Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 70
Examples: Linear objective with box constraints, Linear Programming

Hello, welcome to another module in this massive open online course. So, we are looking at example problems in convex optimization. Let us continue our discussion right.

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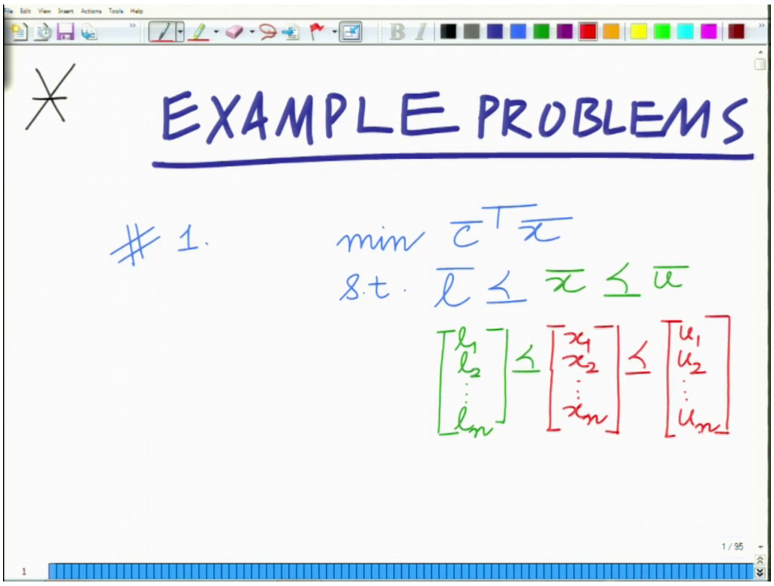

So, we are looking at example problems and let us look at another problem, problem number 1. So, we want to look at meaning this is a simple problem minimize c bar transpose x bar subject to the constraint that l bar is component wise a given vector l bar is component wise less than equal to x bar; which is less than equal to another vector u bar. That is which means that if you have the elements of this vector let us say l is x bar is an n dimensional vector then we have l 1, l 2 up to l n is component wise less than or equal to your vector x 1, x 2 up to x n which is again component wise less than or equal to u 1, u 2 up to u n.

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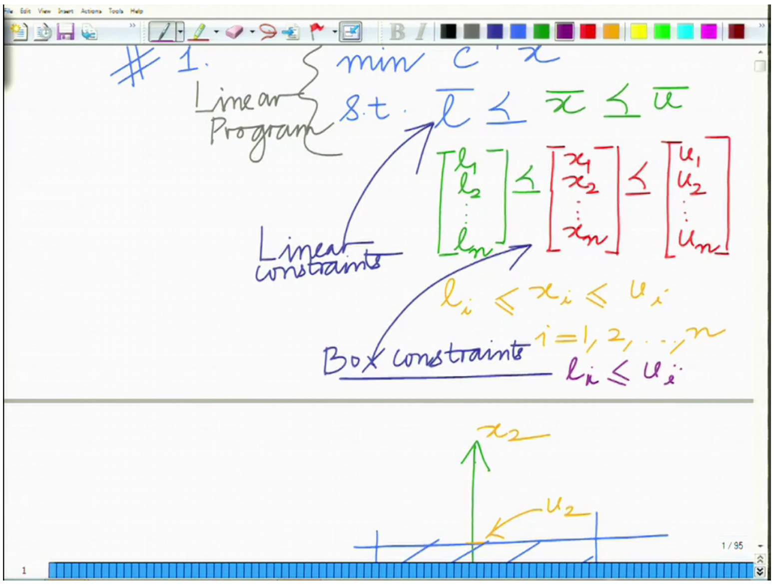

That means each l i has to be less than or equal to x i has to be less than equal to or has to be less than or equal to x i has to be less than equal to these are not there is a single scalar quantity.

So, this is just your normal equality and this has to hold for all i equals 1, 2 up to n and this is also known as box constraints; this constraint these constraints. In fact, a set of n constraints, in a 2 in constraint these are also known as box constraints because, if you look at the 2 dimensional scenario.

