Applied Optimization of Wireless, Machine Learning, Big data
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Lecture - 71
Examples: /1 minimization with /x norm constraints, Network Flow problem

Hello welcome to another module in this massive open online course. So, we are looking doing examples on convex optimization alright; to better understand through various examples alright; so, let us continue the discussion.

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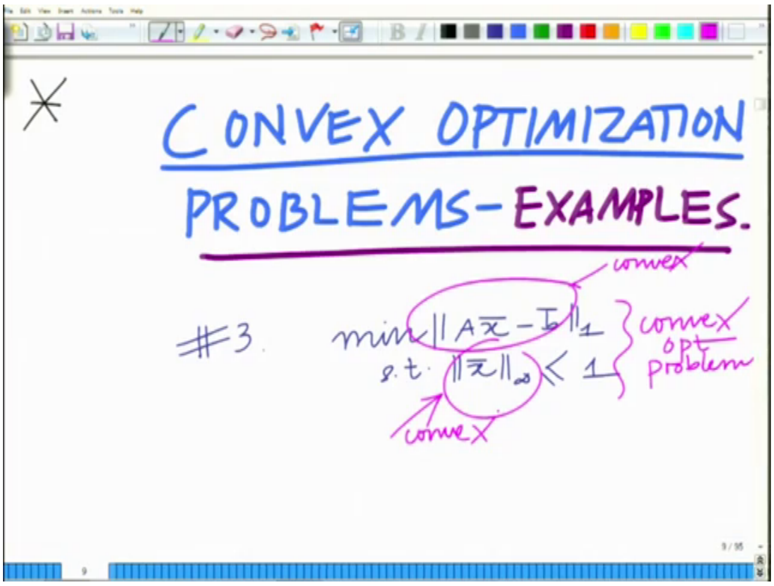

So, we are doing convex optimization problems. So, we are doing convex optimization problems and in particular we are doing a few examples which I think we will focus on or we will illustrate how to formulate these problems and in fact how to formulate them as convex optimization problems; so that we can use that convex solver to solve this optimization problems alright.

So, let us look at an yet another example; example number 3 that is we want to minimize this norm A x bar minus b infinity; subject to the constraint that norm x bar or to minimize x norm x minimize l 1 norm; for norm A x bar minus A A x bar minus b bar subject to the constraint that l infinity norm of x bar is less than norm equal to 1. Now of course, if you look at this; this is convex optimization problem because if you look at this l 1 norm; this is convex and this is convex. So, we have convex objective convex inequality constraint; so, this is the convex optimization problem alright and you can direct solve it.

But we want to sort of; however, it recast it into a home that is more intuitive or more amenable to analysis. And you can see, you will see that this problem which has this looks or seemingly complicated can be cast or can be recast in a very nice form and that is as follows we are going to do the following things.

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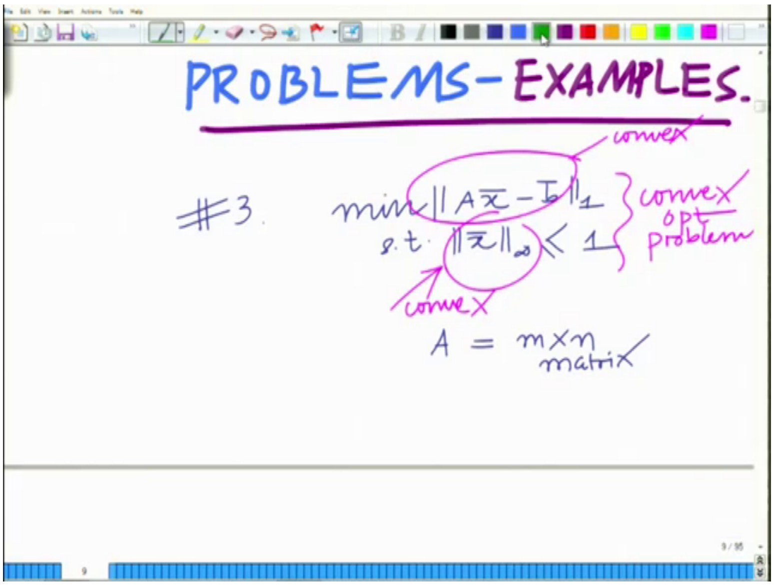

So, you have a A x bar; let us assume A is an m cross n matrix.

