Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 72
Examples on Quadratic Optimization

Hello, welcome to another module in this massive open online course. So, we are looking at Example problems for Convex Optimization. Let us look at another problem that is a Quadratic Optimization all right.

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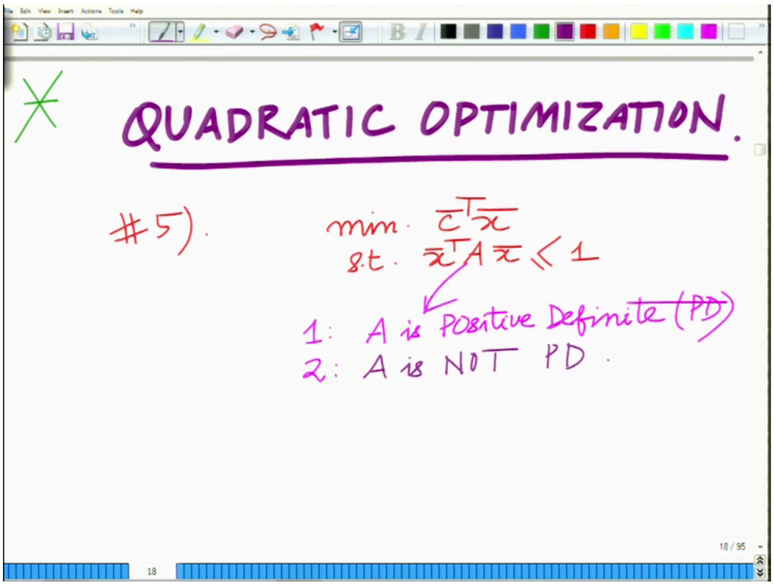

So, we are going to look at solving and this arises fairly frequently that is your Quadratic Optimization, your example number 5. And the quadratic optimization objective is as follows; minimize C bar transpose x bar. C bar transpose x bar subject to the constraint that x bar transpose A x bar less than or equal to and we will consider two cases for this; that is A is positive, A is positive definite that is it is a PD matrix. And 2, what happens the more interesting case is what happens when A is not positive definite ok.

So, this minimize C bar transpose x bar x bar subject to the constraint x bar transpose A x bar is less than equal to 1. Now, what happens in this scenario? Now, let us start with 1; your A is positive definite.

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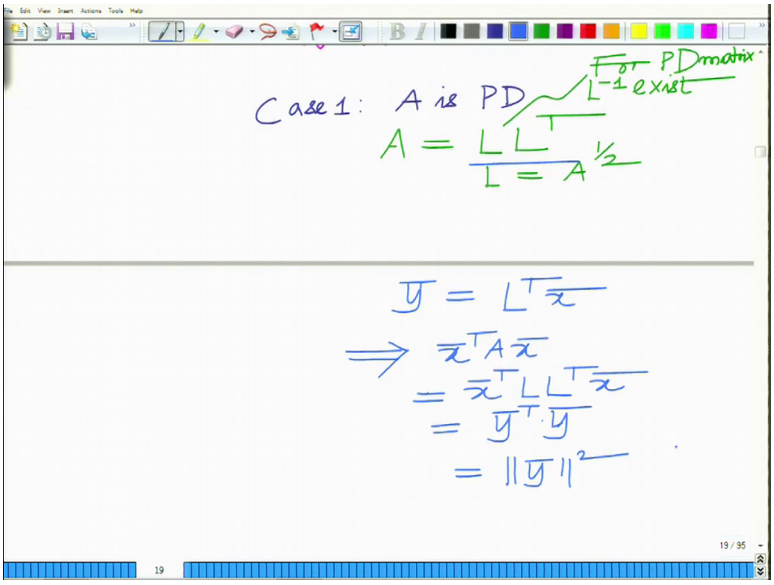

When A is positive definite, you can write A as L L transpose and where, L is you can also call this as the square root of A alright, this is obtained by a Cholesky decomposition all right.

So, any positive semi definite positive definite matrix for that matter also positive semi definite matrix can be decomposed as LL transpose. For a positive definite matrix in addition, this L is invertible ok. So, for PD, matrix L inverse exists ok. So, positive semi definite L inverse might not exist because some of the eigenvalues might be 0 ok. And what happens then what what you can do? It is very simple; you have you set A equals LL transpose. You set y bar equals L transpose x bar.

So, now our constraint if you look at x bar transpose A x bar; this equals x bar transpose L L transpose x bar which is equal to x bar transpose L is y bar transpose and L transpose x bar equals y bar. So, this becomes your norm y bar square. And when? Of course, we have we have set y bar equals L transpose x bar if you look at this, this basically implies that x bar, remember we said L is invertible.

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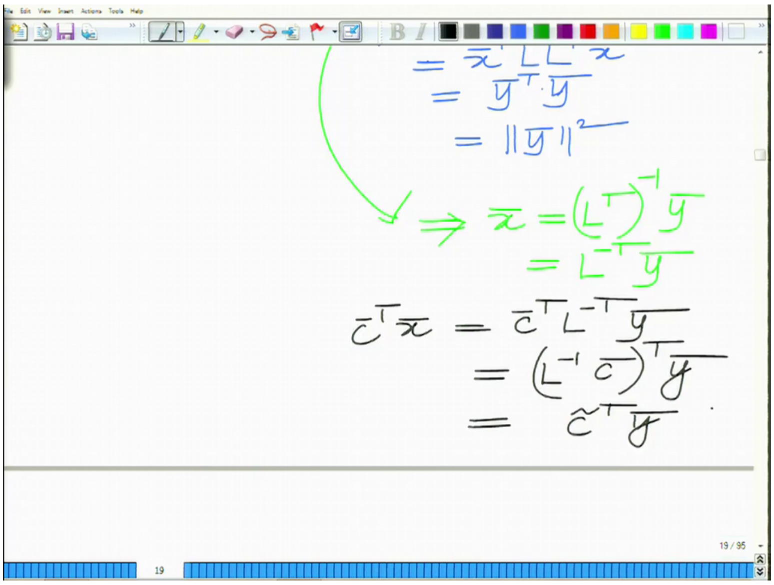

So, this is L transpose inverse y bar which is simply written as L raised to minus T y bar. This is L transpose inverse ok. And therefore, C bar transpose. So, the objective function or the objective function C bar transpose x bar becomes C bar transpose L transpose inverse y bar which is L inverse C bar transpose y bar and you can set this as C tilde transpose y bar ok. So, what is C tilde?

