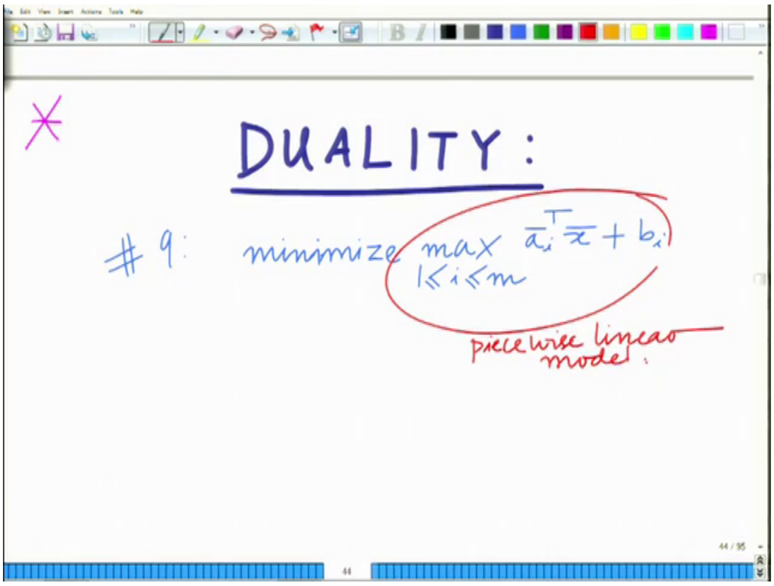
Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

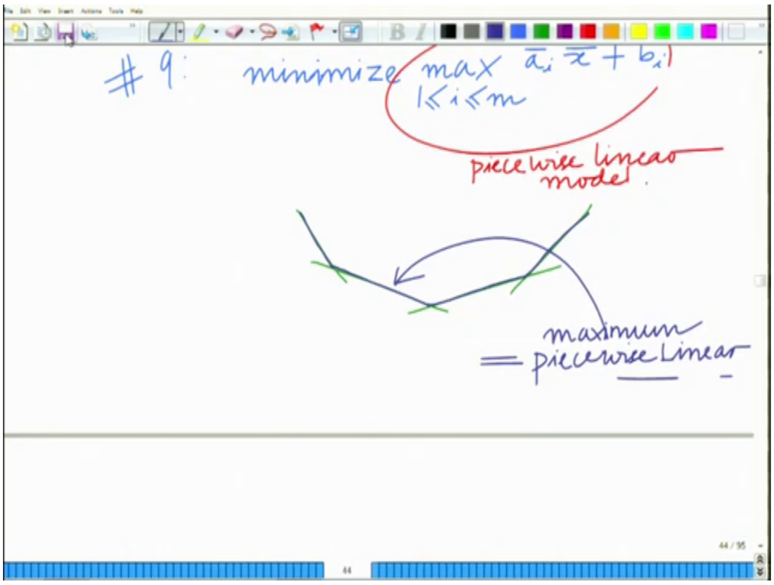
Lecture - 74
Examples on Duality: Min-Max problem, Analytic Centering

(Refer Slide Time: 00:24)



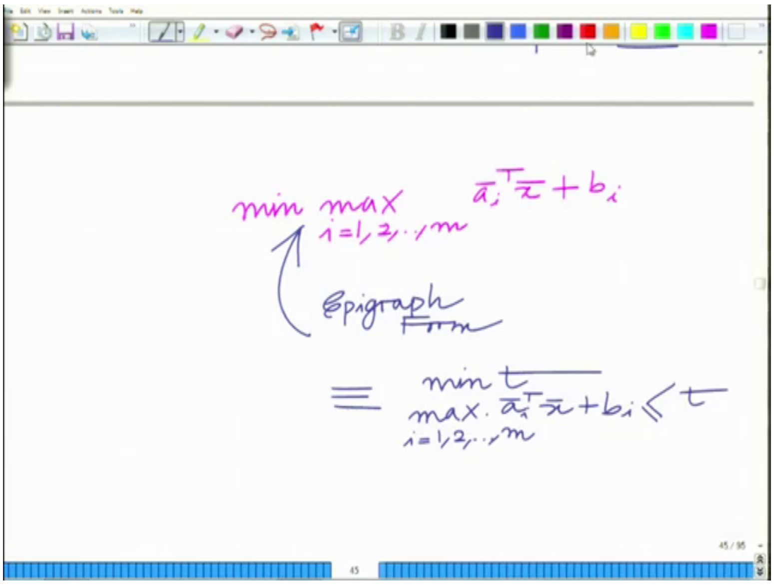
Hello, welcome to another module in this massive open online course. So, we are looking at example problems in duality. Let us continue our discussion alright. So, what you want to look at is duality and some problems to understand this. Well, what we are saying is let us look at this problem, this is for example problem number 9. We want to find that dual of the problem, minimize the maximum of 1 less than equal to i less than equal to m, a bar transpose x bar plus b i. And this is known as a piecewise linear model.

(Refer Slide Time: 01:27)



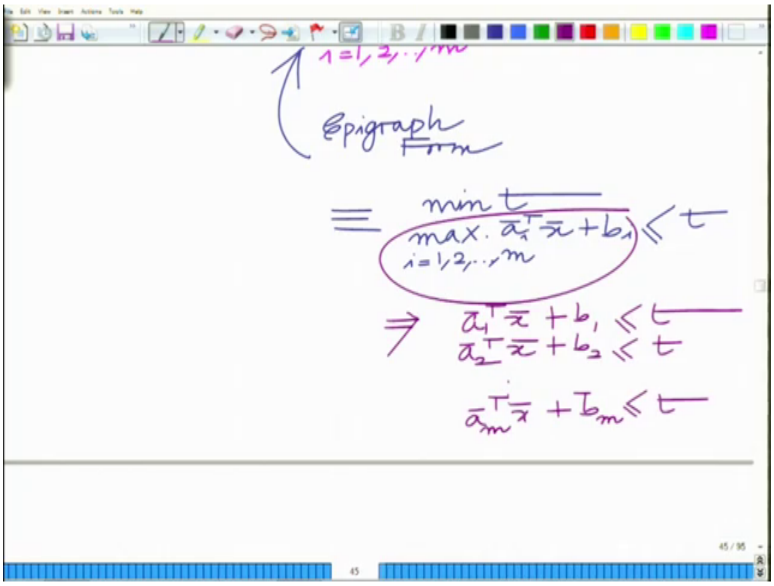
For instance, what you can see each of these represents a line, therefore if you look at these different lines, and now you take the maximum, so the maximum we will have look something like this. So, these are m different lines and this is how the maximum looks like. And you can generalize this n dimensions for hyperplanes all right. So, this is the maximum which you can see is basically not exactly linear, this is piecewise linear ok. So, this is piece this is piecewise linear. So, in each in each segment, you have a linear. So, in each segment, you have a linear characteristic for this. And now what we want to do is we want to minimize.