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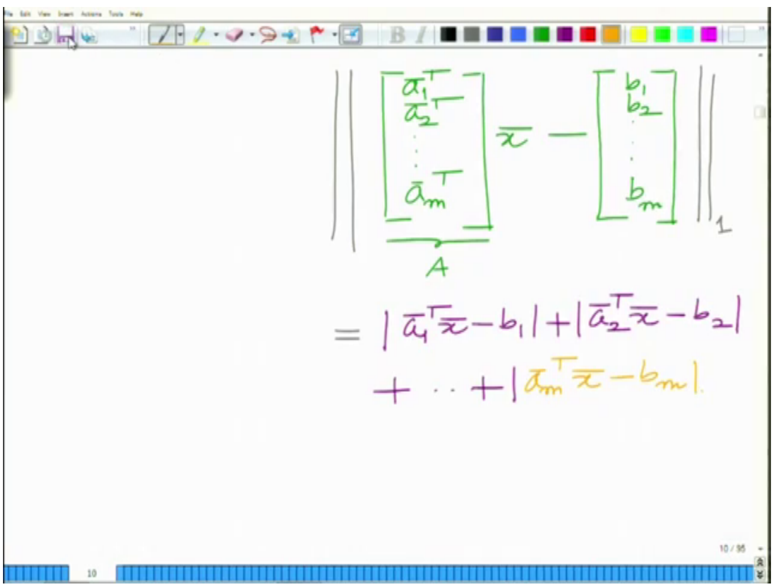

Then you can rewrite this as follows you have this a 1 bar transpose, a 2 bar transpose this will be a m bar transpose x bar. So, this is your matrix A; which has m rows, which I am denoting by m bar transpose a 2 bar transpose, a m bar transpose; a times x bar minus b 1, b 2 up to b m ok. And you are taking the l 1 norm of this ok; this whole thing remember you are taking the l 1 norm nothing, but sum of the magnitudes of this elements of this vector; which is basically you can write this as the sum of the magnitudes, magnitude a 1 transpose x bar minus b 1 plus magnitude; a 2 bar transpose x bar minus b 2 plus so on plus magnitude a m bar transpose x bar minus b m.

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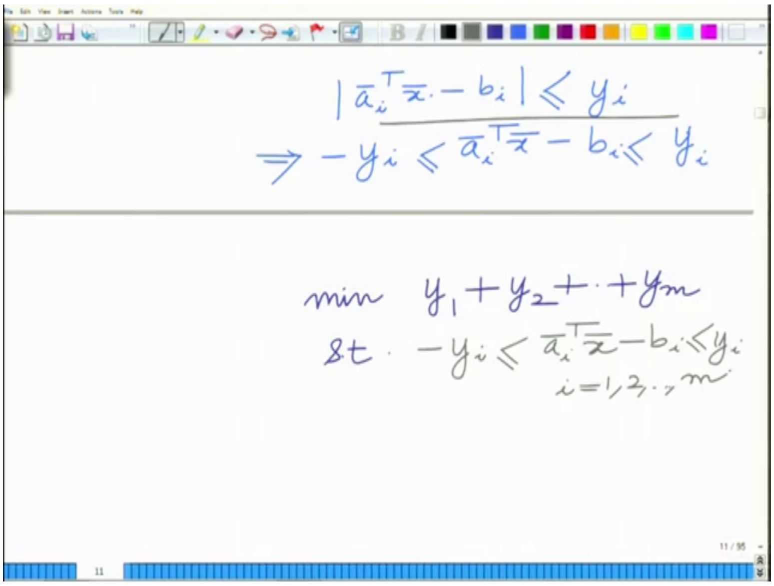

And now what I want to do; is want to introduce the constraint I want to introduce the constraint that is magnitude a i bar transpose x bar minus b i is less than or equal to y i. So, now what as this imply? This implies that minus y i is less than or equal to a i bar transpose x bar x bar minus b i which is in turn less than or equal to or which is less than or equal to y i ok.