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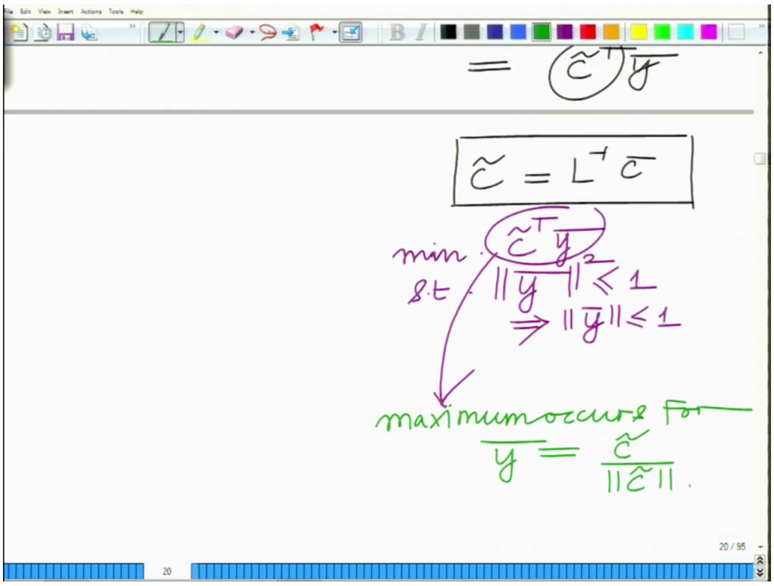

Well, C tilde equals L inverse C bar and therefore, we have recast the objective, we have recast the constraint in terms of y bar. Now, I can write the optimization problem as follows. I can write it as minimize, well I can write it as minimize C tilde transpose y bar. Subject to the constraint norm y bar square less than equal to 1 that is remember, this quantity here is norm y bar square is less than equal to 1, which implies that basically norm y bar is less than or equal to 1.

And therefore, what we are asking is what is the unit norm beam unit norm vector y bar which has maximized, which maximizes C tilde transpose y bar and we know the solution to this problem. The maximum occurs where y bar is the unit norm vector along C tilde ok. So, maximum occurs C tilde divided by norm which is basically remember what is C tilde?

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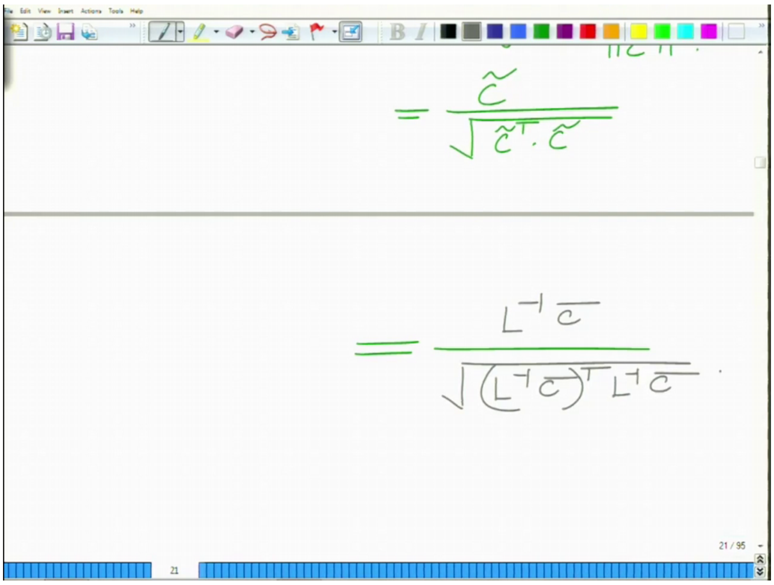

C tilde is so, you can write it as C tilde divided by C tilde transpose C tilde because remember C tilde transpose C tilde is norm C tilde T square and this is therefore, C tilde. What is C tilde?

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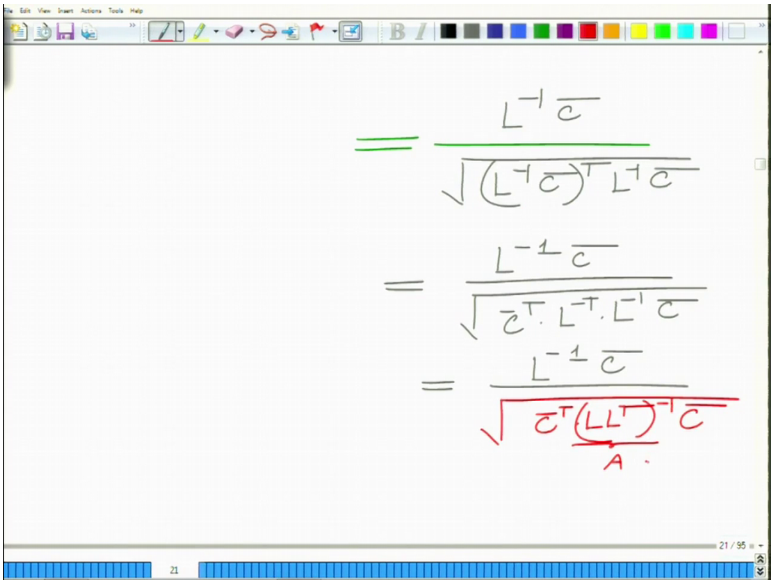

C tilde is L inverse C bar divided by L inverse C bar transpose into L inverse C bar which is basically if you remember; if you look at this L inverse C bar divided by square root C bar transpose L inverse transpose or L transpose inverse L inverse C bar which is L inverse C bar divided by now you can write this as C bar transpose L L transpose inverse C bar and LL transpose is A. So, this has an interesting structure.

