equivalent to finding the null space of the row reduced matrix.

So, we will find the null space of this matrix which we can find as follows this will be well 1 minus 1 minus 2 minus 4 0 2 5 1 into x 1 x 2 x 3 x 4 this is equal to 0.

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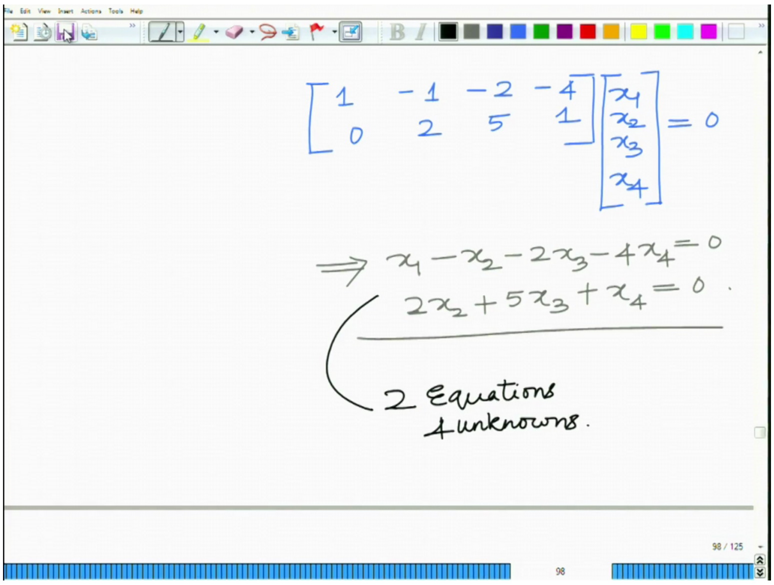

This implies that when we get 2 equations, x 1 minus x 2 minus 2 x 3 minus 4 x 4 equal to 0 and the second equation we will be twice x 2 plus 5 x 3 plus x 4 equal to 0. So, we have 2 equations 4 unknowns. So, observe here we have 2 equations 4 unknowns 4 unknown variable x 1 x 2 x 3 x 4. So, what this means is basically we have 2 free variables we can set 2 of the unknown variables or 2 of the unknown parameters is free variables.

So, let us set x 3 x 4 as free variables and we will express the rest of them x 1 x 2 in terms of x 3 and x 4 ok.

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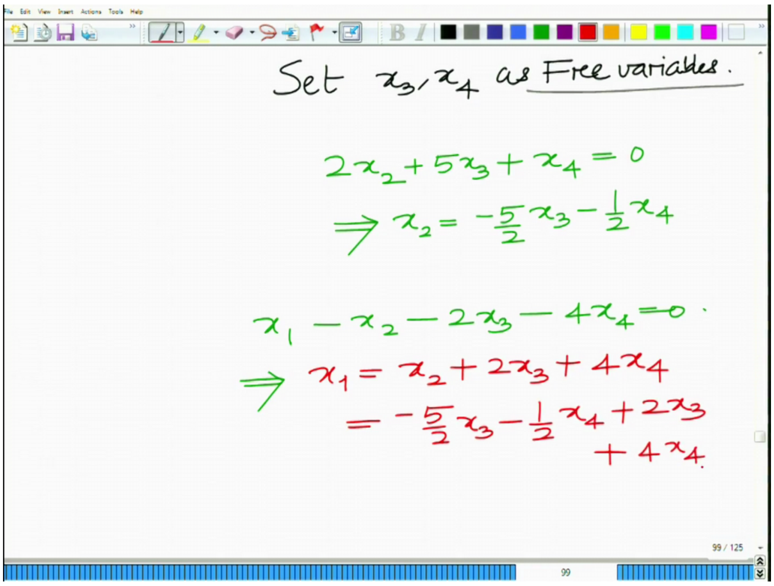

So, set x 1 x 2 or x 3 x 4 as free variables express x 1 x 2 in terms of x 3 x 4. So, we have well we have from the last equation we have 2 x 3 or 2 x 2 plus 5 x 3 plus x 4 equal to 0 this implies that well x 2 equals minus 5 by 2 x 3 minus half x 4 and finally, from the last; from the first equation, we have x 1 minus x 2 minus 2 x 3 minus 4 x 4 equal to 0, which implies if you can look at this is that x 1 equals x 2 plus 2 x 3 plus 4 x 4.