For instance magnitude of a quantity x less than equal to y implies that x lies between minus y and y. So, that is what we review here; minus y i less than equal to a i by transpose x bar minus b i, which is less than equal to y i ok. And therefore, now you have using the now each of this quantities less than or equal to y i, so therefore, using the epigraph form you can write this as summation of y i that is into minimizing summation of magnitudes a i bar transpose x bar minus b i, I can minimize summation of y 1 plus y 2 plus up to y m.

Now subject to the constraint that now we have this constraint, what is that? Minus y i less than or equal to a i bar transpose x bar minus b i which is less then equal to y i; for i equals 1, 2 up to m this is the constraint.

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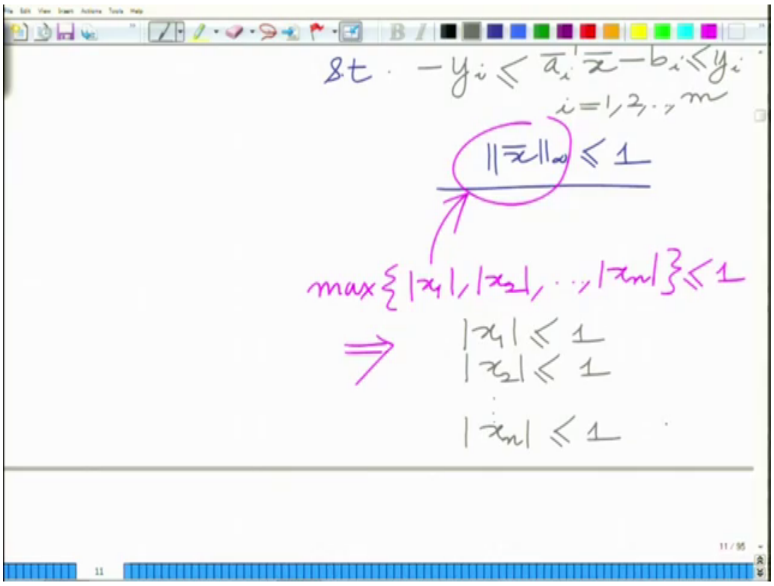

And now you still have the other constraint, the other constraint is remember remains as it is norm x bar infinity less than or equal to 1. Now we have also know that you can write this l infinity norm constraint also as a set of linear constraint. Because look at this, what is this? This is l infinity norm; what is this l infinity norm? This is the maximum of magnitude x 1 magnitude x 2 so on; magnitude of x n because remember x is an n dimensional vector a is m cross n; so, this is less then equal to n.

So, maximum of n quantities less than equal to n; that means, each of those quantities is less than or equal to that is maximum of n quantities less than or equal to something then each of those quantities must be less than equal to the same thing. So, this means that you have well magnitude of x 1 less than or equal to 1, magnitude of x 2 less than equal to 1 so on.

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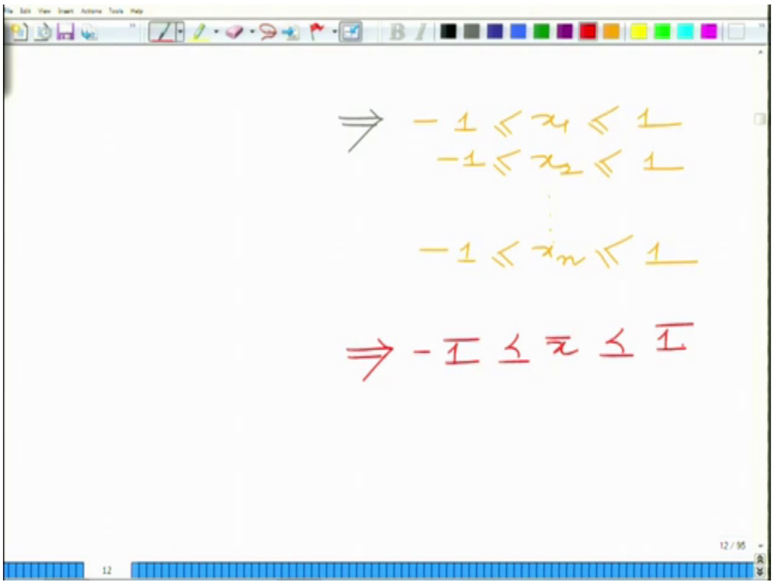

Magnitude of x n less then equal to 1; which implies that minus 1 less than equal to x 1 less than equal to 1, minus 1 less than equal to x 2 less than equal to 1 so on and so forth; minus 1 less than equal to x n less than equal to 1.