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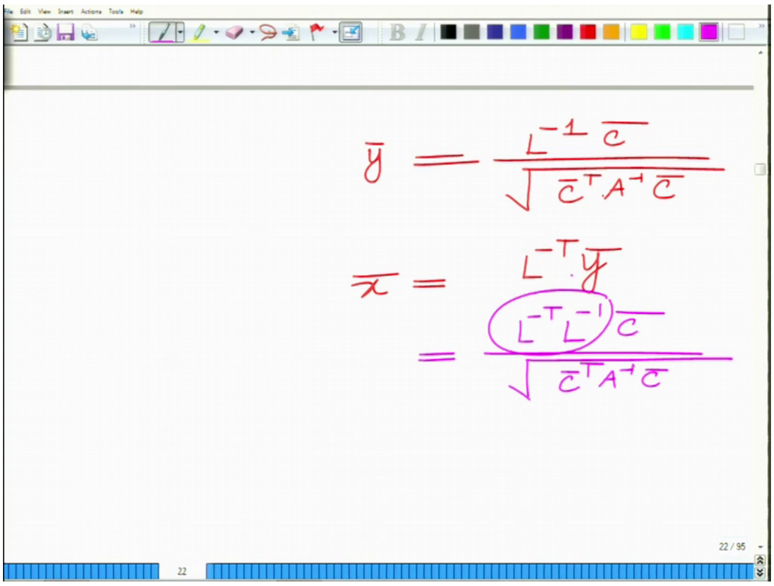

So, you have optimal y bar equals L inverse C bar divided by square root of C bar transpose A inverse C bar ok. And now what is the optimal x bar? Optimal x bar remember is L optimal x bar, if you remember is L transpose inverse y bar which is equal to L transpose inverse L inverse C bar square root of C bar transpose A inverse C bar. Now, if you look at this, this is nothing but L L transpose inverse which is nothing but A inverse.

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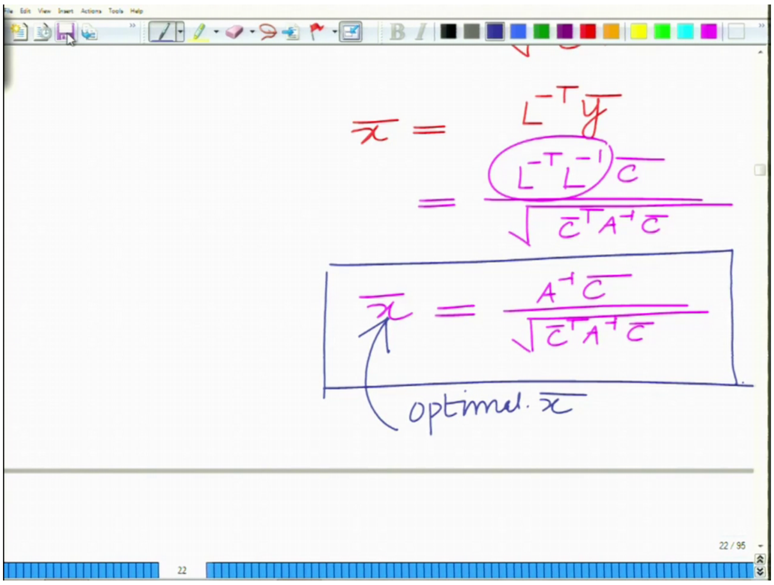

So, this is A inverse C bar divided by C bar transpose A inverse C bar. So, this is the optimal value of; this is a optimal vector x bar. And what is the optimal value of the objective? Optimal value of objective you take this x bar substitute in the objective ok.

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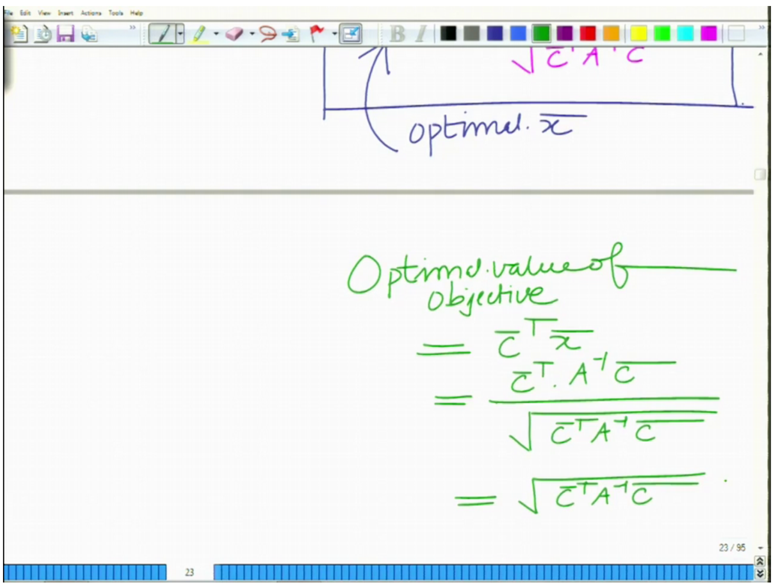

So, optimal equals C bar transpose x bar which is C bar transpose A inverse C bar divided by C bar transpose A inverse C bar which is again you can see square root of C bar transpose A inverse C bar.

So, this is your optimal value of the objective function ok; optimal values of the objective function.

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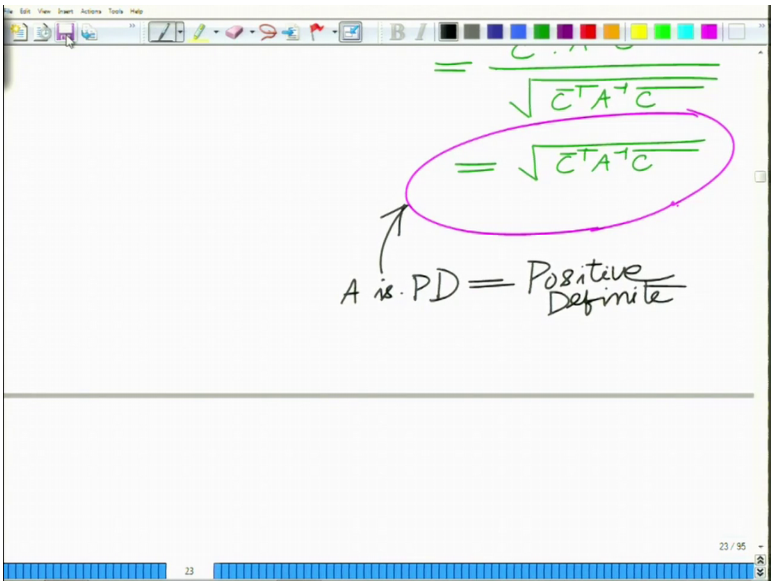

But remember this entire case is when A is PD equals positive. A equals a positive definite matrix. What happens when A is not a positive definite? Then, A phrase not positive definite, then you cannot be decomposed as LL transpose correct or at least L is not invertible which only positive semi definite. So, what happens if A is not PD ok?