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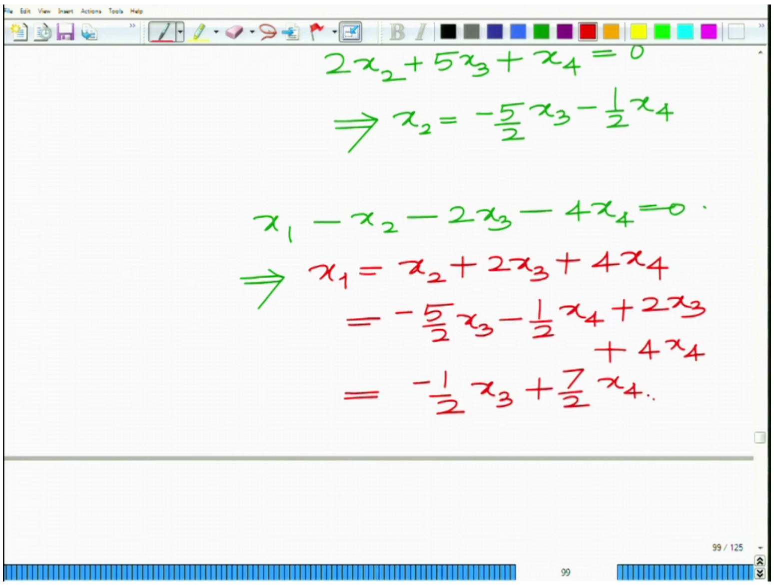

Now, substitute for x 2 that is minus 5 by 2 x 3 minus half x 4 plus 2 x 3 plus 4 x 4, which is basically now, you can see 2 minus 5 by 2 that is minus half x 3 plus 4 minus half that is 7 by 2 x 4 ok.

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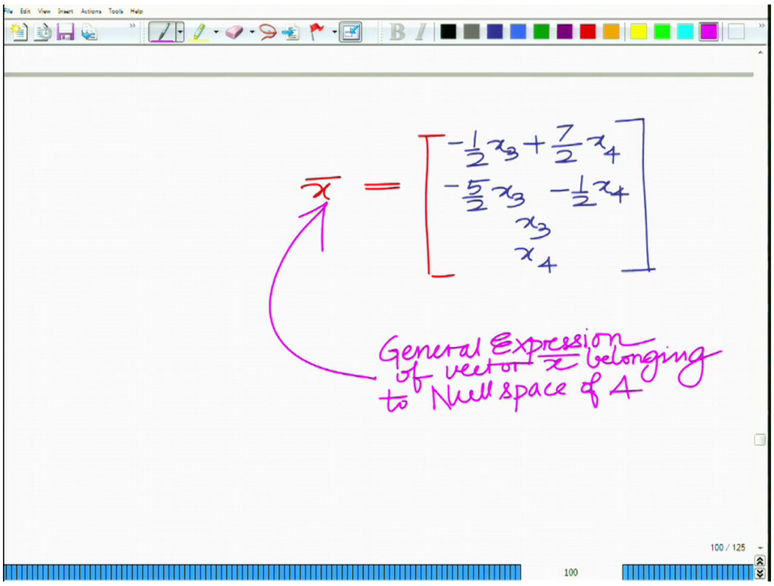

And therefore, the general structure of a null vector belong to null space of x bar will be of the form well minus half x 1 is minus half x 3 plus 7 by 2 x 4 x 2 is minus 5 by 2 x 3 minus half x 4 and this is x 3 and x 4 this is a. So, this is the general structure remember this is the general structure or you can say the general expression of vector x bar belonging to null space of the matrix A. That is if we chose any 2 parameters x 3 and x 4, I can form a vector like this x bar which has this structure all right and that vector x bar will belong to the null space of this matrix.

So, this is the general structure or the general expression for any vector x bar that belongs to the null space of the matrix A ok.