This you can write it in a compact fashion; this you can write it as minus 1 bar less than or equal to this is component wise inequality; less than or equal to x bar less than equal to 1 bar. This is a component wise that is 1 less than equal to each component minus 1 less than equal to each component x i of this vector, less than equal to 1.

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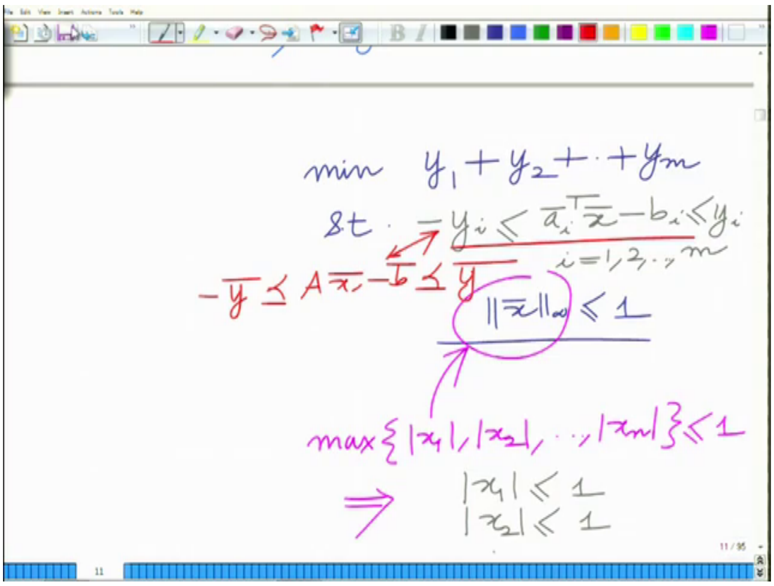

And therefore, now and further if you look at this; you can write this also as a component wise inequality. This you can write as follows, this you can write as minus y bar component wise less than equal to A x bar minus b bar which is component wise less than equal to y bar again ok; so this reduces to this equivalent representation.

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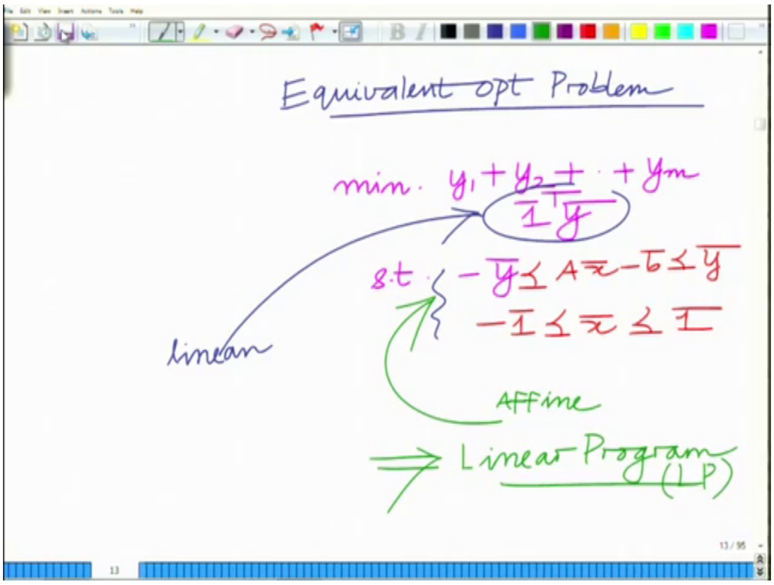

So, now you can write the equivalent optimization problem and that will be you have to minimize y 1 plus y 2 plus up to y m which is basically minimize 1 bar transpose y bar; subject to the constraint minus y bar less than equal to A x bar minus b r bar component wise less than equal to y bar; minus 1 bar component wise less than equal to x bar, component wise less than equal to 1 bar.

And you can see what you can see is this is linear, objective is linear and constraints are all also linear or other affined constraints; implies this is a linear, this is a Linear Program; this is an LP ok. So, what we have been able to show is that interesting or seemingly complicated problem which is basically minimize the l 1 norm of something; subject to the l infinity norm being lower than or less than equal to something. And be written as simple as a very intuitive and an appealing structure; it can be reduced to a simple linear program; linear objective affined constraint ok; so, that what it is.

