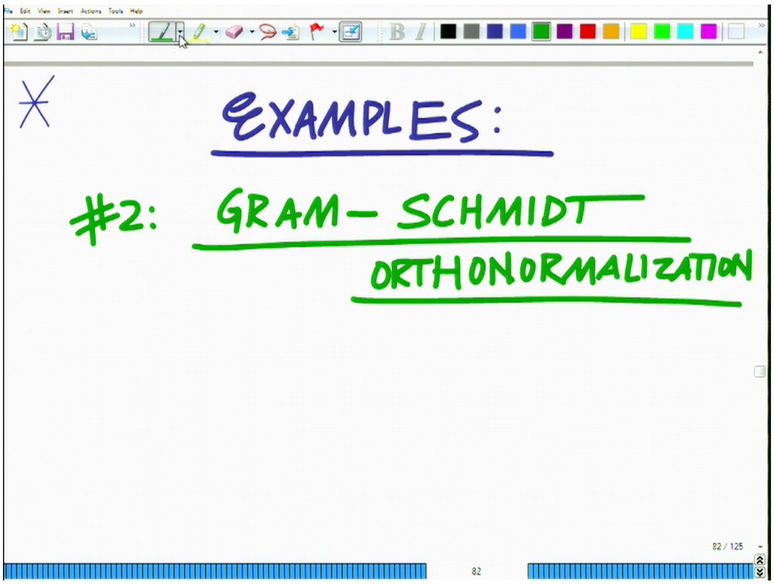
Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 07
Gram Schmidt Orthogonalization Procedure

Hello, welcome to another module in this massive online open course. So, let us we are looking at examples to understand the mathematical preliminaries of optimization. Let us look at another example and this is something that is very important and has a lot of practical utility, this is termed as the Gram Schmidt Orthonormalization Process ok.

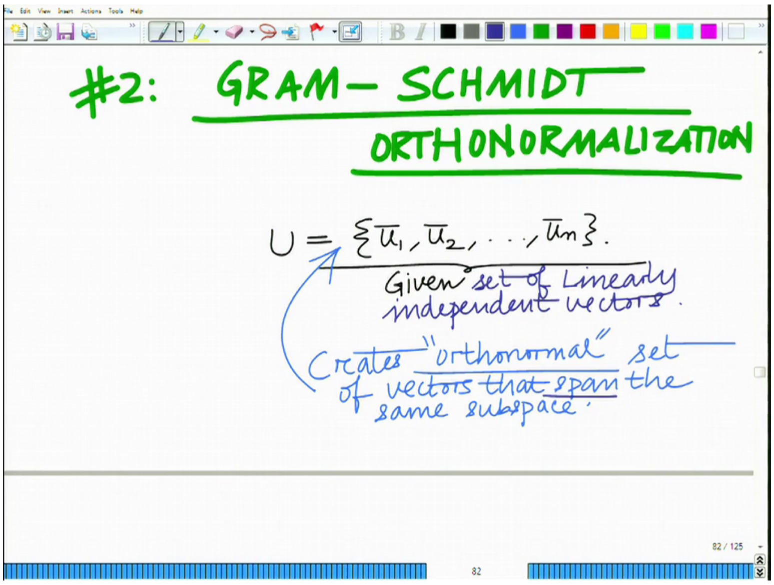
So, we are looking at examples correct?

(Refer Slide Time: 00:39)



And as part of the second example; what we want to look at is the Gram Schmidt procedure for Orthonormalization, Gram Schmidt procedure for Orthonormalization.