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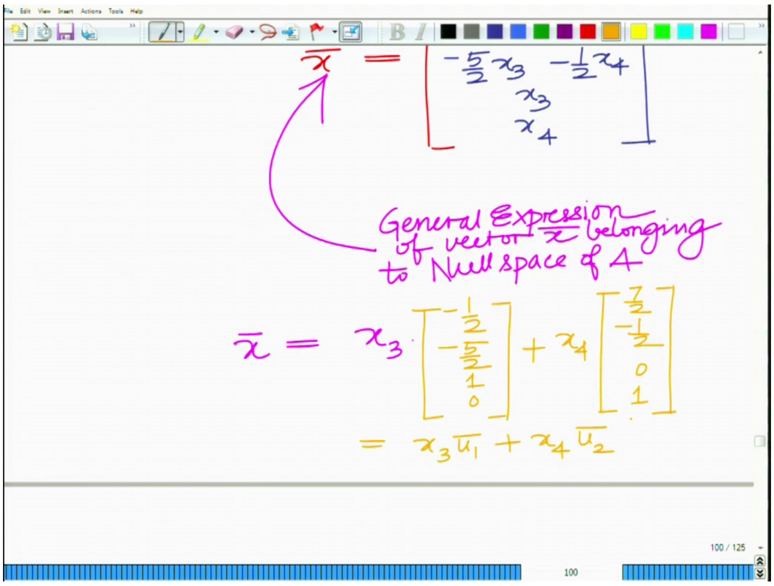

Let us try to simplify this a little bit further so, we have x bar equals now what I am going to do is, I am going to write this as the sum of 2 components x 3 times the vector, I can write this as look at this x 3 times the vector minus half minus 5 by 2 1 0 plus x 4 times the vector 7 by 2, x 4 times the vector 7 by 2 minus half 0 1.

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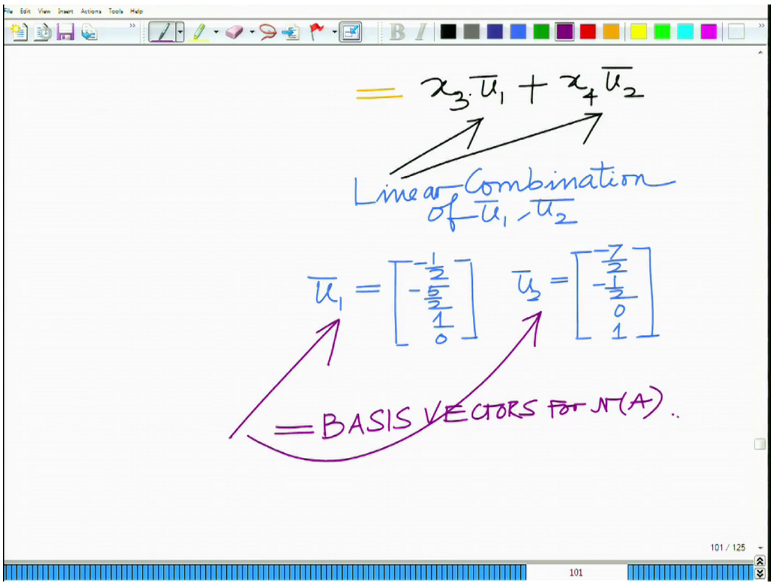

So, I am writing this as the vector x 3 times u 1 bar x 4 times u 2 bar that is this is equal to x 3 times observe this is a linear combination of u 1 bar and u 2 bar this is a linear combination of the 2 vectors u 1 bar and u 2 bar. Therefore, any linear combination of this vectors u 1 bar and u 2 bar belongs to the null space of a all right. So, this null space of A is formed by all linear combinations of these vectors u 1 bar and u 2 bar and therefore, this u 1 bar and u 2 bar are the bases vectors for the null space of the matrix A.

So, what are these vectors u 1 bar you can clearly see u 1 bar equals the vector minus half minus 5 by 2 1 0 this is the vector u 1 bar u 2 bar is the vector minus 7 by 2 minus half 0 1 so, these vectors are basis vectors ok. So, what we are saying is that these vectors, these are basis vectors for the null space of a that is the null space of a is formed by the all possible linear combinations of these vectors u 1 bar and u 2 bar. In fact, you can quickly verify u 1 bar u 2 bar themselves belong to the null space.