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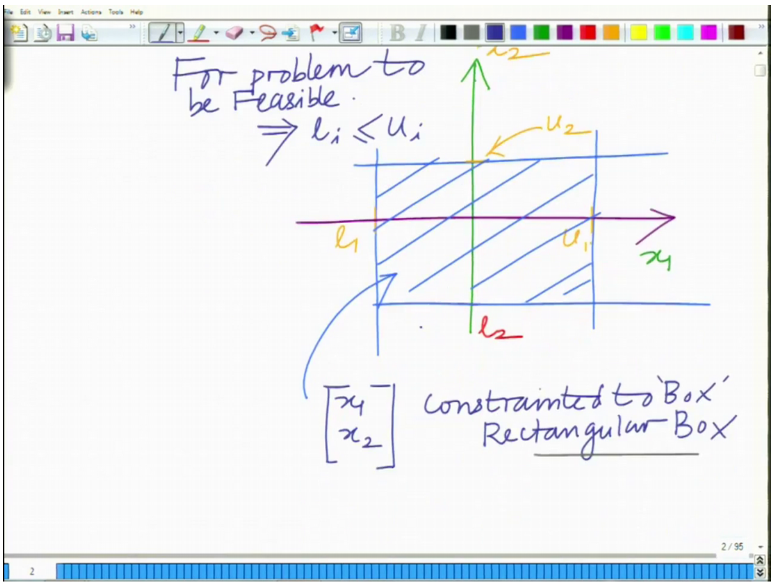

That is if you look at a 2 D plane what we will have is that for l 1 x 1 that is if you look at x 1 and x 2 this will mean that x 1 is less than or equal to u 1 u 1 and x 2 is greater than equal to x 1 is greater than or equal to l 1 l 1 and for the same matter x 2 is less than or equal to u 2 and x 2 is greater than equal to l 2. So, therefore, x 1 x 2 are confined to this box that is greater than equal to l 2 less than equal to u 2 greater than equal to l 1 x 1 greater equal to l 1 less than equal to u 1 ok.

So, they have constrained to this box. So, x 1 that is your vector x 1 x 2 is constrained to this box. Hence, this is also termed as that is in fact the rectangular box correct this is your rectangular box hence is also termed as box type constraint. In fact, the simple optimization problem this can be solved as follows and is a simple. In fact, if you look at it this is simply a linear objective this. In fact, is a linear program alright, it is a simple linear program you have the objective function is linear and the constraints are also linear correct.

So, this is in fact, if you look at this is a simple linear program this is nothing, but if you remember a linear program is nothing but linear objective and linear constraints. And, the solution for this is fairly straight forward it is a simple example.

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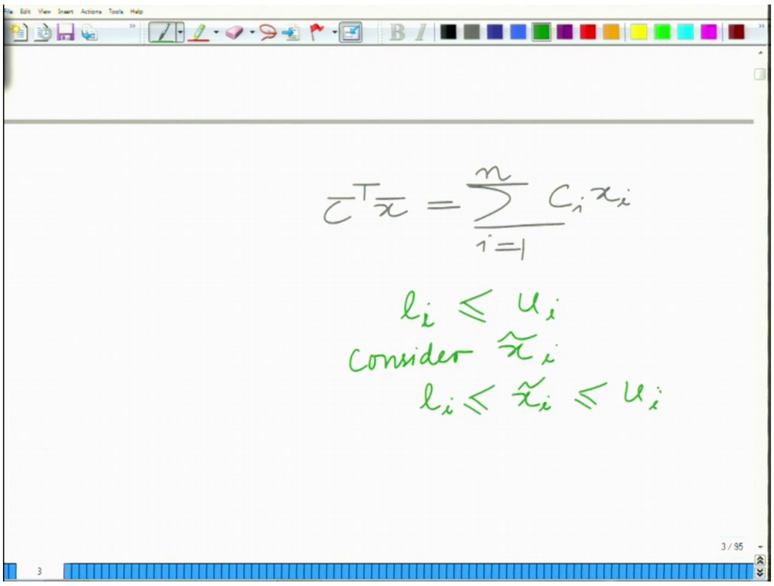

So, if you look at the object you have c bar transpose x bar which is basically i equal to 1 to n summation c i x i. Now, observe this box type constraints make sense only if l i is less than or equal to each u i.

So, we assume here that is l i is less than or equal to u i. If l i is not less than or equal to u i then the constraint no x i can satisfy, x i greater than equal to l i, but less than equal to u i, alright which means the problem becomes infeasible all right. So, problem is feasible only if l i that is each lower bound l i is less than equal to the upper bound u i for each variable x.

So l i so, problem to be feasible; this implies l i is less than equal to u i, otherwise there is no point x which satisfies the constraints no. So, we have l i is less than equal to u i. Now, consider any x tilde i such that l i is less than equal to x tilde i less than equal to u i that is any x tilde which is lying within this interval.