(Refer Slide Time: 02:34)



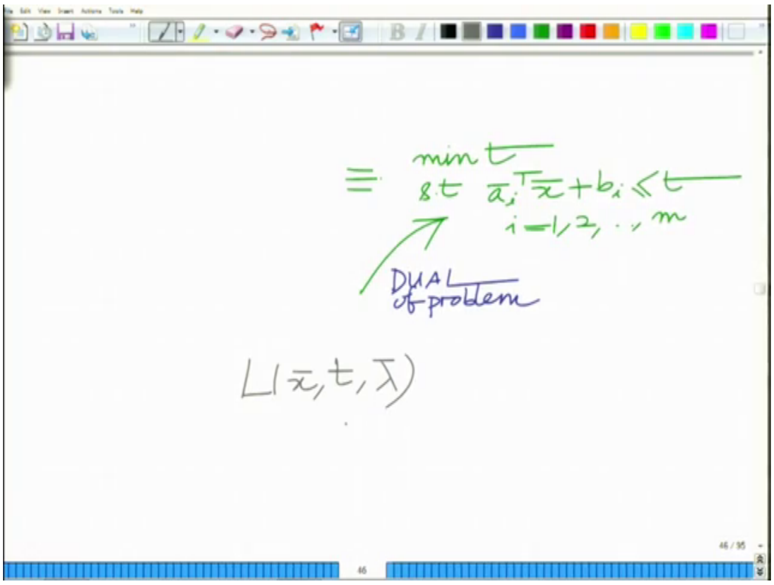
Now, I can write this. Now, to find the dual problem of this what I am going to do is I am going to first you look at minimize the maximum of i equals 1, 2 up to m a bar transpose x bar plus bi. Now, first I am going to write this in the epigraph form. So, using the epigraph form, using the epigraph form, this can be equivalently written as minimize t subject to the constraint that the objective, which is basically maximum of a bar transpose x bar plus b i equal to 1, 2 up to up to m this is less than or equal to t ok. So, this is basically how considerate. Now, obviously this is the maximum is less than or equal to t.

(Refer Slide Time: 03:59)



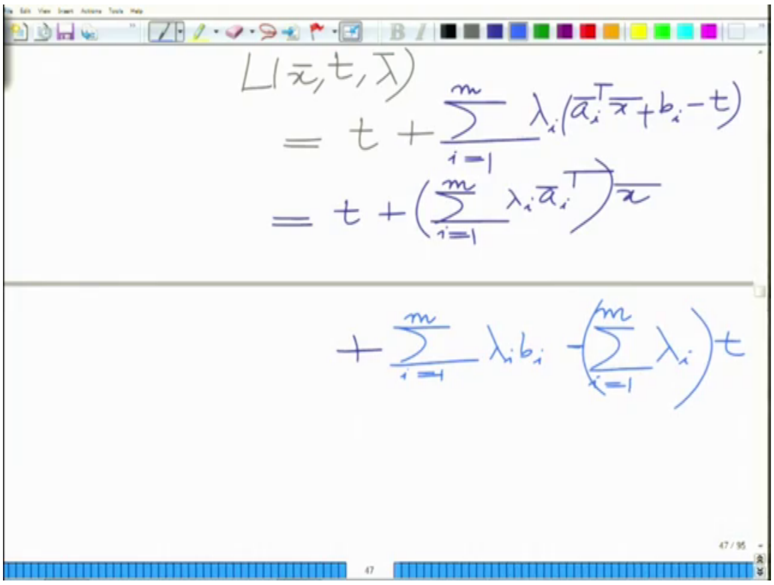
This basically implies that each of this is less than equal to t which means a 1 bar transpose x bar plus b 1 is less than or equal to t a 2 bar transpose x bar plus b 2 is less than or equal to t so on a m bar transpose x bar plus b m bar is less than or equal to t ok.

(Refer Slide Time: 04:29)



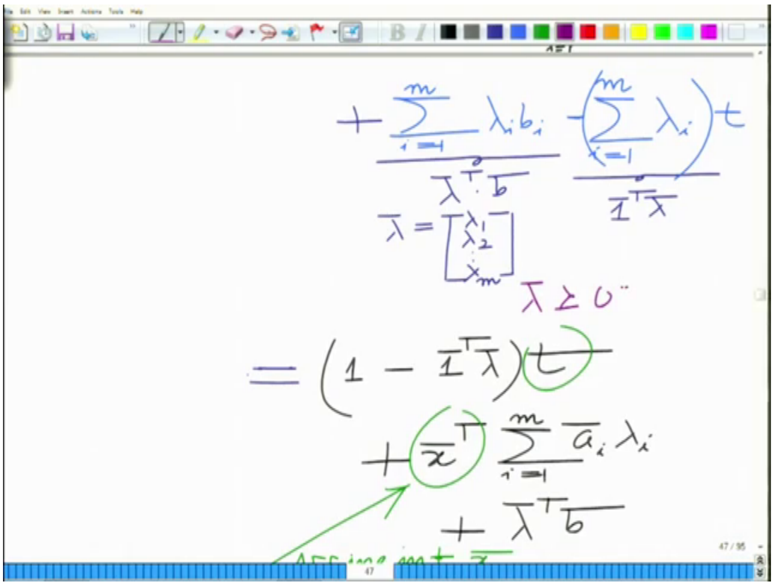
So, I can write this basically as a equivalent optimization problem minimize t subject to the constraint that a i bar transpose x bar plus b i less than equal to t for i equals to 1 2 up to m. This is a much simpler form and there is something that is tractable. Now, I am going to develop the dual for this all right. So, this is the equivalent form and, now we can focus on getting the dual of this problem. And the dual of this problem is obtained as follows, now first you form the Lagrangian L of x bar in fact this is L of x bar comma t comma, since we have inequality constraints that is lambda bar which is basically, now you take the objective t plus summation i equals 1 to m 1 Lagrange multiplier for each constraint that is lambda i into a i bar transpose x bar plus b i minus t.