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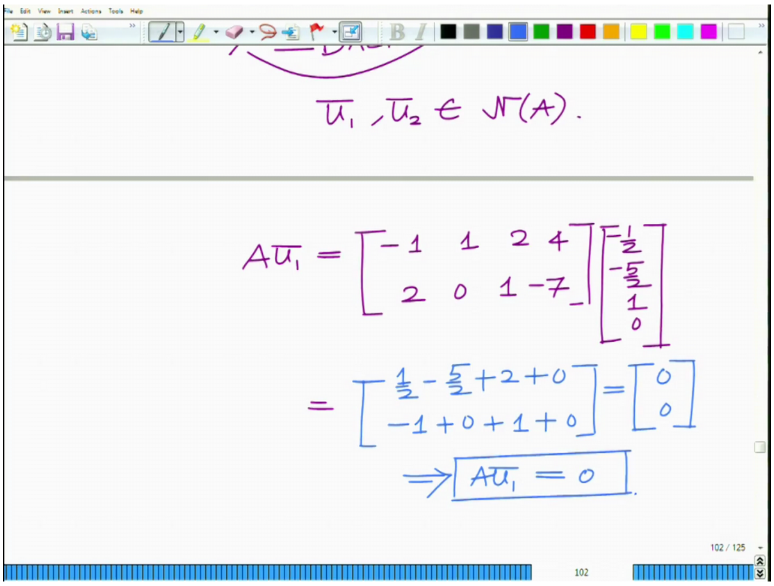

For instance if you consider a u 1 bar u 1 now also remember u 1 bar comma u 2 bar themselves belong to the null space of a you can quickly check this for instance, A times u 1 bar equals multiply this minus 1 1 2 comma 4 2 0 1 comma minus 7 times if you look at this, minus half u 1 bar minus 5 by 2 1 0 this is equal to what will this be equal to? This will be equal to well the first row will be half minus 5 by 2 plus 2 plus 0 second will be minus 1 plus 0 plus 1 plus 0 and you can clearly see this is nothing, but the 0 vectors.

So, this implies A u 1 bar equal to 0 they form the basis naturally u 1 bar u 2 bar itself in that space.

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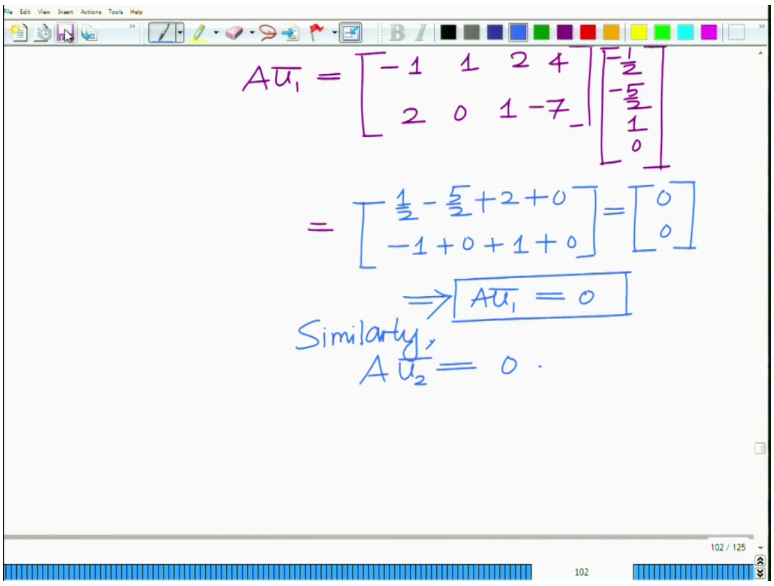

Similarly, A u 2 bar equals 0 you can check that; similarly A u 2 bar is also 0 all right. So, that is the concept of the null space of A matrix that is the set of the space of all vectors x bar, such that A x bar equal to 0 and we justified that this is actually a vector space because if we take any 2 vectors belonging to this space the linear combination also lies in this space ok.