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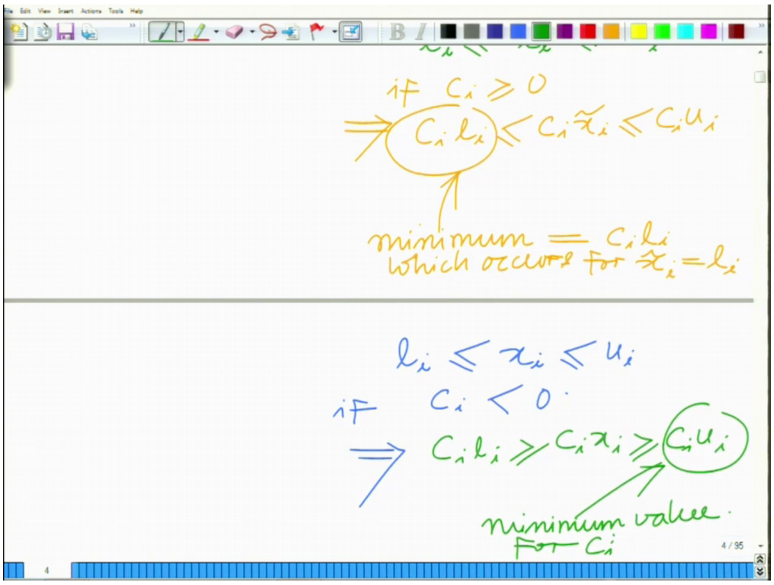

Now, therefore, now this implies something very interesting if c i greater equal to 0, this implies c i inequality remains c i less than equal to c i x tilde i less than equal to c i u i ok.

So, minimum value if x if c i is greater than equal to 0, minimum value of over for x tilde lying in this box is c i l i which occurs when x tilde equals l i which has the (Refer Time: 08:02) x tilde equals l i or x i equals l i. That is the minimum over this interval in this box occurs for l i if c is greater than equal to 0. On the other hand if c i is less than or equal to 0.

Now, now again observe that l i is less than or equal to u i, if that is for any x i which implies if c i is less than 0 in this implies now inequalities get reversed this implies that c i l i greater than equal to c i xi any xi in the centre it is greater than equal to c i u i which means, this is the minimum value, minimum value if c i less than 0 ok.

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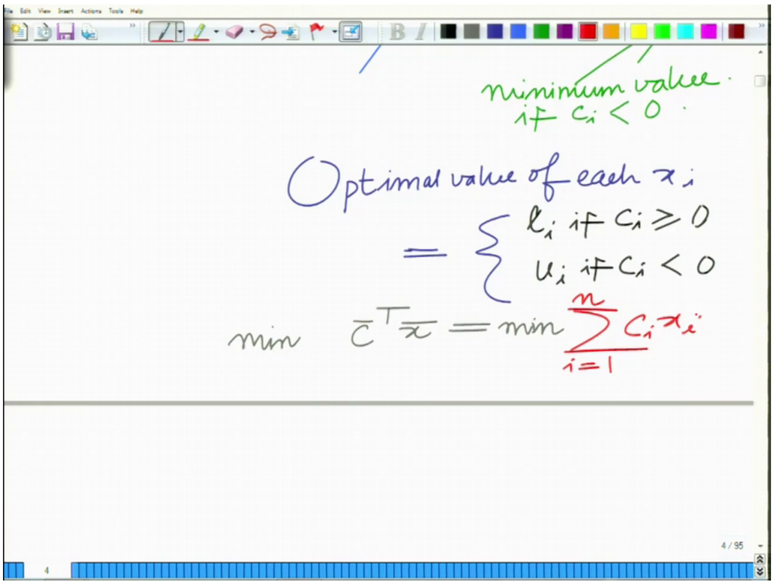

And therefore, the optimal value of each xi now it is clear. Optimal value each xi this is equal to well what is this equal to this is equal to l i if the corresponding c i is greater than equal to 0 and this is equal to u i, if c i is less than 0. And therefore, the minimum up of c bar transpose x bar which is equal to the minimum of summation i equals 1 to n c i xi.