So, it is at and linear program is much more institute because a linear program is I mean the techniques to solve a linear program and various insights into the solution of a linear program are well known alright. So, from something that is sort of difficult to understand that first impression is we are reducing it to something that can be much more easily understood and appreciate alright. And or the both these are remember equivalent form of the same convex optimization problem; the original problem is convex, the recast problem is also convex alright. But the second form is much more intuitive and appealing alright.

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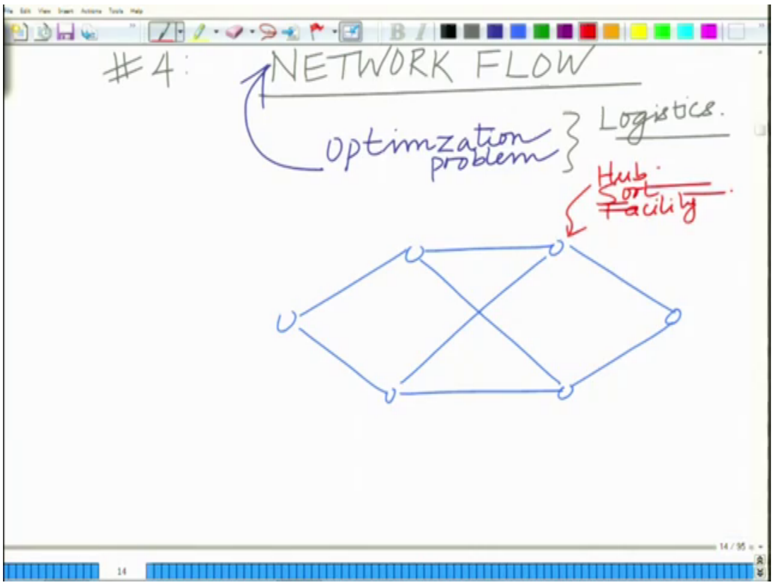

Let us look at another problem which is very interesting from practical perspective; this is termed as network flow, this is a network this at arises in several; this is network flow alright. And this arises several optimization scenarios correct; this is one of the most important kind of optimization problems that arises in various fields such as for logistics management, supply chain management etcetera alright.

So, this something that applies for instance and logistics and the problem is very simple; to consider network for instance let say we have a network of hubs ok; just drawing a simple network these are for instance what you can think of these are hubs or sort facilities. So, this is our network; so each of these nodes this is a hub or a sort facility in for instance in the distribution network; for instance you can consider distribution network of a Etailer ok.

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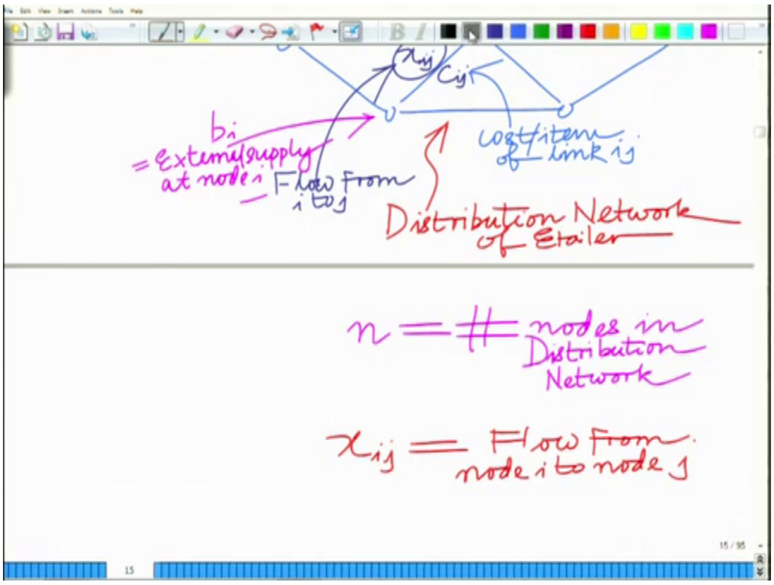

May (Refer Time: 12:41) company with E commerce and Etailing gaining lot of popularity. Remember to distribute these products you need a network of hubs or whatever sort faculties; where you have a lot of this products that bought into sorted and dispatched to other hubs and ultimately of course, delivered to the end user alright.