(Refer Slide Time: 05:39)



Now, what you want to do is you want to group all the terms corresponding to each for instance this can be written as t plus summation of i equals 1 to m lambda i a i bar transpose into x bar plus summation of i equals 1 to m lambda i b i minus summation i equals 1 to m lambda into 1 to m lambda i into t.

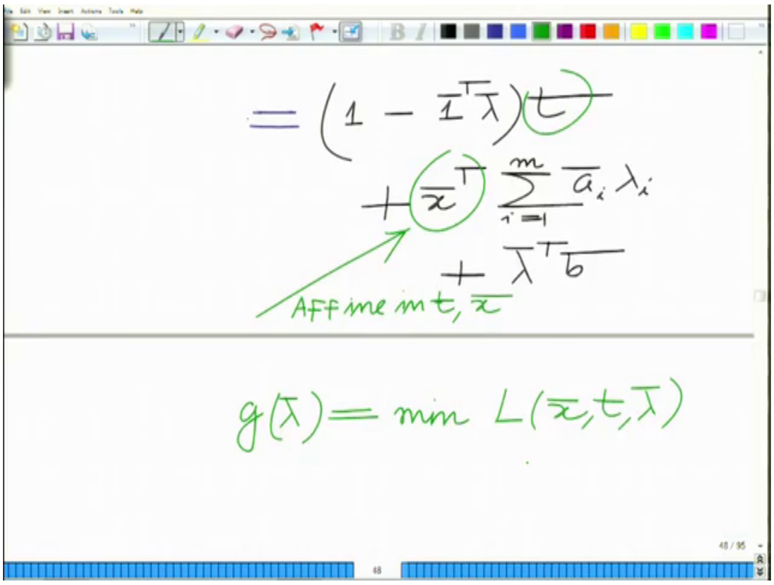
(Refer Slide Time: 06:55)



And now if you look at this, this is nothing but summation of lambda i, this is 1 bar transpose lambda bar this is summation lambda i bi. So, you can write this as lambda bar transpose b bar, i mean I think you know what the definitions of these are these are basically vector of lambda 1 lambda bar is vector of lambda 1 lambda 2 up to lambda m b bar is a vector of b 1, b 2 up to b m and so on.

And therefore, now if you simplify this, what I get is basically I can write this in compact form as 1 minus 1 bar transpose lambda bar into t lets some 1 bar transpose lambda bar is nothing but summation of all lambda i plus you can simply write this take the transpose of this. So, I can write this as x bar transpose summation I equal to 1 to m a i bar into lambda i.

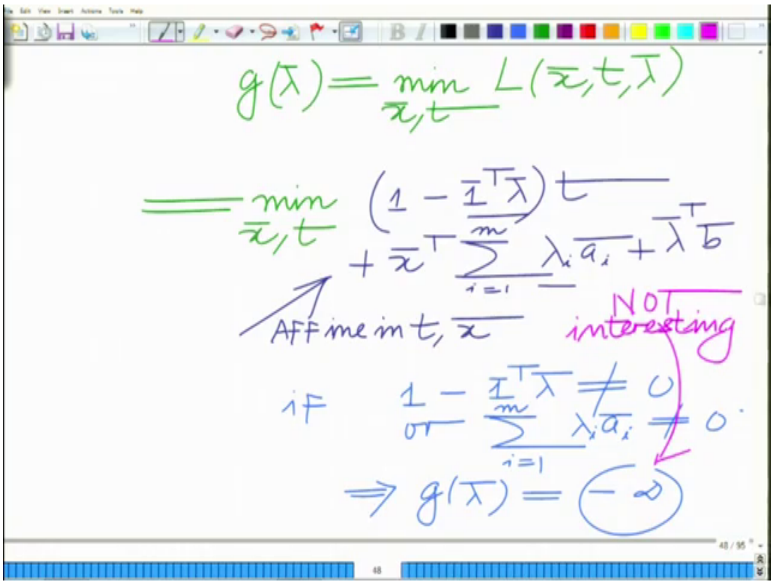
(Refer Slide Time: 07:56)



So, it is very convenient alighted that way x bar transpose well summation i equal to 1 to m a i bar into lambda i bar into lambda a i bar, and lambda into a bar does not matter plus you have this quantity lambda bar transpose b bar this is summation lambda i b i which is basically lambda bar transpose b bar.

Now, observe something very interesting this is linear in t linear in x bar observed or affine this is affine m t comma x bar basically which means it is a hyperplane ok. And hyperplane if the slope is not 0, it goes to minus infinity, which means if either 1 minus 1 bar transpose I mean of course we have to take the minimum of this. So, now to get I am sorry there is one more step that is the dual is give basically getting the minimum of with respect to x bar t comma lambda bar which is basically.

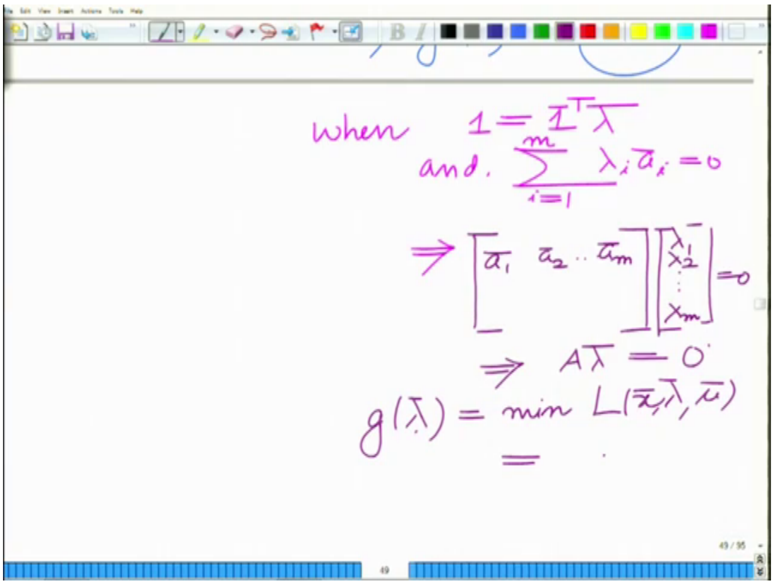
(Refer Slide Time: 09:08)



Now, you write this a little elaborately this is minimum by the way or x bar comma t or the primal variables x bar comma t 1 minus 1 bar transpose lambda bar into t plus summation or plus x bar transpose summation i equal to 1 to m lambda i a i bar plus lambda bar transpose b bar. Now, you observe this is affine in t comma x bar all right which means it is a hyperplane. If the coefficients of t or x bar are not 0, it tend to minus infinity ok.