(Refer Slide Time: 01:20)

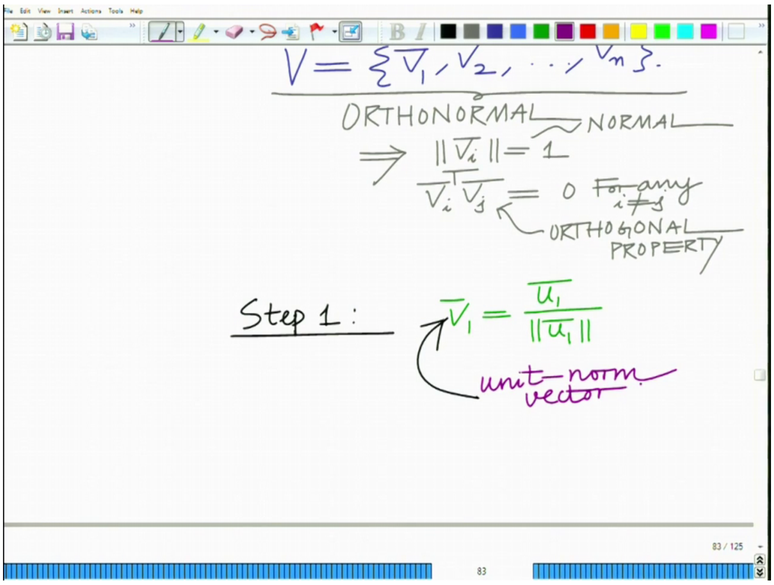


And what this does is for a given set of linearly independent vectors; u 1 bar, u 2 bar un bar. So, this is a set of linearly given set of linearly independent vectors. What the Gram Schmidt Orthonormalization procedure does; is it creates an orthonormal set creates, I am just going to explain in a moment what this means it is it creates an orthonormal set of vectors that span the same surfaces span the same subspace in the sense that; linear combinations of this vectors can be used to generate any vector in that subspace all right.

So, both these subspaces spared by these sets are the same. And what is the meaning of this term orthonormal? An orthonormal set of vectors is a set in which each vectors has unit norm that is, the normal right. We said the process of making the norm of vector unity is normalization. And orthogonal represents the fact that the vectors in this set all the vectors in this set are pair wise orthogonal to each other.

So, that makes it an orthonormal set of vectors.

(Refer Slide Time: 03:22)

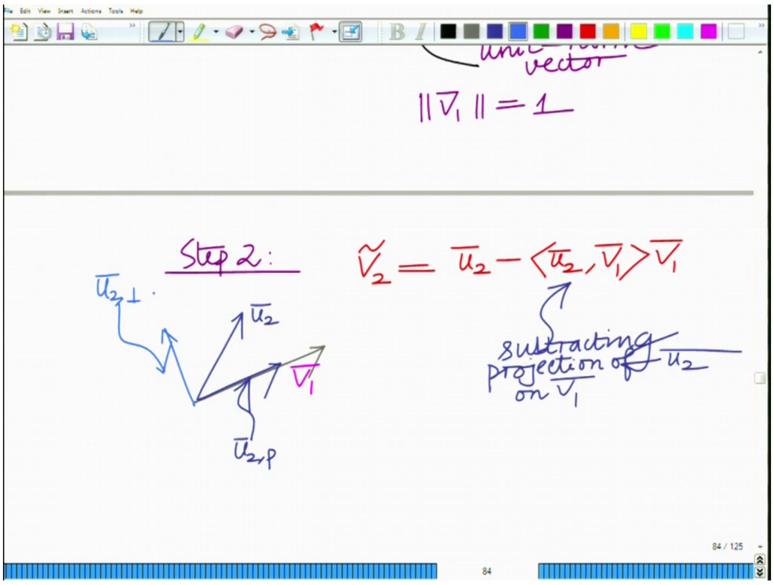


So, we have an orthonormal bases, which is frequently very convenient to consider for as the basis of a vectors space. So, V 1 bar V 2 bar V n bar. Now this is orthonormal implies; norm V i bar each vector norm V i bar equals to 1 this is basically your normal property. And if you look at any pairs of vectors V i bar or the real vectors V i bar transpose, V j bar equal to 0 if i naught or for all i naught equal to j, for any i naught equal to j all right. And this basically represents the orthogonal property.