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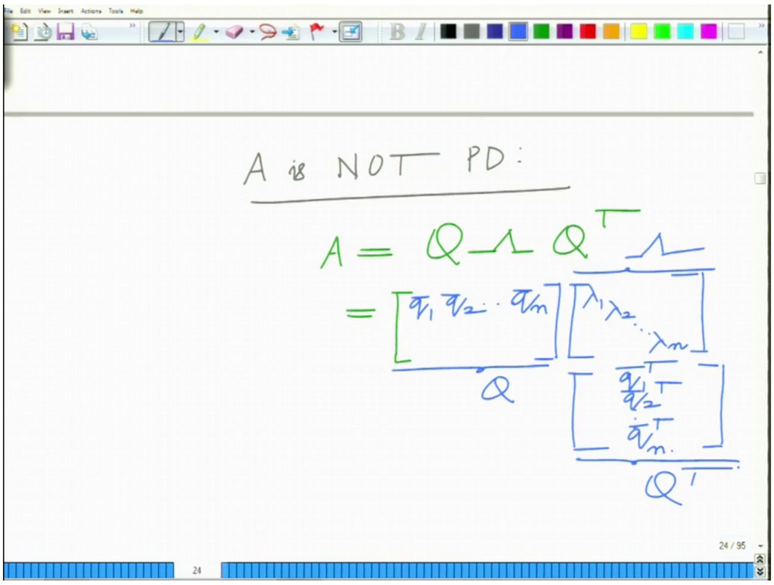

So, that is an interesting scenario A is not; A is not positive definite, what happens in that scenario is let us say you have an eigenvalue decomposition of A which is Q lambda Q transpose. So, I can write this as remember the eigenvalue decomposition. I can write this as a matrix of eigenvectors q 1 bar q 2 bar q n bar, if this is an n cross n matrix times the matrix of eigenvalues lambda 1 lambda 2 lambda n times transpose of q that is q 1 bar transpose q 2 bar transpose q n bar.

So, this is your Q, your diagonal matrix of eigenvalues lambda and this is your Q transpose and this is basically your matrix of eigenvectors ok. This is your matrix of eigenvectors.

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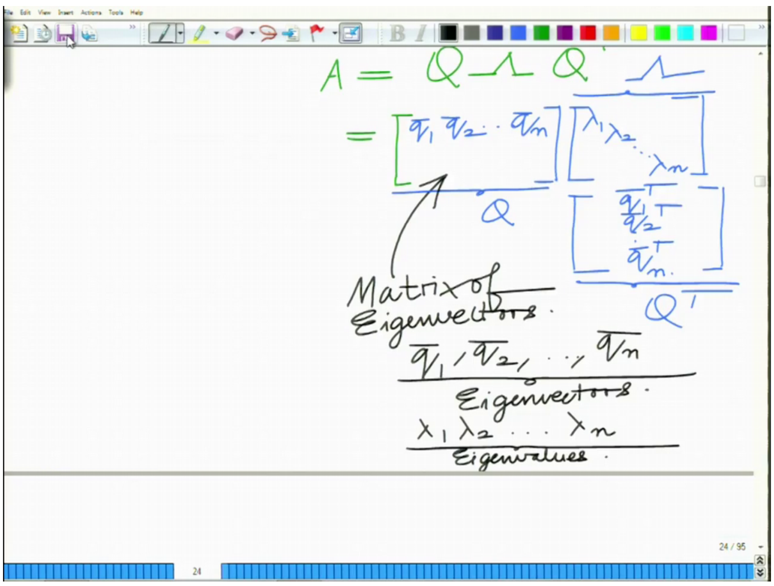

So, basically you have q 1 bar q 2 bar up to q n bar. These are eigenvectors and lambda 1 lambda 2 lambda n these are these are eigenvalues; the corresponding eigenvalues.

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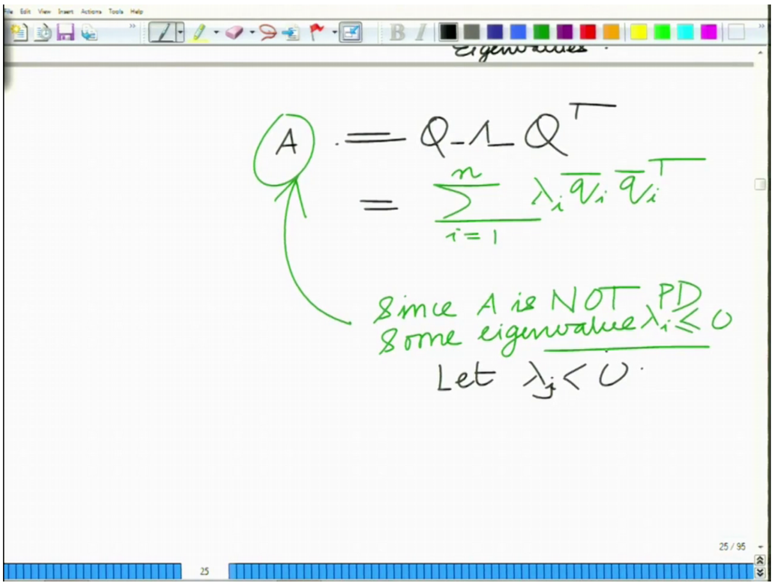

Now, I can write this as A equals Q lambda Q transpose and you can multiply it out and you can say I can write this as summation i equals 1 to n lambda i q i bar q i bar transpose

Now, since A is not PD, not PD you have some eigenvalue lambda i is less than or equal to 0 ok. Now, remember if all eigenvalues lambda i are strictly greater than 0 then the matrix becomes positive definite right. So, here we are assume, so, here because there is not PD then it must be the case that some lambda i is less than or equal to 0. Let us assume for simplicity that some lambda is less than 0 particular lambda i. Now, let say lambda j be less than 0; particular lambda j be strictly negative, what happens in that scenario? Now, let us go back and let us again here in this case set y bar equals Q transpose x bar; this is the matrix of eigenvectors.

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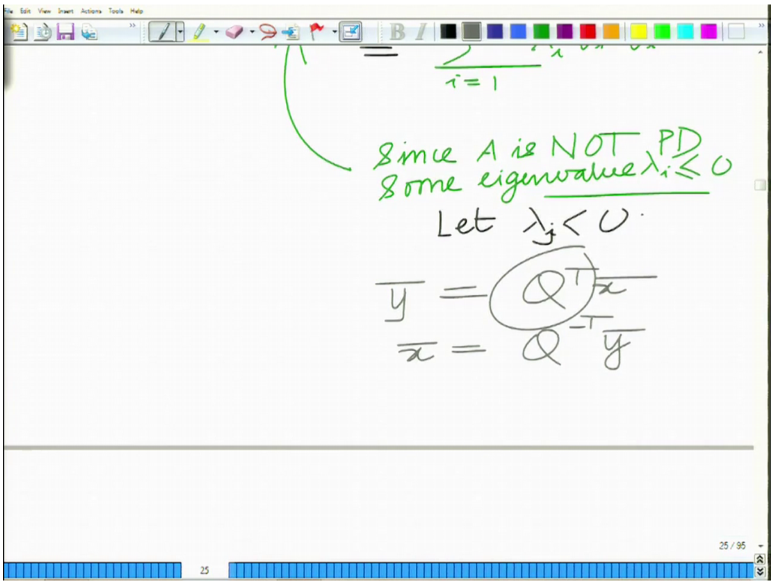

So, I can always write x bar equals Q inverse transpose y bar ok.