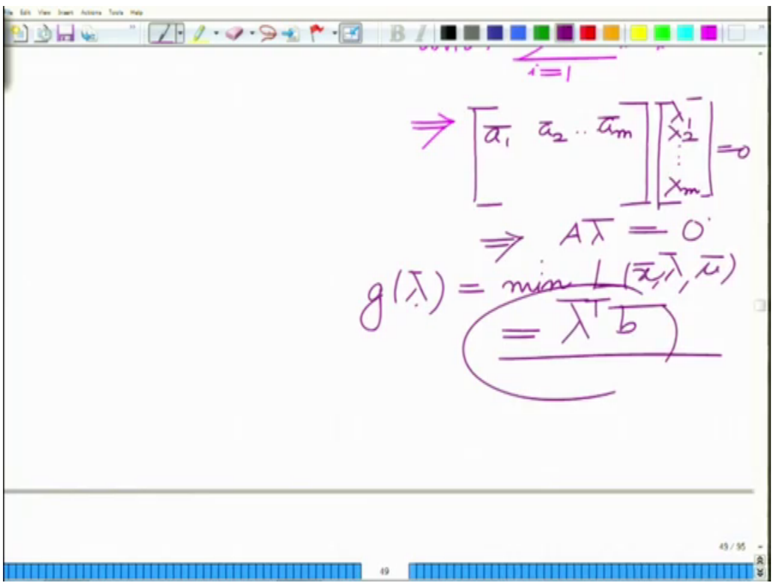
So, if 1 minus 1 bar transpose lambda bar is not equal to 0 or summation i equal to 1 to m lambda i a i bar is not equal to 0, that means g of lambda bar equals infinity minus infinity. Now, minus infinity is still a lower bound, but as we have seen previously minus infinity is not an interesting lower bound ok. So, this is not something that we would like to work with that mean minus infinity is also a lower bound, but the original optimization problem, so but this is not very interesting for the simple reason that minus infinity is the lower bound for any optimization problem any minimization. So, this is not interesting.

(Refer Slide Time: 11:03).



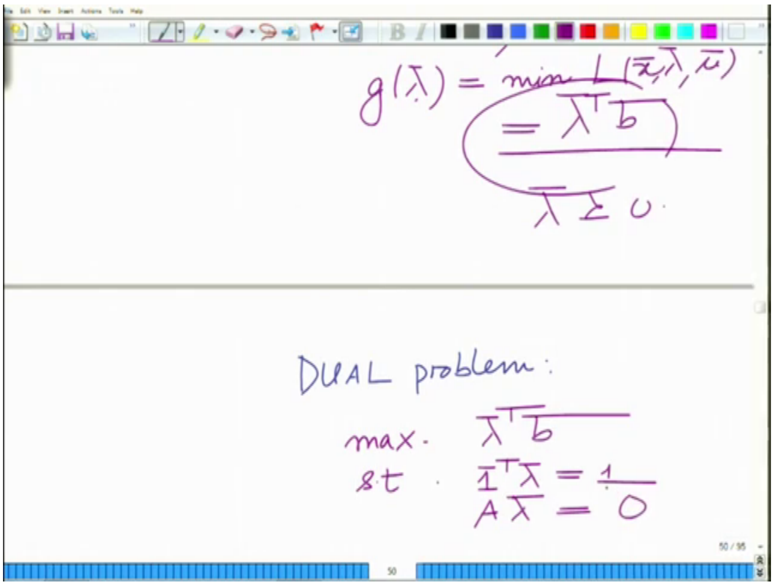
So, what is interesting on other hand when the coefficients are 0 that is 1 equals 1 bar transpose lambda bar summation of all lambda is 1, and summation i equals 1 to m lambda i a i bar equal to 0, which means if you write this as a matrix in matrix form you have the vector a 1 bar, a 2 bar up to am bar times lambda 1, lambda 2 up to lambda, and this equal to 0 this implies that A times lambda bar equal to 0. If 1 equal to 1 times lambda bar and A times lambda bar equal to 0, then the minimum is the minimum of x bar comma lambda bar comma mu bar. And this minimum you can see is nothing but lambda bar transpose b bar.

(Refer Slide Time: 12:16)



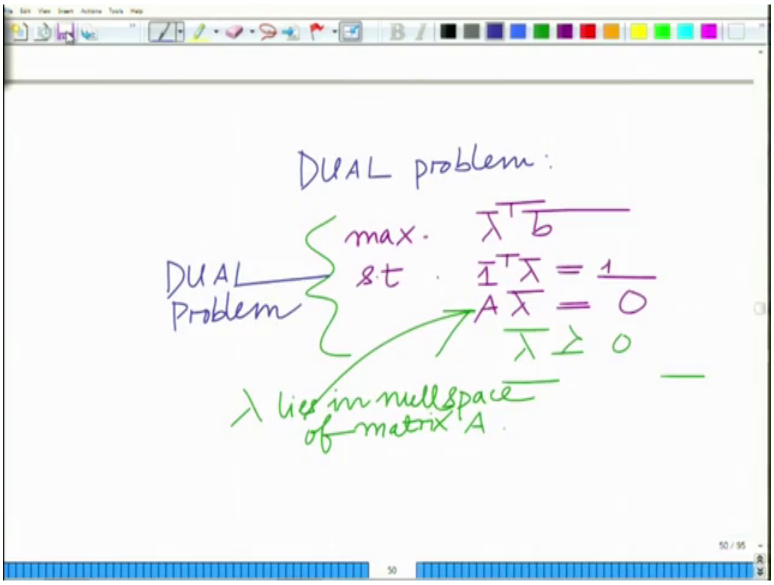
So, this minimum for that scenario is lambda bar transpose is lambda bar transpose lambda, but the minimum is lambda bar transpose. Of course, now in the original when you formulate the Lagrangian, you have to also ensure that remember these are lambda bars or component wise greater than equal to 0. So, this is true minimize lambda bar transpose b bar provided lambda bar is component wise.