So, you have this network of connected hubs alright and your products entering and then the product leaving or commodities which are entering in commodities which are leaving each hub; so we have flows between these hubs. So, x ij this denotes the we can call it this as basically the flow from hub; i or load i and each flow has a cost C ij is the cost per item of link or the path between i j.

So, you have C ij which is the cost per item; x j is the number of items that represents the flow between load i and load j. So, these are basically your sort facilities or basically this is the network of sort facilities or hubs ok. Now in addition what you will have is you will have a supply correct; so you will have external supply b i. So, if you look at each hub or each load you might have some things that is supplies flows; that are coming from other loads, flows are going into other loads and in addition you might have an independent supply b i alright.

So, this b i indicates the supply that is coming into load i; if b i is positive on the other hand its b i is negative it means that that supply alright that commodities are leaving from that alright. So, b i is the external supply at node i it is positive; it is coming in, negative if it is going out ok; this is external supply at node i ok. So let us say n equals the number of lodes in the distribution network; in this distribution network it is a very elegant problem n equals number of nodes in the distribution network; we have already seen x i j which is the flow from node i to node j let me just define that again.

Because that is an important quantity flow from node i to node j; we have already seen c i j is the cost from cost per item form node i into node j. And let us denote the upper bound and lower bound of the because remember each sort facility have a certain limit in the flows corresponding to, but because there might be certain links or certain parts right; which are where are restrictions in terms of the a in terms of the flow the total outflow in flow products alright.

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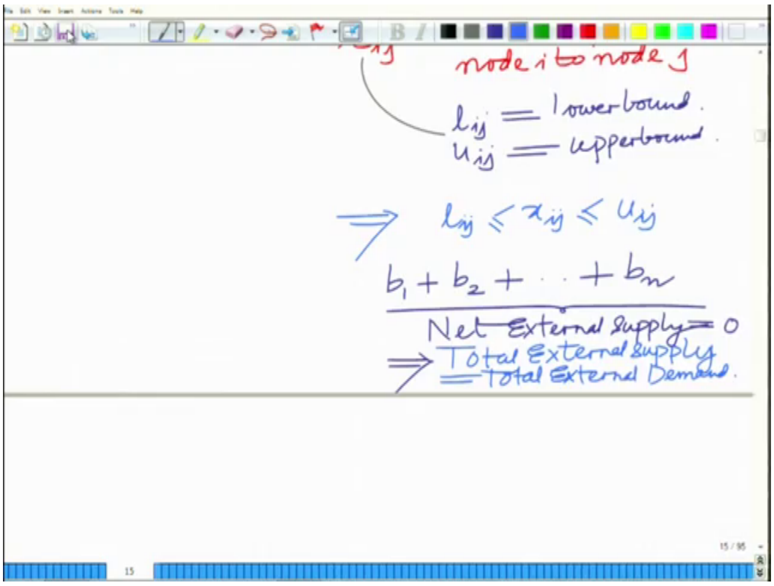

So, we want to have this flow corresponding to this flow we have the lower bound x ij; l ij equal to lower bound and u ij equals upper bound ok; which basically implies its very simple which basically implies that each flow x ij has to be between l ij and u ij ok.

So, this is your lower bound and u ij is your upper bound ok. And we will enforce another condition that is if we look at the total external flow b 1 plus b 2 for all nodes. The total external supply equal to total external this is the total flow that is total external flow or you can think of this as net external supply equal to 0, which implies that total external supply equals total external demand; that is what it means is very simple its very intuitive that is this products of commodities that are entering at some node are eventually leaving at some other nodes.

So, it cannot happen that this commodities are entering; a large number of commodities entering and only few commodities are leaving, which means that this commodities are getting lost or it does not mean that only some commodities are entering a large number of commodities are leaving which means commodities are some somehow being magically generated alright.

So, just means that commodities are entering and whatever commodities are entering at the various hubs are eventually leaving the network at possibly the same or different hubs alright that depends on the flow ok. So, the net external supply that is if you look at b 1 b 2 b n some of the bi's are positive some of the bi's are negative. Because at some point we have supplied entering some point we have the supply leaving alright. So, the net external supply alright must be 0 which means the total external supply equals the total external demand.