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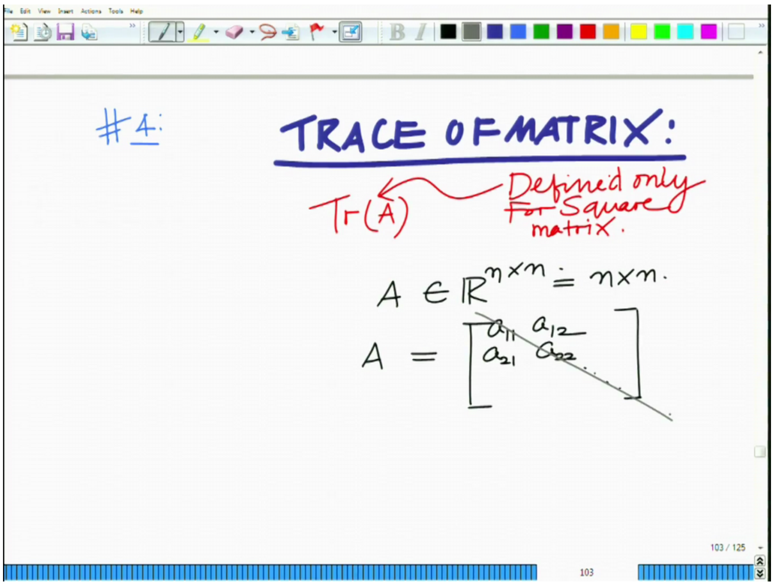

Let us look at another example, example number 4 and this is regarding the trace property, this is regarding the trace of a matrix remember that briefly if you just recall the trace of any matrix is defined only for a square matrix, this is defined and trace of A matrix remember is the sum of the diagonal elements. So, if A is an n cross n square matrix, trace of A is basically the sum of the that is trace of A is nothing, but the that is if a equal to this matrix this is an m cross n matrix, the trace of the matrix equals the sum of the diagonal elements; trace of the matrix is sum of the diagonal elements.

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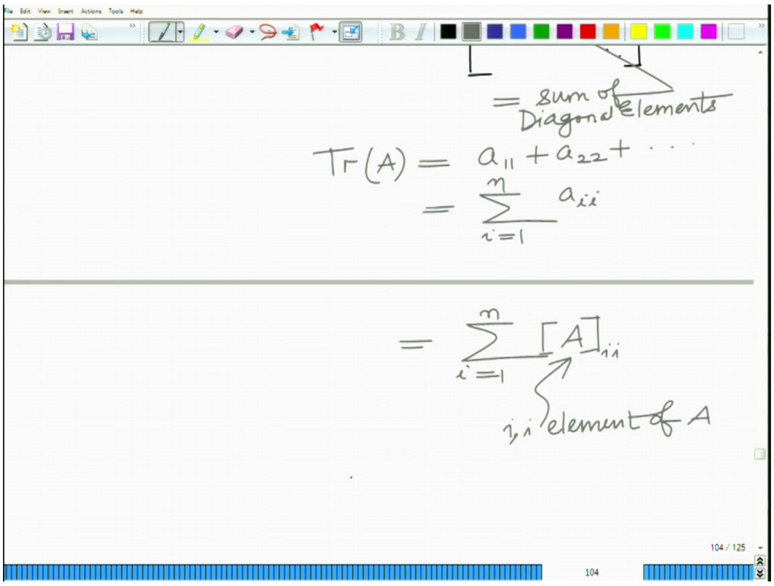

So, trace of A equals a 11 plus a 2 2 plus so, on that is basically if you look at this i equals 1 to n a i i and that can also be denoted as summation i equals 1 to n take the element A i i this denotes basically i comma i element of A this notation basically denotes the i comma i element of the matrix A. So, the trace a trace of a square matrix A is basically the sum of the diagonal elements of the matrix A all right a 11 plus a 2 2 a 3 3 plus a 3 3 and so on ok. Now if you look at.