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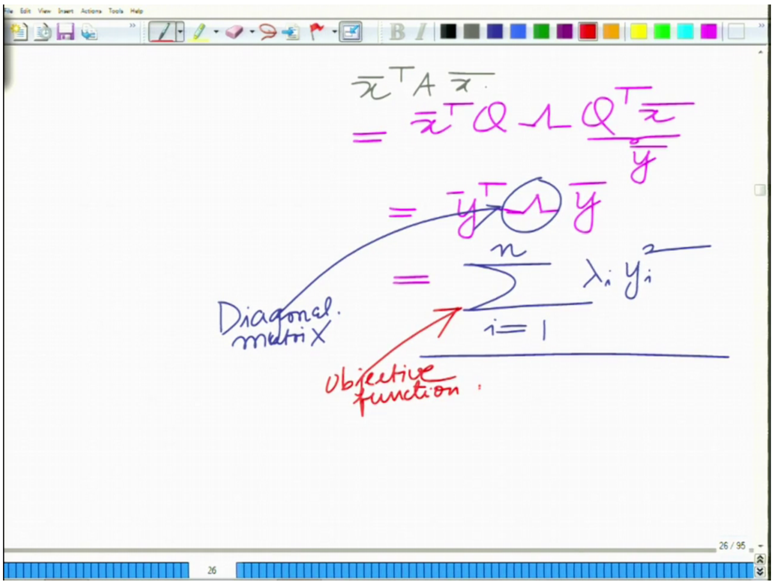

And therefore, now if I look at x bar transpose A x bar this is equal to well x bar transpose Q lambda Q transpose x bar. Now we know Q transpose x bar equals y. So, this will become y transpose y bar transpose lambda which is a diagonal matrix into y bar. And now if you multiply this, you can see this is simply because lambda is a diagonal matrix; remember this is a diagonal matrix. Therefore, what you will get is summation i equals 1 to n lambda i y i square.

So, this is what you get ok; this is the objective function. Now, similarly what happens to the constraint? This this is basically your or this is basically your constraint I am sorry this is basically your constraint.

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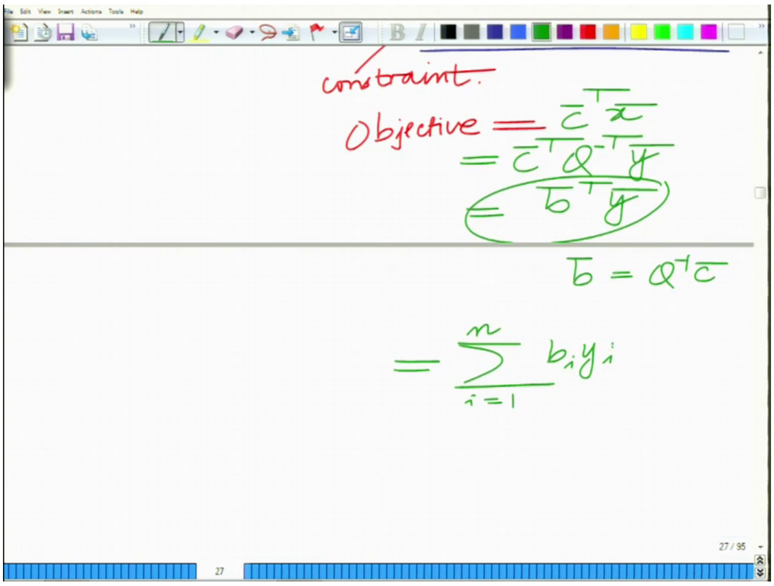

Now, I mean the constraint is this is less than or equal to 1. Now what happens to the objective? Objective equals what happens to the objective? Objective will be well C bar transpose x bar which is C bar transpose Q inverse transpose y bar. Let us say this is some b bar transpose y bar, where, b bar equals Q inverse C bar and this is naturally this is summation i equals 1 to n bi y i. So, I can recast this optimization problem.

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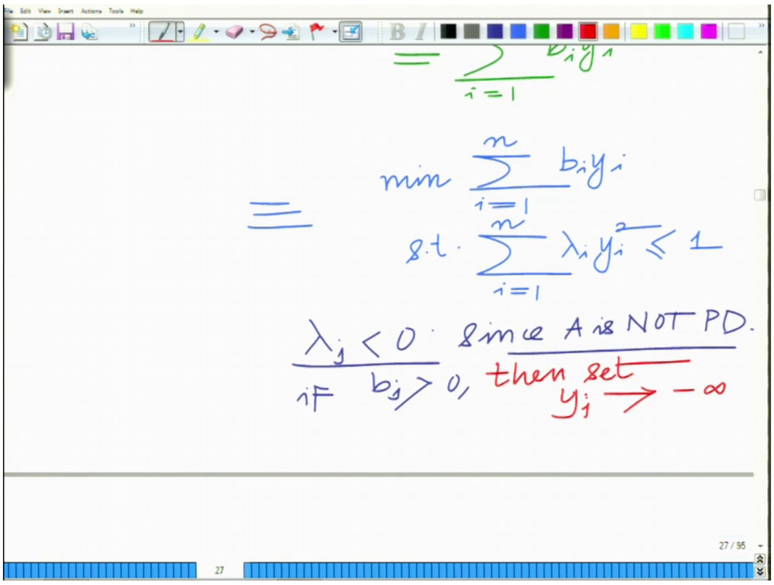

So, this is equivalent to original optimization problem is equivalent to minimize summation i equals 1 to n bi yi subject to the constraint summation i equal to 1 to n lambda i y square less than or equal 1. Now, we have to solve this optimization. Now, let now we know or we are assuming that one particular lambda j is less than 0 ok, since is not positive definite.