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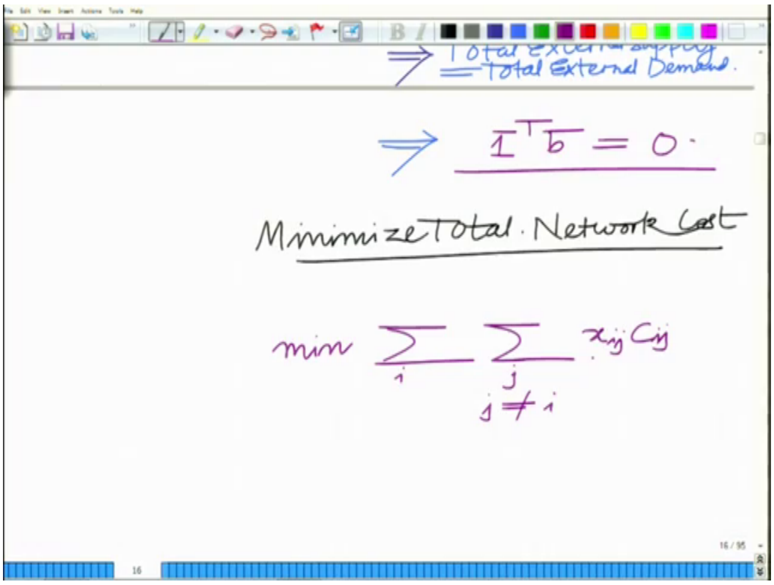

Which also means that if you represent this sufficiently; this means that 1 bar transpose if you look at this vector of supply vector 1 bar trans b bar equals 0. Now what we want to do is we want to formulate this network flow problem; which is minimized the total cost of the network.

And it is very simple; now what is the total network cost? Total network cost is if you look at all links i, j alright and j not equal to i; if you look at the flow x ij and the cost per item c ij; this is your objective alright this is the total cost total cost.

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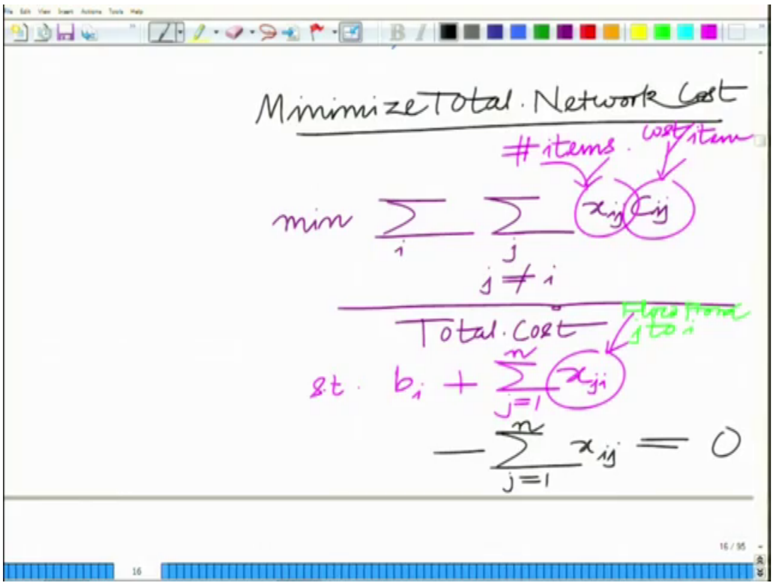

Because if you look at this; this is your cost per item and this you can think of as number of items ok. So, this is basically the total cost of running your distribution network subject to the constraint, now this is the important subjective constraint if you look at each node b i which is the total external supply plus the total inflow; if you look at each node i j equal to 1 to n, x; ji that is supply from, that is flow from node j to i.

Now, flow from j to i minus the flow from node i to all nodes j this must be equal to; this is the important constraint. Basically says that total external supplied each hub and the total flow from all the other hubs to a particular node i must equal to total flow of commodities or goods from node i to the other nodes or a just makes this enforcing the constraint that none of the goods are either being are not unaccounted for at hub i; alright they are goods; that goods are not being generated or goods at not disappearing.