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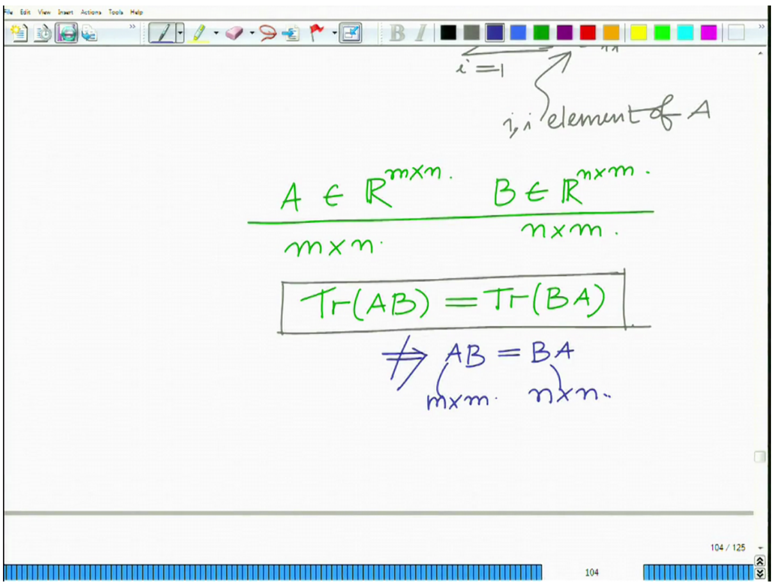

Now, the property that we are interested in (Refer Time: 22:33) I have not yet got to the property that problem that we want to; we want to show that for any 2 matrices let us consider 2 matrices a of the same size A belongs to m cross n and B not same size, but B belongs to n cross m. So, A transpose and B have the same size rather.

So, A is m cross n and B is n cross m. So, basically so, that we can multiply A and B then we can show that trace of A B equals trace of B A this is an interesting property. So, you want to show that for 2 matrices A which an of size m cross n and n cross n trace of A B that is sum of diagonal elements of AB is same as the sum of diagonal elements of a BA. Now note this does not imply that AB equals BA. One should not confuse that simply trace of AB equals trace of B A does not imply that AB themselves AB itself is equal to B A. In fact, AB most general AB typically is not equal to because if you look at AB, AB is of size m cross m B A is of size n cross n. So, the matrices if m is not equal to n the sizes themselves are different. So, AB is not equal to B A.

So, the look at this is m cross m this is n cross this is an n cross n matrix B A now let us start with this proof this a very interesting property. So, and this is very helpful in several simplifications.

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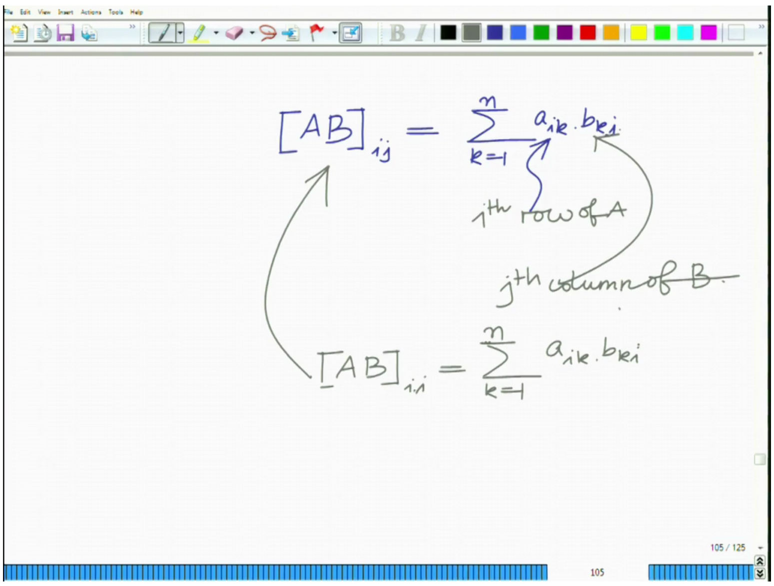

Now, first let us start by looking at the i jth element of the matrix product AB ij, you can show this is equal to summation k equals 1 to n of a i k b k j that is the i jth element of the product is summation of all elements in the kth row of a and jth column multiplied by corresponding elements in the jth column. So, this basically is kth row of all elements in the kth row of A and these are elements in the that is you take the you take the kth row of

A jth column of B I am sorry this is jth column of B and do an element by element multiplication and then take the sum that is the i jth element of AB.