So, they are orthonormal in the sense; all the vectors in the set are orthogonal to each other and each vector has unit norm all right and they span the same subspace. So, how does this procedure work? Well the procedure works a various steps the Gram Schmidt Orthonormalization procedure and that can be described as the follows ok.

So, what we do is; we start with the first vector is the first step. You can think of this as step one, what we do is we create a unit norm vector that is; we create first V 1 bar equals well u 1 bar divide. So, in each step we create a set of orthonormal vectors. So, the first step is we create a vector V 1 bar equals u 1 bar divided by norm u 1 bar. So, you can observe that this is unit norm because in fact, what we are doing is we are normalizing vector u 1 bar by it is norm so this is unit norm ok.

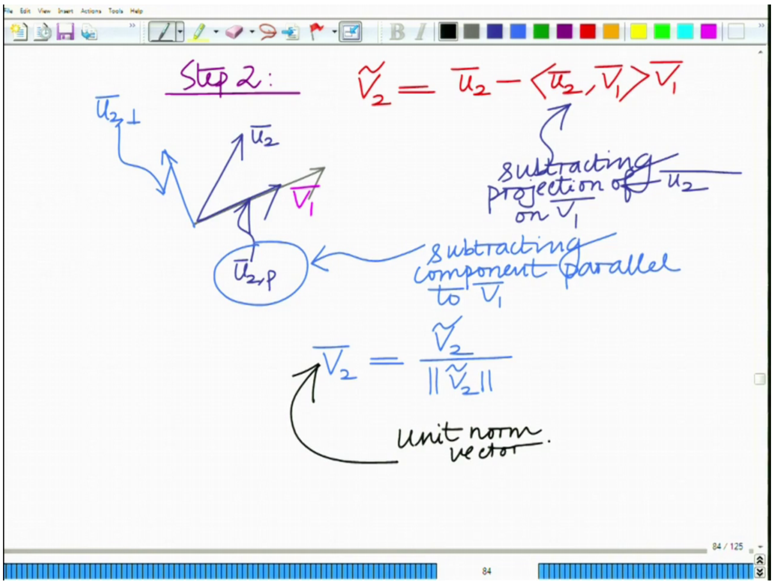
(Refer Slide Time: 05:41)



So, that is implies norm V 1 bar equals one so far satisfies the criterion for Gram Schmidt Orthonormalization. Now in step 2 what we do is; we want to create a vector that is orthogonal to V 1 bar remember. So, we look at V 2 tilde equals u 2 bar minus, what we do is we will subtract the projection of u 2 bar on V 1 bar that is what we are doing.

And so what we are doing here is we are subtracting projection of u 2 bar on V 1 bar that is we have these 2 vectors remember, let us say this is V 1 bar and now you have this vector u 2 bar and this can be represented as the sum of 2 components, one is the projection, which we can term as the parallel component u 2 bar the parallel component. You can write as u 2 bar P and the other is the perpendicular component and this can be written as u 2 bar perp.

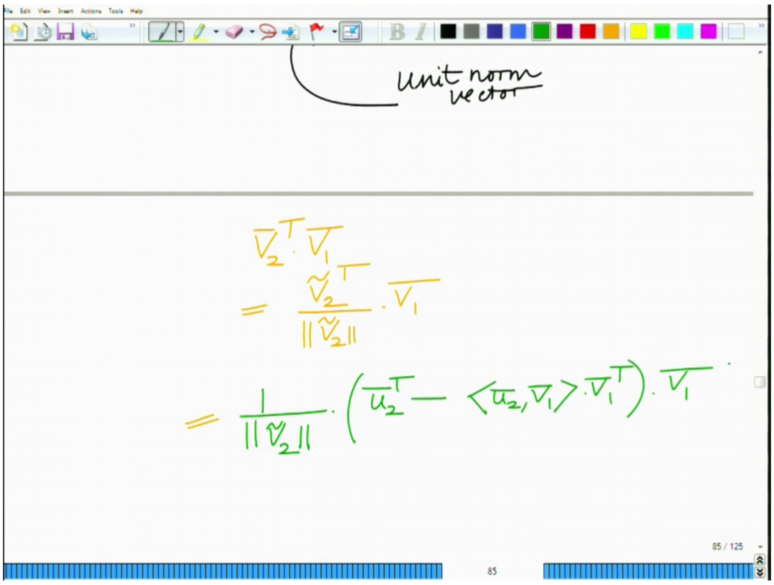
(Refer Slide Time: 07:40)