So, one particular lambda is less than 0 alright because A is not a positive definite matrix. Now, what happens? Now, if that corresponding bj is greater than 0, now let us assume 2 cases. If bj greater than 0, then set y i to be a very large negative value ok. Now, since lambda j is negative or said that corresponding yj to be a very large negative value.

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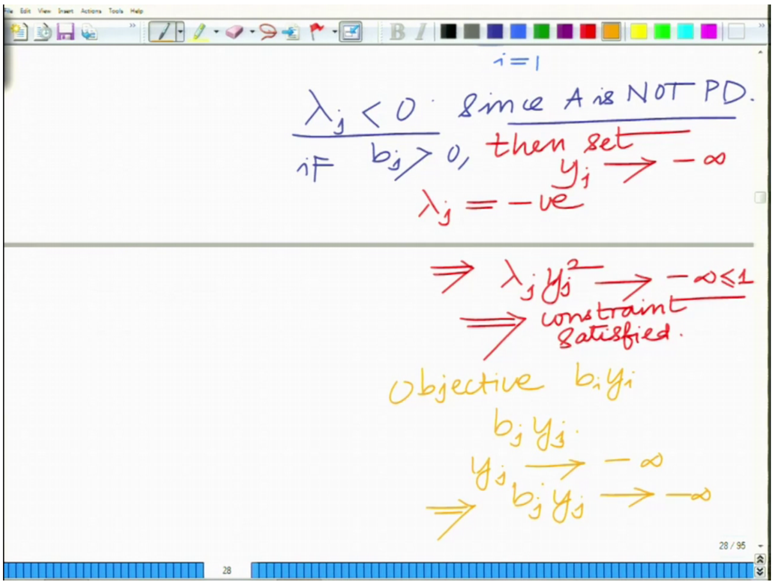

Now, since lambda j is negative implies lambda j into yj square equals negative or tends to minus infinity, implies the constraint is satisfied.

So, what is happening? Because lambda j is negative y j square is also positive. It was always positive. So, lambda j y j square as y j is tending to minus infinity y j square tends to infinity right. So, therefore, lambda j to yj square again tends to minus infinity ok. So, the constraint is always satisfied all right. So, this is always going to be less than or equal to 1. But if you look at the objective, the contribution of the contribution of yj, this will be bi into y i, y i tends to infinity or this will be bj into yj ok; yj tends to minus infinity implies what does the yj tends to? Well, yj tends to minus infinity implies since bj is greater than 0; b j yj tends to minus infinity since bj is greater than 0.

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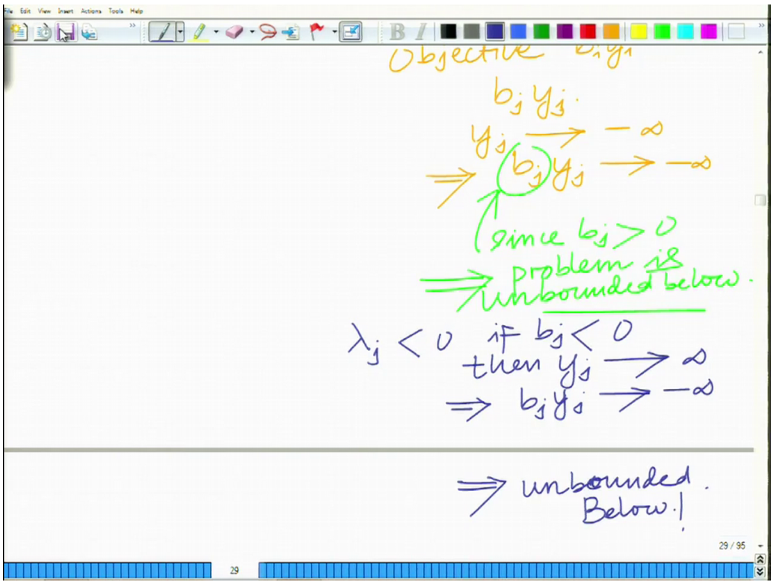

; Implies problem is unbounded, this is unbounded below. So, basically by setting y j equals minus infinity by making yj as small as possible, I can make the optimization objective as for as possible.

Now, consider another scenario. If once again lambda j is less than 0, if bj is also less than 0, then set yj to a large positive value. This again implies constraint is always satisfied because lambda j to y js negative which is or tends to minus infinity, but bj yj tends to b j y j tends to minus infinity implies again this is unbounded below alright.

So, basically what you have? So, if lambda is one of the eigenvalues is less than 0 and the corresponding bj is not equal to 0, basically it is either greater than 0 or less strictly greater than 0 or less than it, but not equal to 0, then the optimization problem is unbounded below that is you can make the optimization objective as small as you. And you can consider all other cases. What happens when bj is 0, what happens when lambda j is 0 and so on and but this is the most interesting case alright. So, what happens when he is positive definite, what happens when he is not positive definite?

So, A being positive definite has a very important role to play in this problem alright. Let us look at another example; example number and this is a very interesting example, example number 6.

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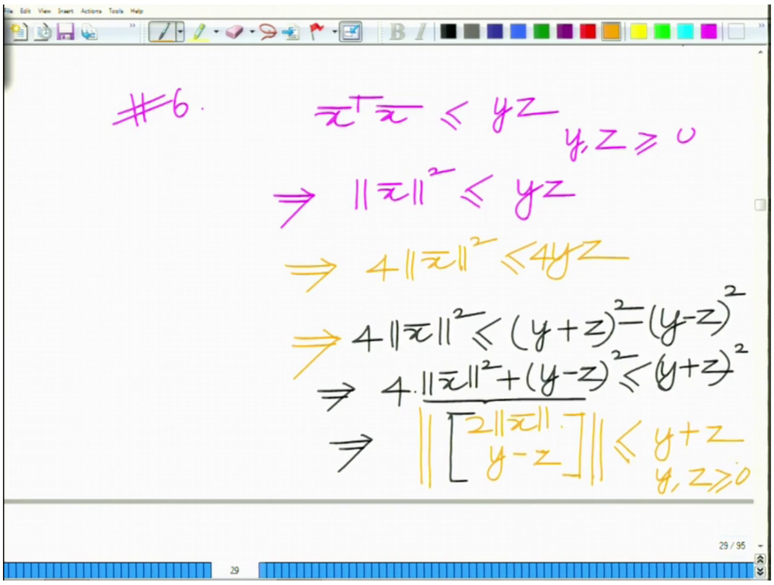

So, what we want to start with is we want to start and show that if you consider any vector x bar transpose x bar transpose x bar x bar and we have x bar transpose x bar less than equal to y z or y comma z greater than or equal to 0. Well, this implies that norm x bar square less than or equal to y z; this implies 4 norm x bar square less than or equal to 4 times y z, but 4 times y z is basically y plus z wholes y plus z whole square minus y minus z whole square.