(Refer Slide Time: 12:47)



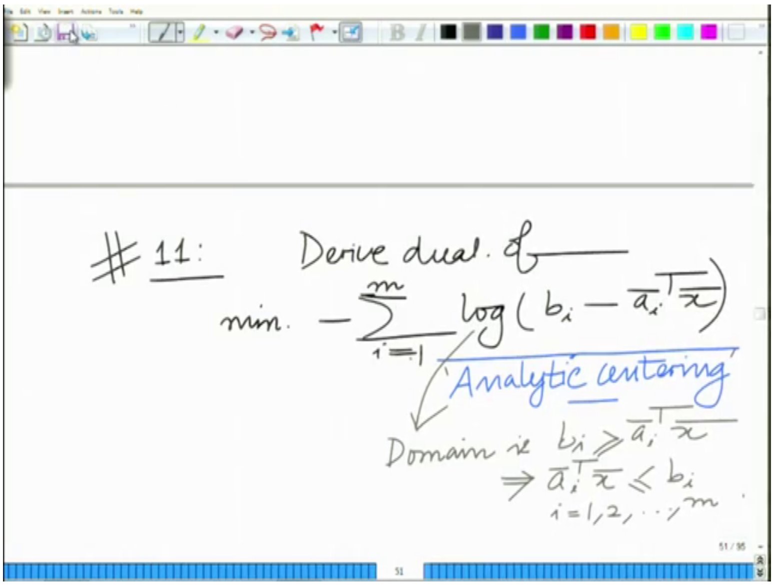
So, lambda bar is component wise greater than or equal to 0. And therefore the dual problem can be formulated as, now interestingly that is maximized g lambda bar that is maximized the dual function which is maximize lambda bar transpose b bar. Subject to the constraint that 1 bar transpose lambda bar equals 1 summation of all lambda i equals 1 A times lambda bar equals 0 lambda lies in the null space of a all right. So, remember look at this is an interesting condition.

(Refer Slide Time: 13:41)



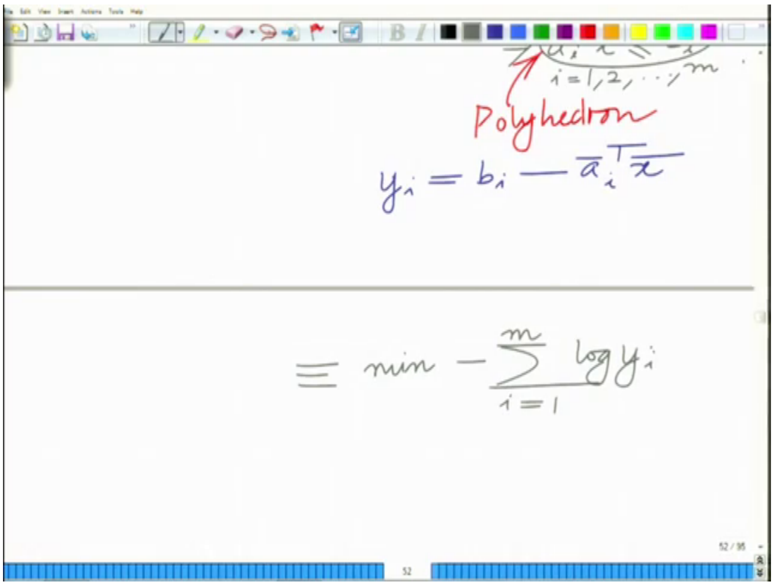
This means that lambda lies in null space lambda lies in the null space of matrix A and further non negativity of the Lagrange multipliers associated with the inequality constraints that is lambda bar greater than equal to 0. This is the dual problem dual problem for the given the original min max problem that is minimization of the piecewise linear function all right can be written in a linear program in this one all right.

(Refer Slide Time: 14:34)



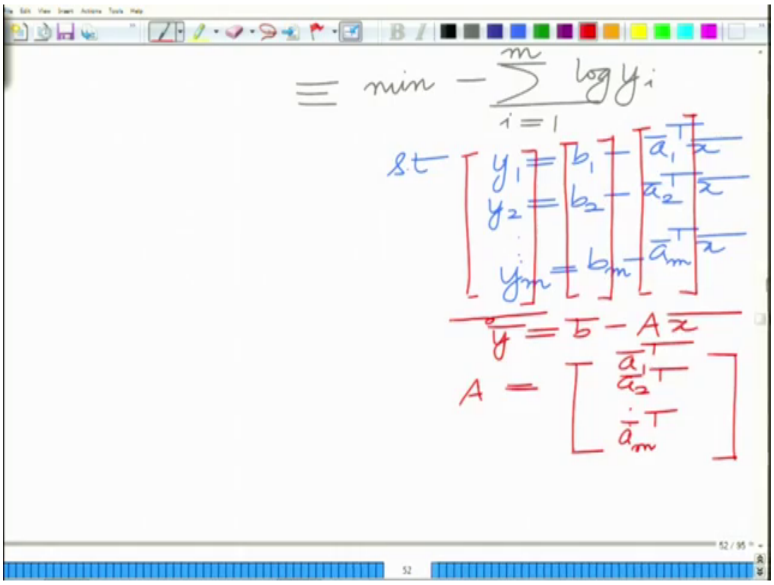
Let us look at another interesting application. We want to derive the dual minimize the negative summation of log of b i minus a i bar transpose x bar. This problem is arises frequently in practice this is termed as analytic this is the analytic centering also termed as the analytic centering problem all right. So, going to minimize this summation i equal to 1 to m log b i minus of course the domain of this is b i greater than equal to a i bar transpose x bar. So, this implies a i bar transpose x bar has to be less than or equal to b i for i equals 1, 2, m that is the domain of x bar ok. So, you have basically this is an intersection of as we know half spaces. So, this is a polyhedron.

(Refer Slide Time: 16:12)



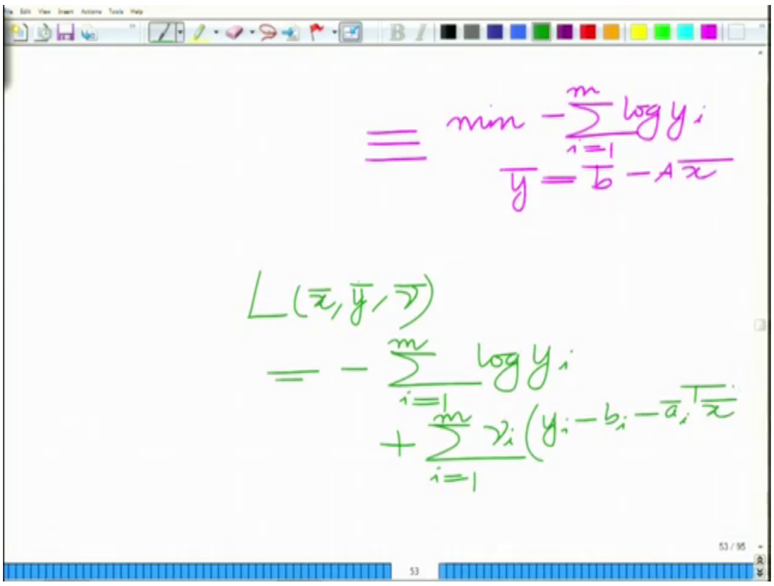
So, the domain is basically a this is basically a polyhedron. And to develop the dual again what we will do is we will use a simple substitution. We will substitute y i equals b i minus a i bar transpose x bar. So, therefore, the optimization problem can be equivalently written as minimize minus summation i equal to 1 to m log y i that is the log natural logarithm of y i subject to the constraint that each y i equals b i minus a i bar transpose x bar.