So, what we are doing is, we are subtracting this component along V 1; to create something that is parallel to V 1 bar. So, subtracting and retaining whatever is perpendicular to V 1, but that creates basically to the orthogonality property ok. And that is what the Gram Schmidt Orthonormalization procedure is achieving and of course, now we have to ensure unit norm. So now, V 2 bar equals V 2 tilde divided by norm of V 2 tilde, and this makes it unit norm.

And you can see orthogonality as follows consider just a quick demonstration of orthogonality.

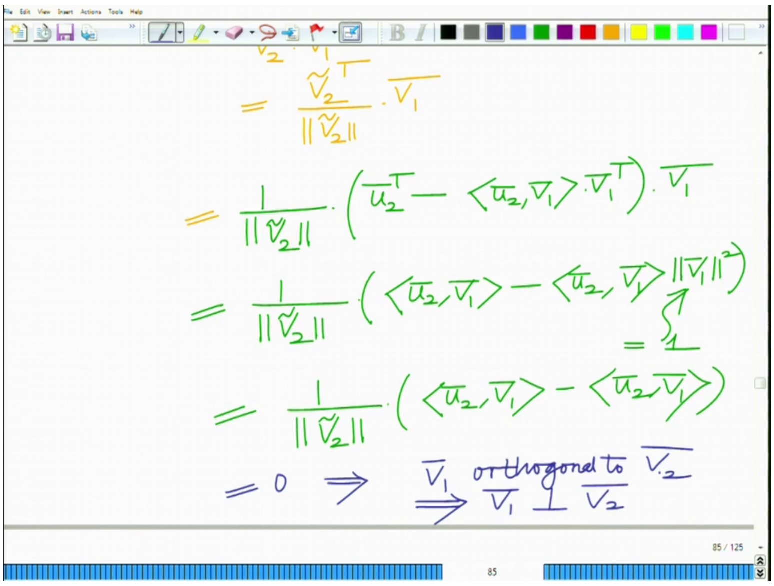
(Refer Slide Time: 08:45)



Consider V 2 bar we have already seen V 2 bar is unit norm V 2 bar transpose is V 1 bar equals V 2 tilde transpose divided by norm V 2 tilde times V 1 bar, because V 2 bar equals V 2 tilde divided by norm V 2 tilde something that we have already just seen.

Now, this is 1 over norm V 2 tilde times, well V 2 tilde is nothing but what we have seen. V 2 tilde transpose is well as we just seen that is u 2 transpose u 2 bar transpose minus u 2 bar V 1 bar inner product into V 1 bar transpose times V 1 bar.

(Refer Slide Time: 09:41)