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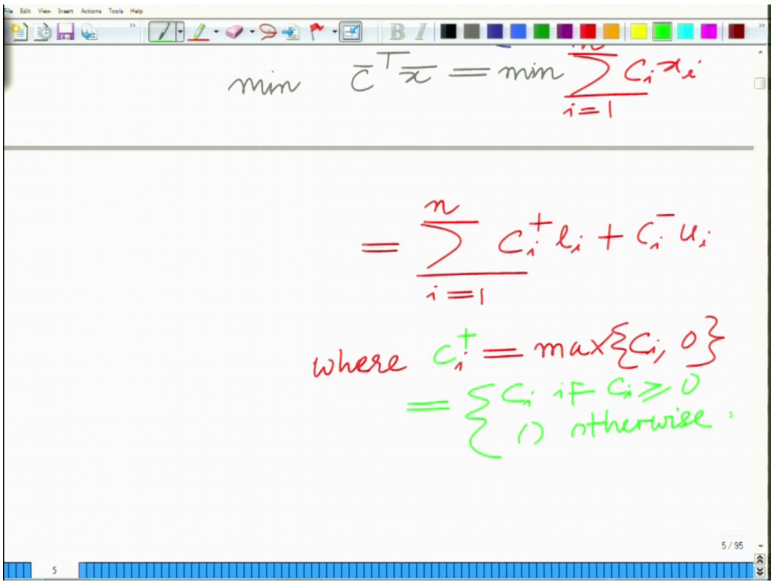

This is equal to summation you can also write it as summation i equal to 1 to n c i plus times l i plus c i minus times u i, where c i plus you can also write this as maximum of c i comma 0 which is equal to I am sorry this is not c i bar the c i plus this is equal to c i if c i greater than equal to 0 and 0 otherwise.

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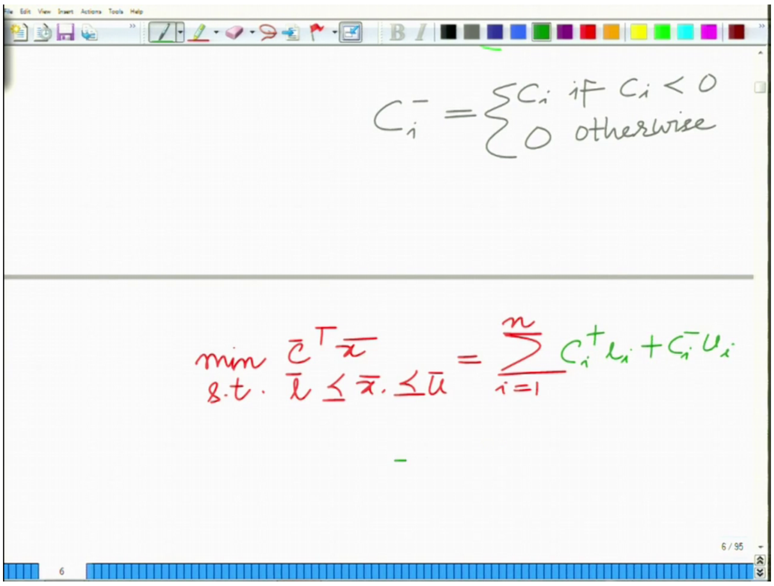

On the other hand, similarly c i minus c i minus equals c i if c i less than equal to 0 0 otherwise and you can also write it in a compact fashion as follows; you can also write this as therefore, minimum c bar transpose x bar subject to your constraint that is the box constraints l bar component wise less than or equal to x bar component wise less than equal to u bar. This is equal to summation i equals 1 to n c i plus l i plus c i minus u i which is equal to you can write this as l bar transpose c bar plus plus u bar transpose c bar minus where c bar plus contains all this contains all positive elements of c bar or contains only non negative.

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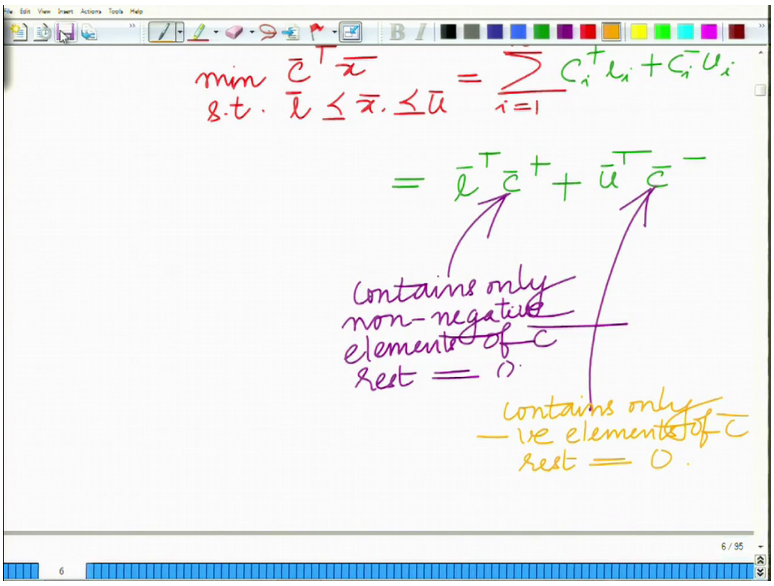