(Refer Slide Time: 17:08)



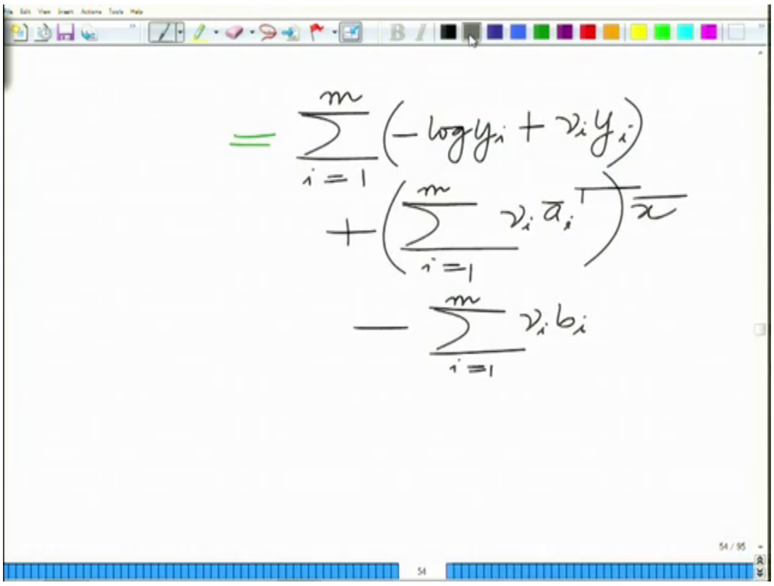
Subject to the constraint that y 1 equals b 1 minus a 1 bar transpose x bar y 2 equals b 2 minus a 2 bar transpose x bar, and y m going to b m minus am bar transpose x bar. And in fact, you can stack these things as a vector when you stack these things as a vector, what you get is the following thing, you can write this as y bar equals b bar minus matrix A or matrix A times x bar. What is A? A is the matrix a 1 bar transpose a 2 bar transpose up to am bar ok.

(Refer Slide Time: 18:26)



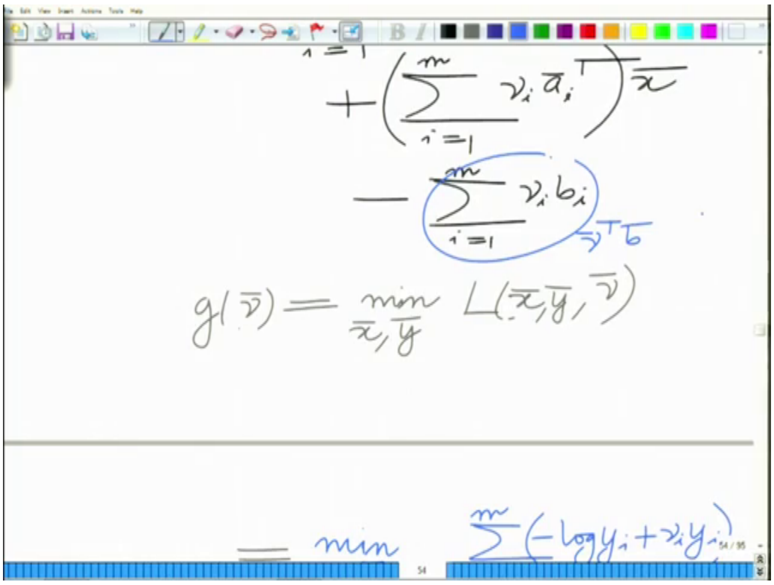
So, I can write this. So, basically I can write this equivalently as minimize minus summation i equals 1 to m log y i subject to the constraint y bar equals b bar minus A x bar. Now, what I am going to do again to develop the dual, I am going to write this I am going to form the Lagrangian and from that minimize it over the primal optimal primal the primal variables ok. So, the Lagrangian is lambda x bar comma y bar comma, now there are equality constraint. So, I am going to use nu bar which is basically the objective minus summation i equal to 1 to m log y i plus summation i equal to 1 to m, 1 Lagrange multiplier for each equality constraint that is y i minus b i minus a i bar transpose x bar.

(Refer Slide Time: 19:44)



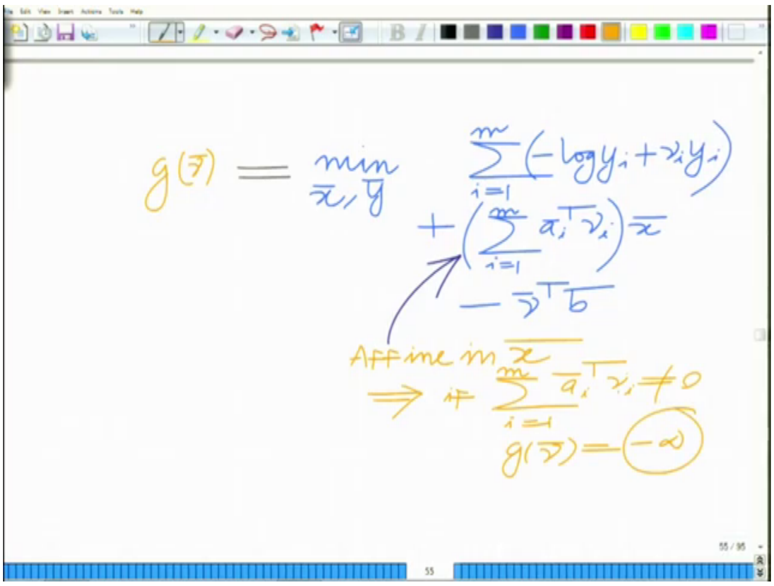
Now, once again collecting all the terms what you will observe is this will become summation i equals 1 to m first collecting the terms corresponding to y i i equal to 1 to m minus log y i plus nu i times y i plus once again collecting all the terms corresponding to x bar this will be a summation i equals 1 to m nu i a i bar transpose times x bar nu i into a i bar transpose into x bar, and minus summation i equal to 1 to m nu i b i this is going to be minus b i plus a i bar transpose x bar. So, this is going to be plus nu i a i bar transpose minus b i. So, this is basically the Lagrangian ok. And now we have to take the infimum or the Lagrangian all right with respect to the primal of primal the very primal variables where that is basically y bar and x bar.