Now, what is AB i i that is the diagonal element, diagonal element is simply take the ith row of kth row this is simply I am sorry this is the ith row of A; k equals one to n this is simply a i comma k into b k comma i because we are looking at the i ith element. So, this is ith row of A ith column of B ok.

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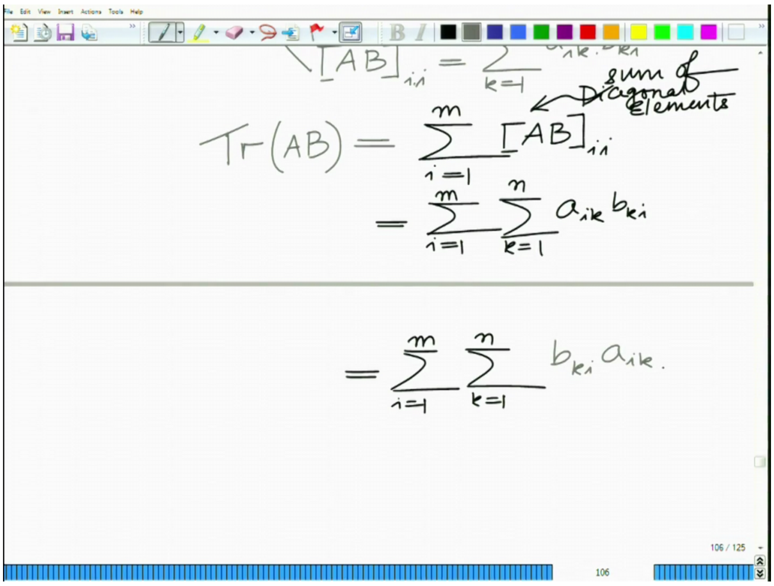

And now trace of AB recall trace of AB is nothing but sum of the diagonal element. So, that is basically you have to take the summation over i equals m of all the diagonal elements AB i i correct this is the sum of the diagonal elements, which is equal to now substitute this expression for AB i i that is the summation i equal to m summation k equal to 1 a ik b ki. Now what I am going to do is I am just going to write it in a different this product. So, i equals 1 to m summation k equals 1 to n now I am going to first write b ki a ik b ki is b ki times a ik. So, I am going to write b ki a ik.

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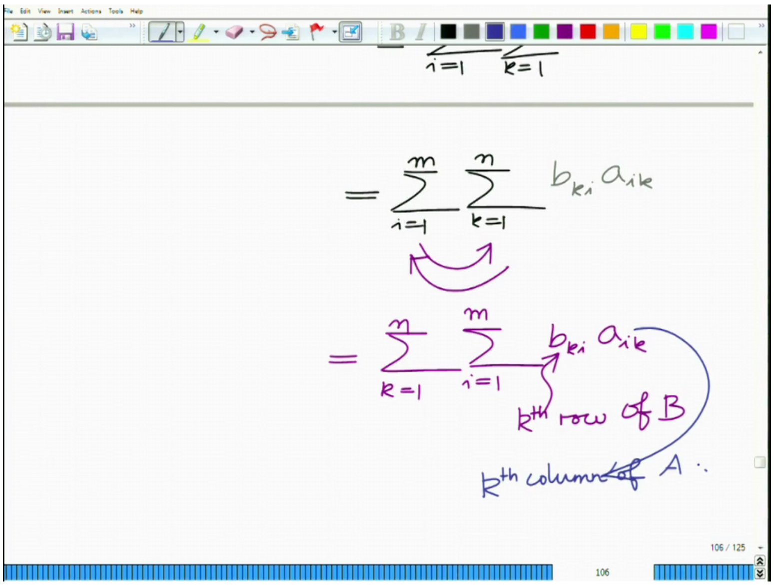

Now, I am going to interchange the summation the order of summation. So, I am going to interchange the order of summation that becomes k equal to 1 to n summation i equals 1 to m. Now I have b ki a ik now look at this b ki a is basically nothing, but this is basically kth row of B; kth row of and a ik is correspondingly the kth column, you are taking an element wise product of kth elements of kth row of B and kth column of A.