So, this implies 4 times norm x bar square less than or equal to y plus z whole square minus y minus z whole square and this implies that well, if you bring this over here, 4 times norm x bar square; norm x bar square plus y minus z whole square less than or equal to y plus z whole square. This implies; now, if you look at this as a vector; this is nothing but the norm of the vector. This is the norm of the vector 2 norm x bar y minus z because the norm of this vector is 4 norm x bar square plus y minus z whole square. So, the norm of this vector or the 2 norm of this vector to be more specific is less than or equal to y plus z because y comma z is greater than or equal to 0 that is what we have from the original constraint ok. So, this condition is an interesting property.

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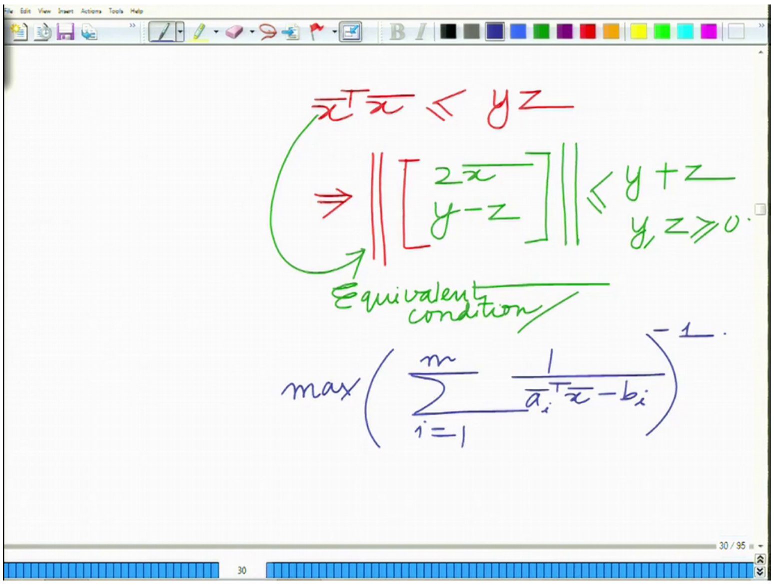

This condition that x bar transpose x bar is less than or equal to y z can be equivalently written as norm of this vector, norm of this vector to x bar y minus z is less than or equal to y plus z, where y comma z is greater than 0 ok. So, this is an equivalent condition that we have derived. This is an equivalent condition. Now, what we have to do? Now, let us use this condition. Now, let us say you want to maximize the harmonic mean that is let us say i equal to 1 to m 1 over a i bar transpose x bar minus bi inverse; this is the harmonic mean of these quantities; harmonic mean of so, this is. What is this?

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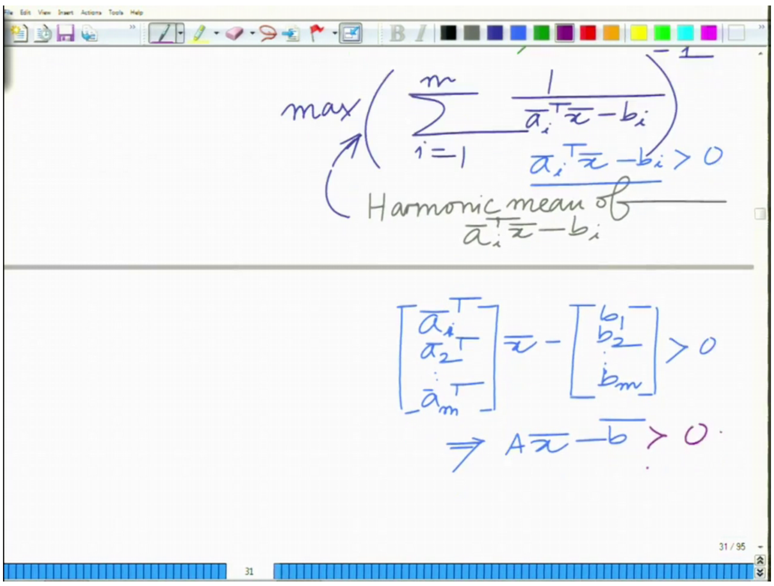

This is the harmonic mean, harmonic mean of the a i bar transpose x bar minus b i. So, maximize this now, of course, the condition here is also that these quantities are non negative; A bar transpose x bar minus bi is greater than 0; is greater than 0 which of course, now if you stack this in the form of a matrix, you can write this as a 1 bar transpose, a 2 bar transpose, a m bar transpose, x bar minus b 1, b 2 up to b m greater than 0 and this implies that if you look at this as matrix A, this implies that matrix A x bar minus b bar correct. This is component wise, this matrix is component wise greater than 0 ok. So, this is an equivalent way of writing this ok. Now maximize this quantity.

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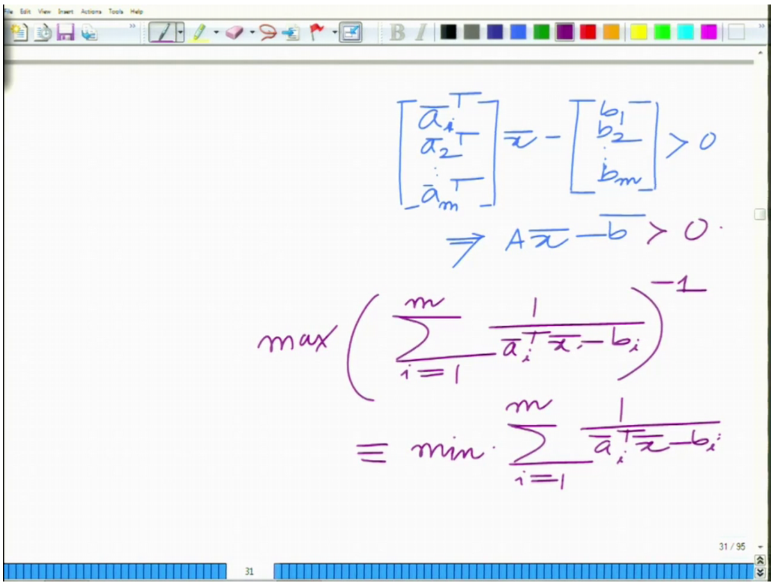

Let us go back to this quantity for a moment. 1 over a i bar transpose x bar minus bi inverse; this is equivalent to minimizing the reciprocal because everything is non-negative. This is equivalent to minimizing summation 1 over m summation equal to 1 over m a i bar transpose x bar minus bi. Now, let us write this in an epigraph form.