(Refer Slide Time: 21:07)



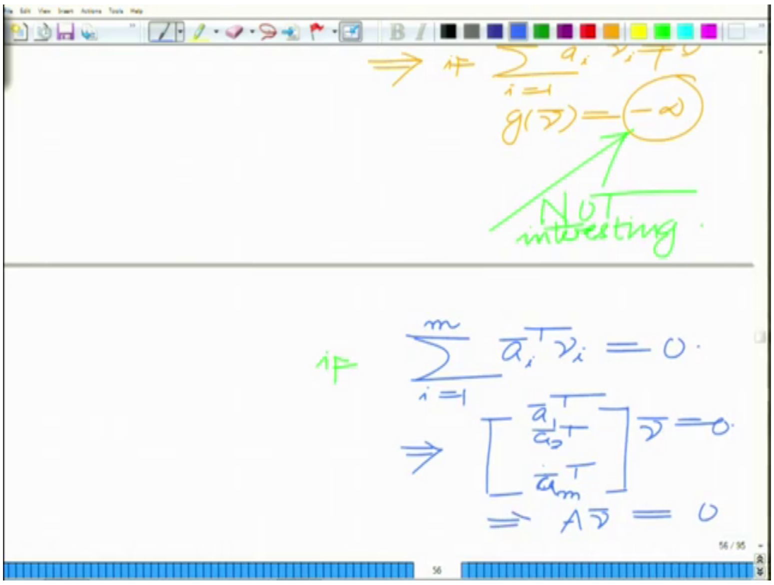
So, now what we are going to do is g of nu bar is the minimum or x bar comma y bar or x bar comma nu bar which is equal to the minimum or x bar comma y bar well we have the original way I am going to just simply write it minus summation i equal to 1 to m or i equal to 1 to m minus log y i plus nu i y i plus summation i equal to 1 to m a i bar transpose nu i into x bar minus. Now, of course summation of nu i b i I can simply write this as nu bar transpose b bar just for brevity which is minus nu bar transpose b bar.

(Refer Slide Time: 22:18)



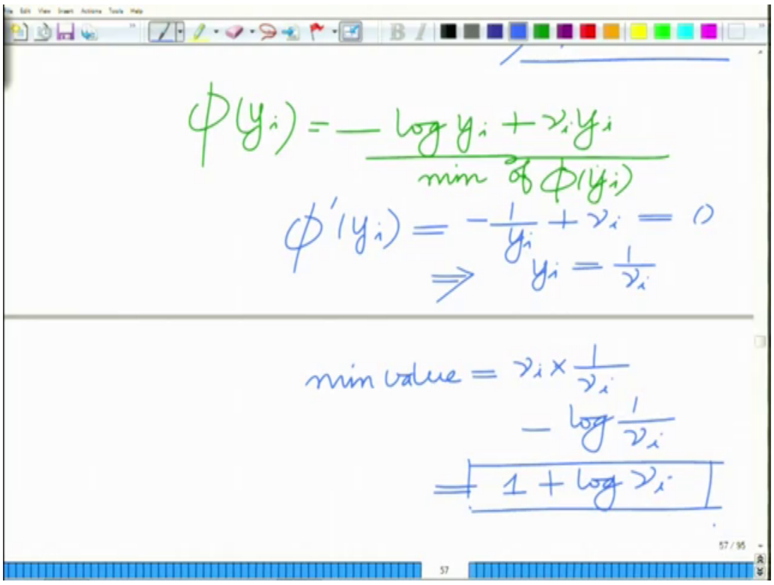
Now, again you can see the following thing. This is affine in x bar affine in x bar implies if summation i equal to 1 to m a i bar transpose nu i not equal to 0, then you are g of nu bar that is the minimum minus infinity that is the minimum of this, because it is an affine function minimum is 0 which is not interesting, once again this is not interesting.

(Refer Slide Time: 23:04)



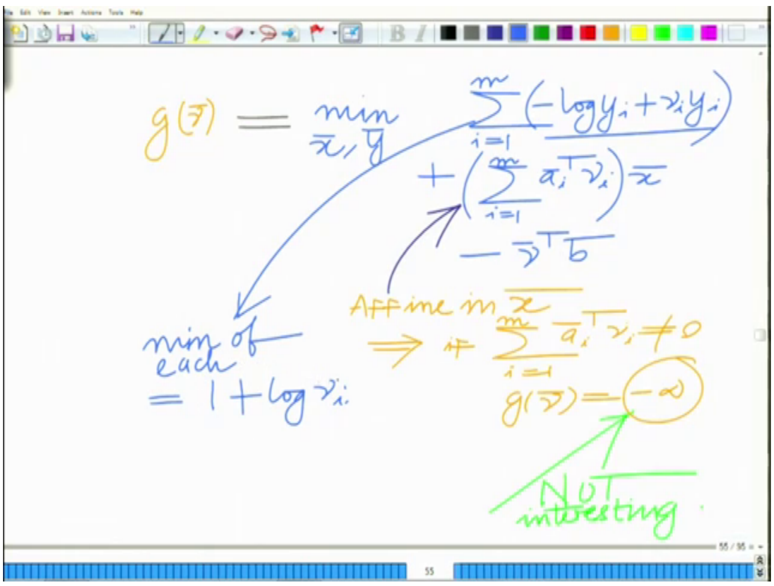
Therefore, on the other hand if summation i equal to 1 to m nu i a bar transpose nu i equal to 0. This implies if you look at this a m bar transpose into nu bar equal to 0 or basically A times nu bar equal to 0, then of course this is not an affine function. So, we would like to consider this condition. So, now, we would like to consider this condition A times, because we still have this other part which is now let us come to this part minus log y i plus nu i yi.