So, this is nothing but the; if you look at this is nothing but the k kth element of BA. So, if you look at this is nothing, but you are taking B A kth row and kth column and this is basically k kth element of B k.

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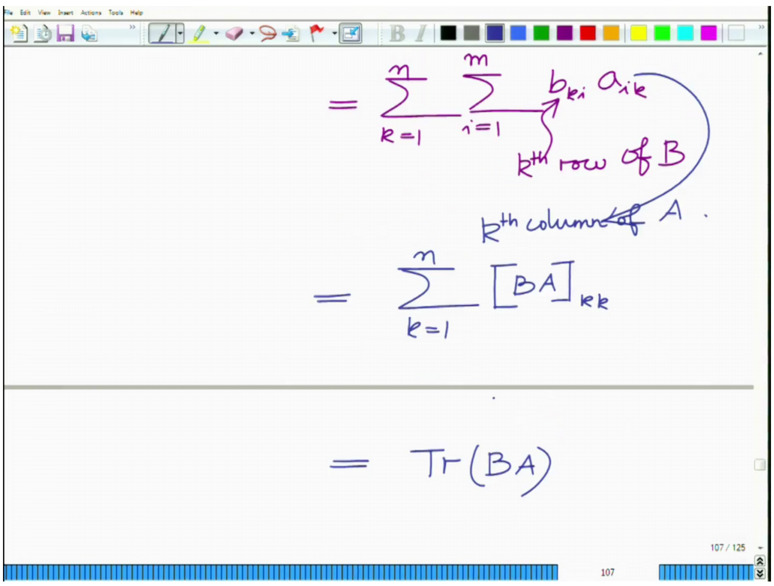

And therefore, what you can see is basically this is summation i equals 1 to m you take the matrix product B A and this is the k kth which is the diagonal k th diagonal element of B k and you are taking the sum over all I am sorry sum over k equal to 1 to n you are taking over the sum of all the diagonal elements k equal to 1 to n BA kk which is nothing, but trace of BA. So, this is nothing, but the trace of B A correct this is nothing, but the trace of BA.

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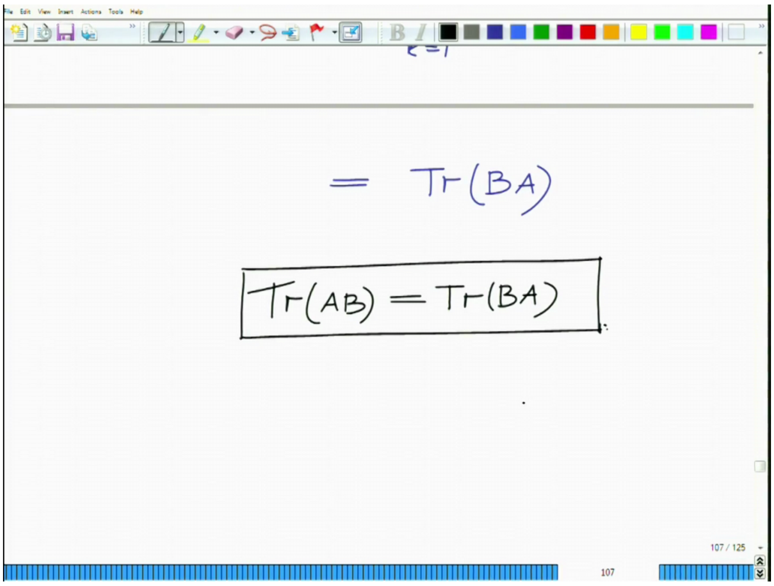

And therefore, what we have is trace AB and this is very handy property, trace AB equals trace of trace AB equals trace of B A. And this is very handy property because in general for matrices you do not have a commutative property that is AB is not generally equal to B A, but trace of AB equals trace of the matrix B A this is I am interesting property of matrices, which will come handy in several problems or several optimization problems, where you have to manipulate matrices or manipulate the product of matrices all right. So, let us stop here and we will continue with some other problems in the subsequent modules.

Thank you very much.