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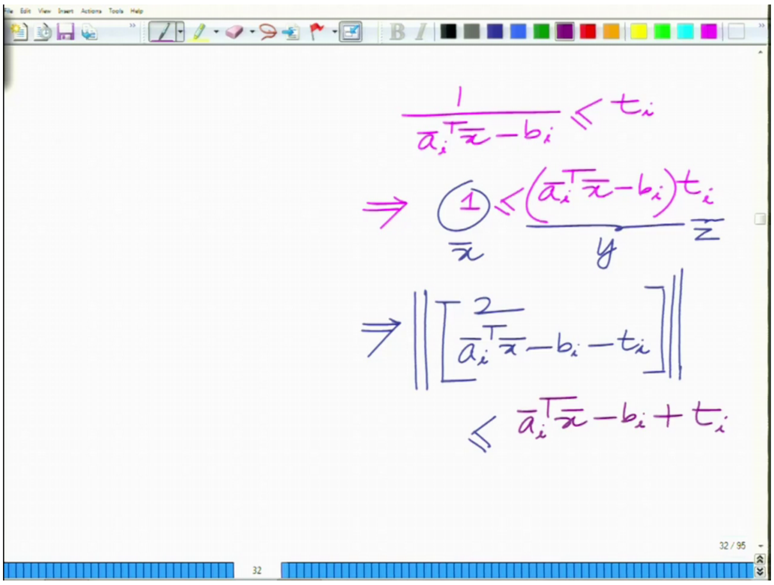

Let us use this variable ti ai bar transpose x bar minus bi less than or equal to t i; then I can minimize sum of ti's and this implies that now 1 is less than or equal to ai bar transpose x bar minus bi into ti and now we have an interesting parallel.

So, this is your x bar, this is your y, this is your z and I can equivalently write this condition as twice x bar correct twice x bar which is twice; y minus z which is basically ai bar transpose x bar minus bi minus ti, the norm of this quantity is less than or equal to y plus z that is basically ai bar transpose x bar minus bi plus ti.

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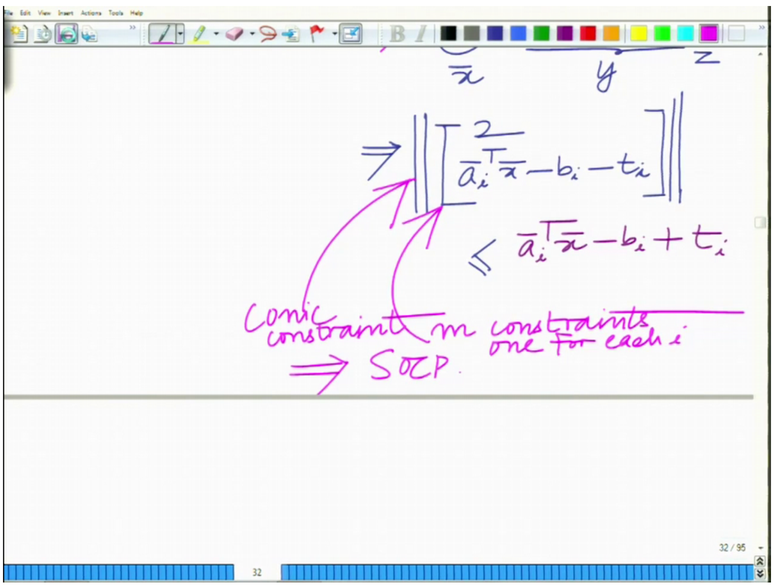

So, I can write each constraint, so, I will have m constraints; one for each i and if you look at this something very interesting about this. You might recognize this is a cone, second order cone constraint because if we look at this is a norm of something less than or equal to something that is linear on the right. So, this is a conic constrain. So, the resulting optimization problem will be a second order cone program.

So, this is very interesting implies SOCP and therefore, this can be written as a equivalent optimization problem.

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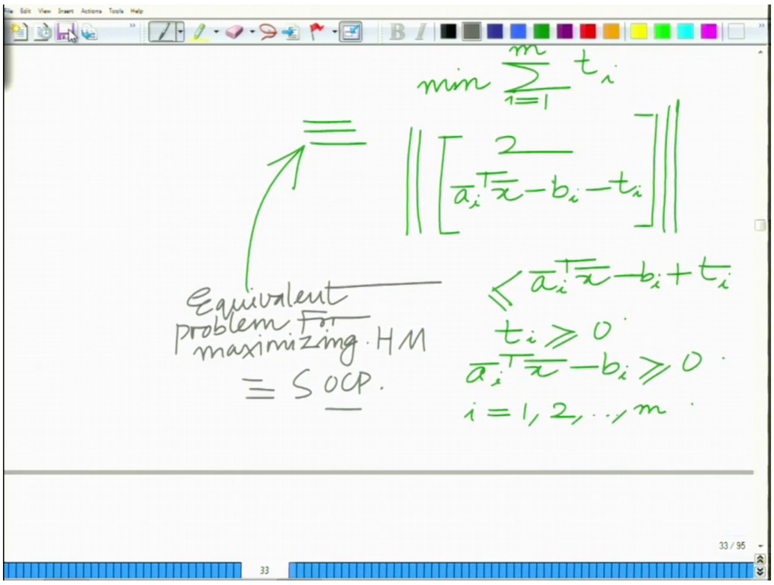

This is minimize summation i equals 1 to m ti subject to the constraint that norm 2 ai bar transpose x bar minus bi minus ti is less than or equal to a bar transpose x bar minus bi plus ti ok, where each ti is greater than or equal to 0.

And of course, we have the constraint ai bar transpose x bar minus bi is greater than or equal to 0, for i equals 1, 2 yeah and this is the equivalent problem for maximizing the harmonic mean, which is can be written as a second order cone program which is very interesting. HM and this can be written as a SOCP that is your second order cone program. So, it has very interesting practical application.

So, once you write this as a second order cone program, you can use the convex solvers readily available to solve this thing alright, since the minimizing the harmonic mean itself. I mean it is a complicated problem which is not very easily sort of doable using the normal technique alright. So, by writing it as an equivalent convex optimization problem, this can be rather readily solved in this form alright ah. So, let us stop here and continue in the subsequent.

Thank you very much.