(Refer Slide Time: 24:05)



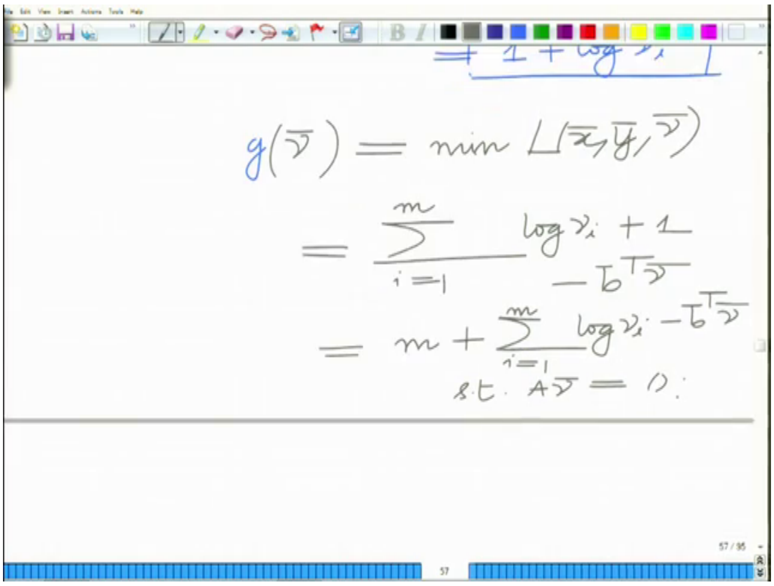
Let us look at what is the minimum of this consider minus log y i plus nu i times yi. Let us call this phi of yi. Now, we have to find minimum of this or minimum of basically phi of y. So, to do that first differentiate this with respect to phi y i that is minus y i plus nu i equal to 0 which implies the minimum occurs for well y i equals 1 over nu i and what is the minimum, minimum value equals nu i into 1 over nu i minus log 1 over nu i which is basically 1 plus log nu i. So, in summary, if a times nu bar equal to 0, the minimum occurs for the minimum is basically the minimum of each log y i plus nu i y i is basically, so, the minimum of each of this equals basically 1 plus log nu i

(Refer Slide Time: 25:36)



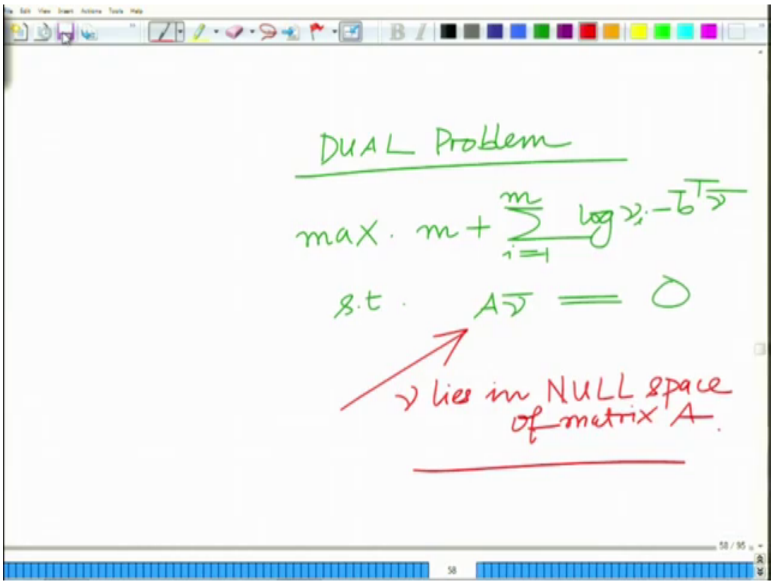
And therefore, the net minimum is this 1 plus summation 1 plus log nu i minus nu bar transpose b bar.

(Refer Slide Time: 26:02)



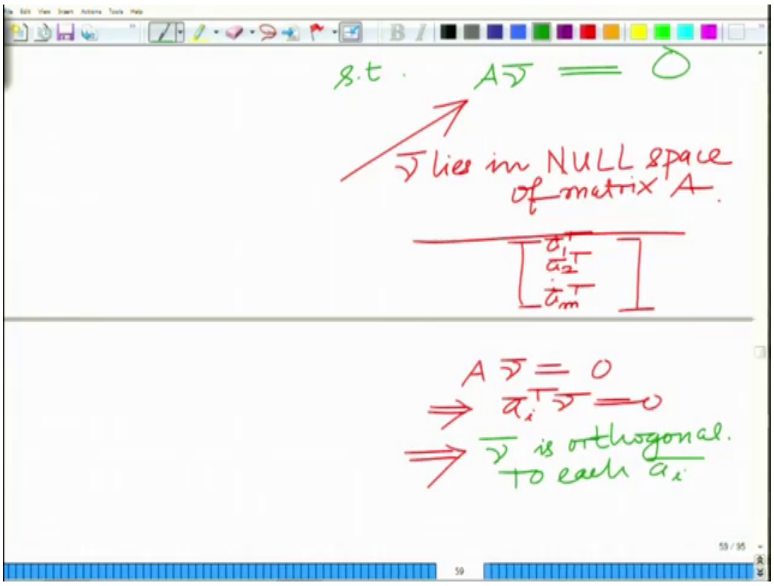
So, basically what you have is g of lambda bar equals minimum of L of x bar comma y bar comma nu bar I am sorry this is g of nu bar this is g of nu bar which is basically which is you can see summation i equals 1 to m log of nu i plus 1 minus well b bar transpose nu bar or nu bar transpose b bar which is if you take the constant 1 that is basically m plus summation i equals 1 to m log nu i minus b bar transpose nu bar. And of course, subject to the constraint remember A into nu bar equal to 0 otherwise it is minus infinity which is not interesting.

(Refer Slide Time: 27:05)



And therefore the dual problem is the dual problem of this analytical center is maximize the dual function which is m plus summation i equal to 1 to m log of nu i minus b bar transpose nu bar subject to the constraint that A times nu bar equal to 0 that nu lies in the null space of A ok. It is very interesting nu lies in the null space of nu lies nu lies in the null space of A.

(Refer Slide Time: 28:04)



And you can see basically what A is, A is a 1 bar transpose, a 2 bar transpose a m bar transpose. So, A nu bar or nu bar lies in the null space a nu bar equal to 0 implies basically each a i bar transpose nu bar equal to 0, which means basically nu bar is orthogonal in nu i bar is orthogonal nu bar is orthogonal to each a i bar. So, nu bar lies in the null space of the matrix all right. So, these are some examples of various problems view up problems convex optimization problems and how to formulate their dual problem which is often which often yield very useful insights. And these are often very useful in practice especially for practical application all right. So, let us stop here, and continue in the subsequent modules.

Thank you very much.