So, the external supply plus the flow from all the nodes right must be equal to the external flows; some of all external flows from any part. And this must hold for all particular loads or all particular hubs in your or all particular sort facilities in your distribution unit.

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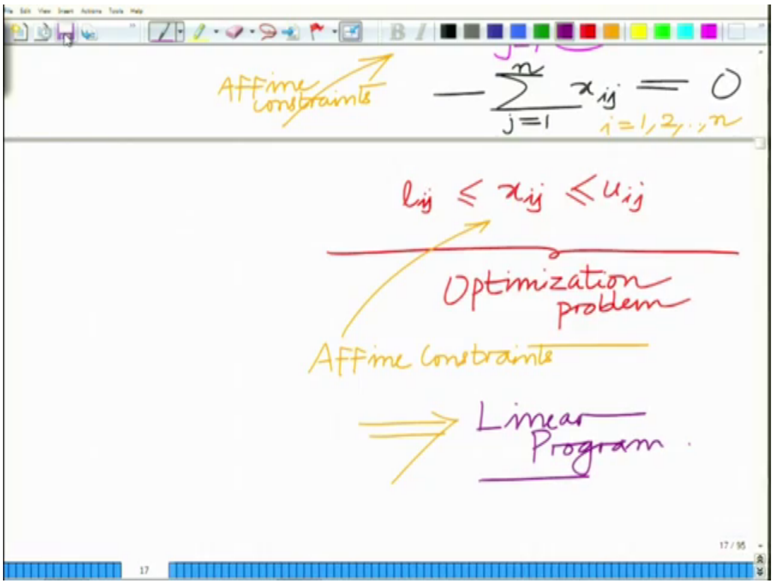

And we have the bounds that we have already seen l ij less than or equal to x i j less than equal to u ij alright; that is your constraint. And of course so this is your optimization problem and if you look at this; this is your optimization problem, this is your optimization this is your optimization problem.

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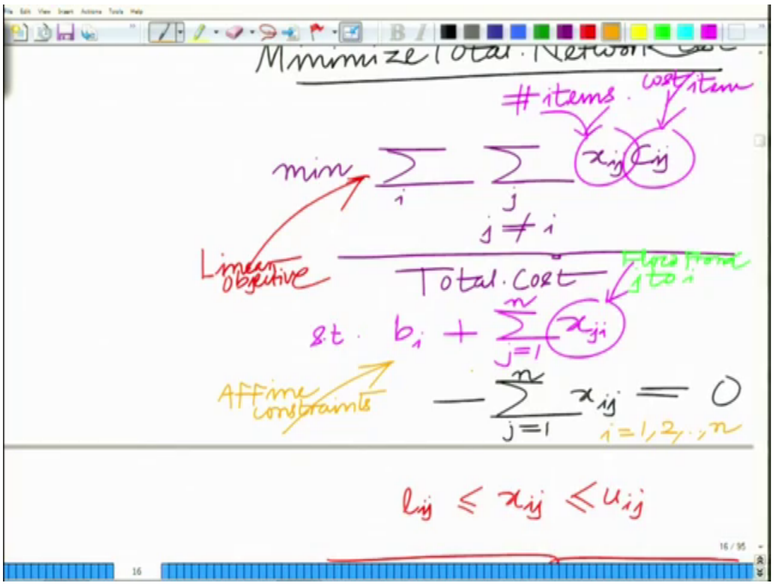

And you can see again linear objective; linear objective of course, this must hold for all i ok; this is basically linear or affined constraints. Again affined constraints; implies this is also and very interestingly and you can see the practical nature of this problem, we are looking at a large distribution and even easily extend it to any number of because the statement is general.

So, you can have a large distribution network which tens of thousands of hubs; what this says is as long as total external supplies is equal to the total external demand, which generally holds alright and you have no the links the cost associated with each link which is varying in a linear fashion alright with the number of items. Even formulate this is a linear program and there would this complex problem minimize of the cost of a supplied chain or cost of a distribution network can be formulated as a linear program.

And therefore, convex optimization has several practical applications; in fact, this problem itself can be applied in a practical context to minimize the cost associated with the network alright. So, this is the very practical problem and there are several such problems which are severe or significant practical relevance alright. So, we will stop here and continue in the subsequent modules.

Thank you very much.