The rest are 0 and this also contains only negative elements of c bar c bar minus rest of 0. So, that is the optimal value of this problem. So, you can say it c bar transpose x bar the minimal value is c i plus into l i plus c i minus into u i; that is where c i plus equals c i if c i is greater than equal to 0 and 0 otherwise ca minus equals c i if c i is less than 0 and 0 all right.

So, this is although the idea is very simple. It shows you formally how to come up with the solution of a simple objective convex optimization problem like this one which is a linear program alright. Let us proceed to a slightly more sophisticated example which the solution to which might not be very obvious and that is the following.

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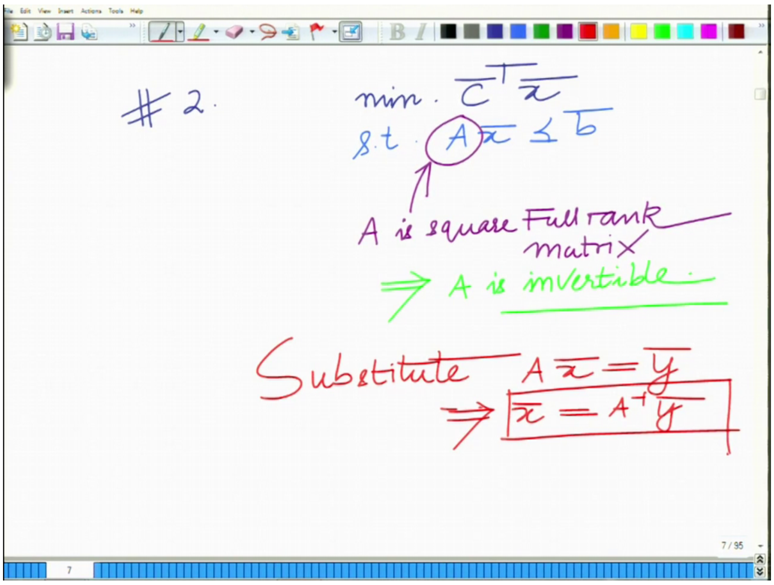

Say problem number 2; we want to minimize the following again, linear objectives what transpose x bar subject to the constraint A x bar now this is a is not box wise there is a component wise inequality A x bar component wise less than equal to b bar ok, you can see objective is linear, constraints are linear. Therefore it is a linear program, but slightly more sophisticated and it depends on the solution, now very much depends on the nature of A to make the problem simple.

We will say A is A is a square full rank matrix. This implies that A is invertible ok; A is a square full rank matrix which implies that A is invertible and we need to solve this optimization problem. For this what we do is we substitute x bar equals y bar. So, we will set A x bar equals y bar which implies that we will convert this into an optimization problem in terms of y ok. We are introducing this new variable y. Now, since A is an invertible matrix, there is a 1 to 1 correspondence between x bar and y bar. So, which means x bar equals A inverse y bar ok. So, from x bar 1 can find y bar from y bar 1 can find x bar. Since, A is invertible there is a 1 to 1 correspondence ok.

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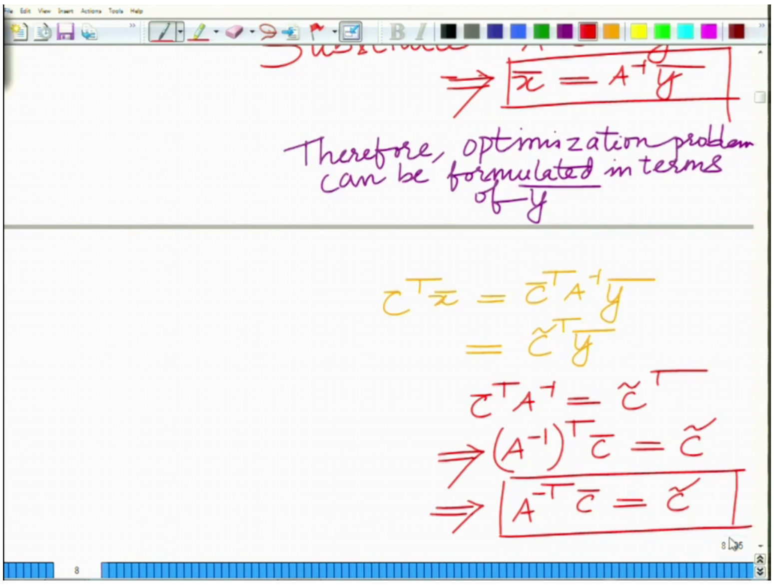

Now, we will write the equivalent optimization problem in terms of y bar. Therefore, optimization problem can be formulated in terms of y bar. So, we have the objective c bar transpose x bar which is c bar transpose A inverse y bar and I can formulate this as c tilde transpose y bar, where, c bar transpose inverse equals c tilde transpose which implies taking the transpose on both sides A inverse transpose c bar equals c tilde.

So, I combined the inverse and transpose I am simply going to write A minus A inverse transpose which means, A transpose inverse or inverse transpose both of these are the same thing A inverse transpose c bar equals c tilde ok.

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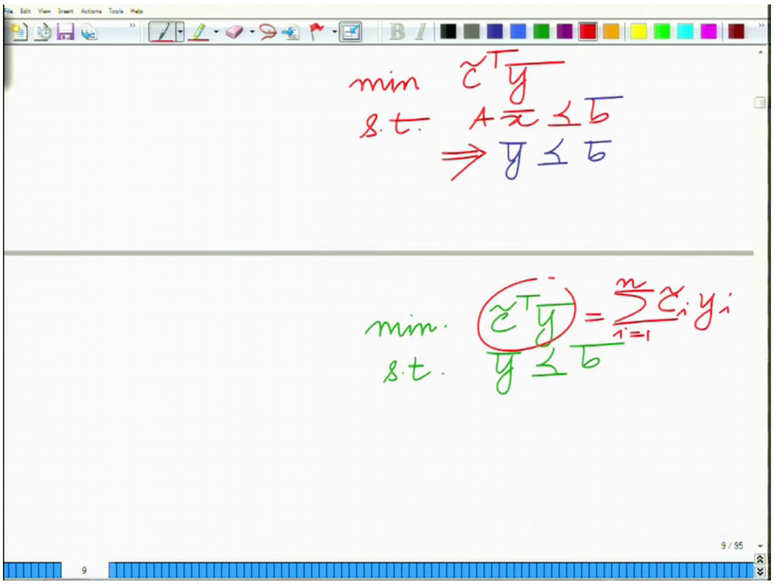

So, the objective becomes minimize c tilde transpose y bar subject to the constraints. Subject to the constraint A x bar component wise less than equal to b bar, but A x bar equals y bar. So, this constraint will be y bar is component wise less than or equal to b bar. So, now, we have a very nice we have a much more insightful and simpler optimization problem. Minimize c tilde transpose y bar in terms of y bar in terms of the new variable y bar subject to the constraint y bar is component wise less than or equal to b bar and this will basically be summation i equals 1 to n c tilde i y i ok.

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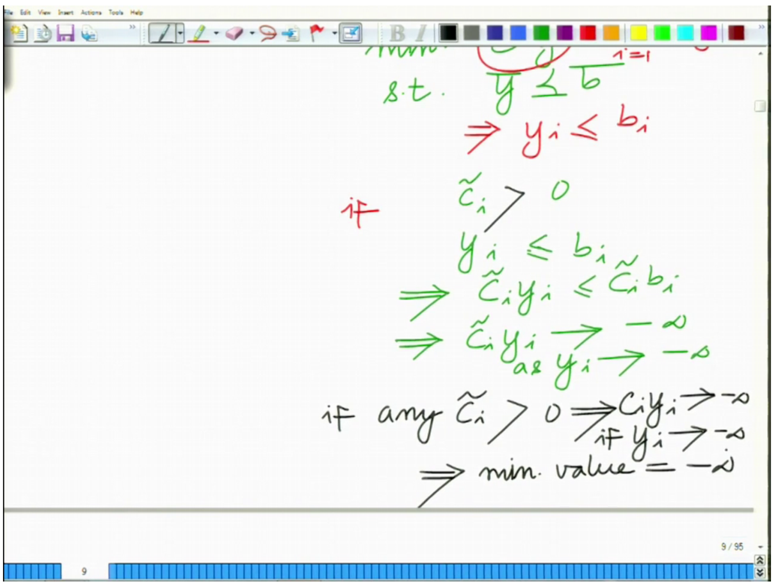

And the constraint will be component wise constraint this implies that each component of this vector y y i is less than or equal to each component of vector b that is b i ok. Now, it is easy to see what the solution of this optimization problem is now. If once again you go back to the now one can see what the we have reduced it to an equivalent form from which it is simpler to infer the solution. Now, you can again rely on the principles we have seen previously if c i tilde.

Now, see if c i tilde is greater than equal to 0, we have y i less than or equal to bi which implies c i y i less than or equal to c i bi which implies c i y i or rather c i tilde, I am sorry c i tilde say i tilde y tends to minus infinity as y i tends to minus infinity ok. So, implies that I can make the objective right by tending that particular y i. So, if any c i tilde is greater than equal to 0, I can take that y i to minus infinity objective becomes minus infinity. So, it is unbounded below ok.

So, if any if any c i tilde greater than equal to 0 implies or to be more specific, if any c i tilde is greater than 0, there are not greater than equal to 0. If any tilde is any c i tilde is greater than equal to 0 implies c i y i tends to minus infinity, if y i tends to minus infinity; in implies minimum value equals minus infinity.

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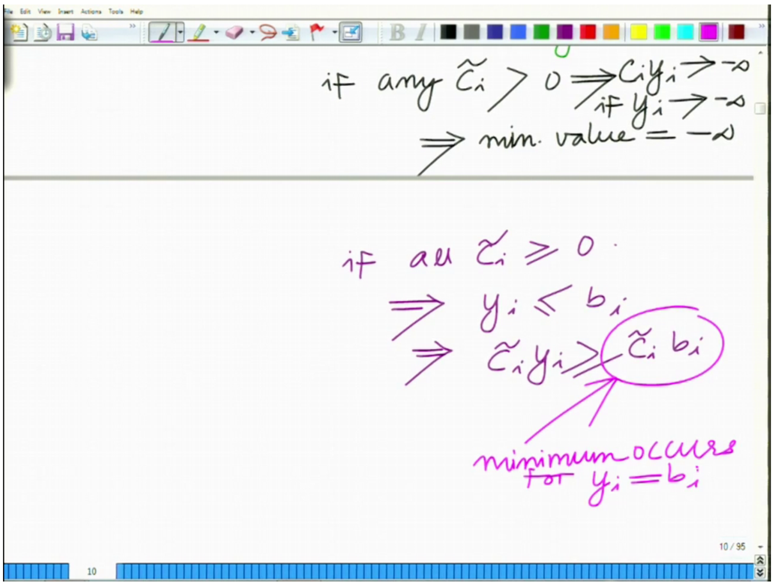

Now, if all c i tilde are less than or equal to 0, if all c i tilde; now you observe that yi less than or equal to bi implies c i because c i tilde is negative c i yi c i tilde y is greater than or equal to c i tilde b i. So, the minimum occurs for c i tilde bi.

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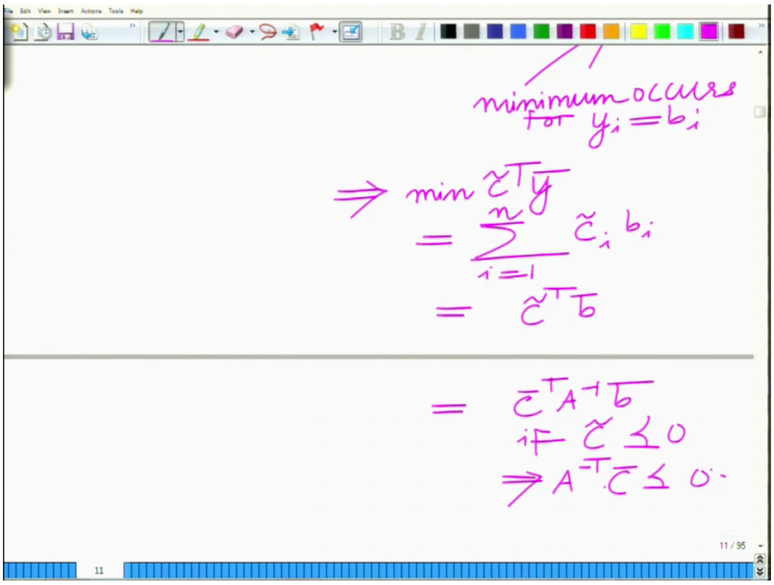

And therefore, the met minimum implies minimum c bar transpose or rather c tilde transpose y bar equals well, summation i equals 1 to n c i into bi which is equal to c tilde bar transpose b bar and we know what is c tilde transpose d tilde transpose is c bar transpose A inverse b bar, but this is only if each c i tilde is component wise less than 0, a component wise less than 0 or component wise less than or equal to 0, component wise less than or equal to 0 which implies c tilde equals A inverse transpose c bar is component wise less than equal to 0.

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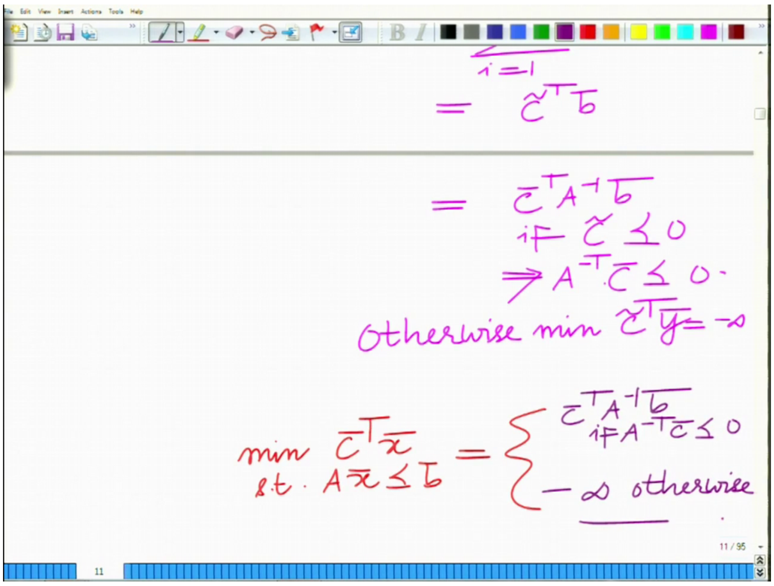

And otherwise minimum c tilde transpose y bar equals minus infinity because if any component of c tilde that is any component of A inverse transpose c bar is greater than equal to 0 greater than 0 corresponding y i can tend to minus infinity and objectivity to minus infinity. Therefore, the minimum value, therefore, now summarizing this minimum value of this is equal to c bar transpose inverse b bar if A inverse transpose c bar is component wise less than equal to 0; this is equal to minus infinity otherwise.

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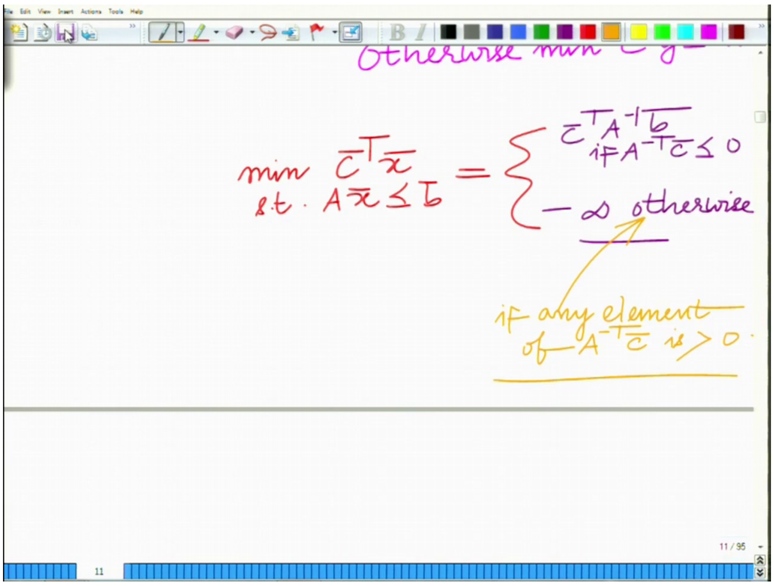

That is otherwise means, if any element if any element of A inverse transpose c bar is if any element of A bar inverse transpose c bar is greater than 0 alright. So, that is basically the solution to this optimization, what optimization problem. What is interesting about this? This is very insightful what is a seemingly complex optimization problem, can be given I mean the solution to the seemingly complex optimization problem can be found in a very elegant fashion and it yields a lot of important insights alright.

So, these examples hopefully helped you better understand the different aspects of different facets of convex optimization problems, how to solve them and the valuable insights that they are alright. So, let us stop here.

Thank you very much.