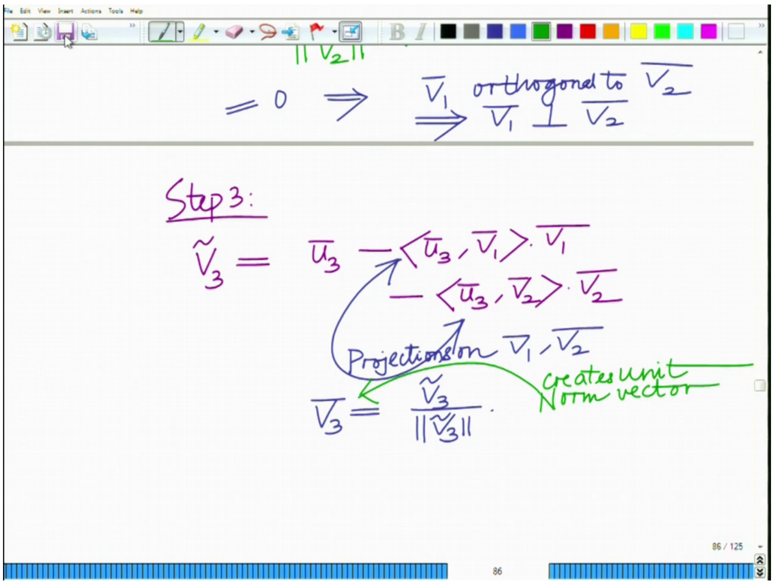


Now, if you expand this what you obtain is; this is known of V 2 tilde u 2 bar transpose. V 1 bar that is the inner product of u 2 bar comma V 1 bar minus the inner product of u 2 bar comma V 1 bar times V 1 bar transpose V 1 bar is norm V 1 bar square, but this is nothing but this is equal to 1. Since this is equal to 1 over norm V 2 tilde times u 2 bar V 1 bar minus u 2 bar V 1 bar, which you can now see is basically nothing but is equal to 0 implies; V 1 bar orthogonal to V 2 bar.

And this can also be represented as V 1 bar perpendicular to V 2 bar, because remember we said the cosine of the angle between these 2 vectors is related to the inner product. If the inner product is 0 angle, which means the cosine of the angle is 0 all right, which means the angle is theta is 90 degrees and therefore, the vectors are perpendicular to each other ok.

And now so, basically now we have created V 1 bar V 2 bar. So, basically at every step we are creating a set of orthonormal vectors. Now expect 3 you can clearly see how we can generate.

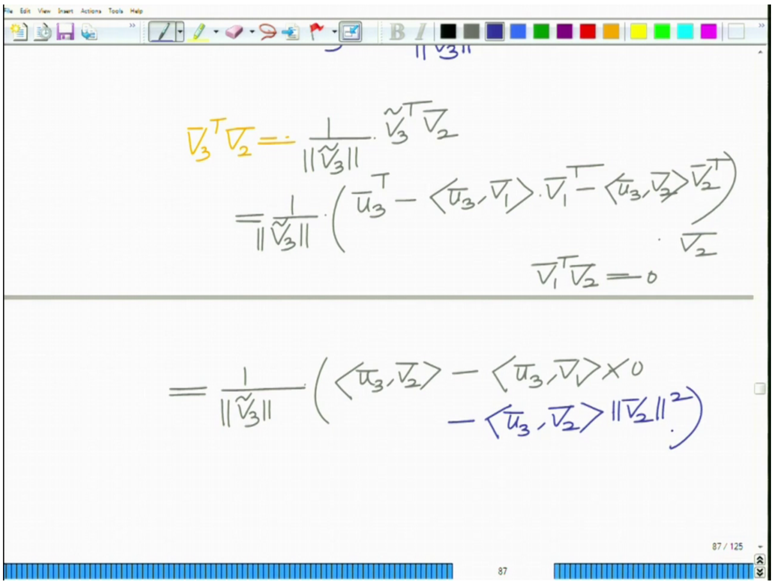
(Refer Slide Time: 11:14)



We have V 3 tilde equals well u 3 bar minus, remove the projection of u 3 bar on V 1 bar minus, remove the projection of u 3 bar on V 2 bar. So, this inner product is basically giving the projection of the unit norm vector. Inner product with the unit norm vector just the projection ok. So, these are basically projections on V 1 bar comma V 2 bar and these are being subtracted. And now we create a unit norm vector by V 3 bar equals V 3 tilde divided by norm V 3 tilde.

So, what this does is this creates a unit this creates a unit norm vector ok. And now you can quickly check quickly check the orthogonality property once again.

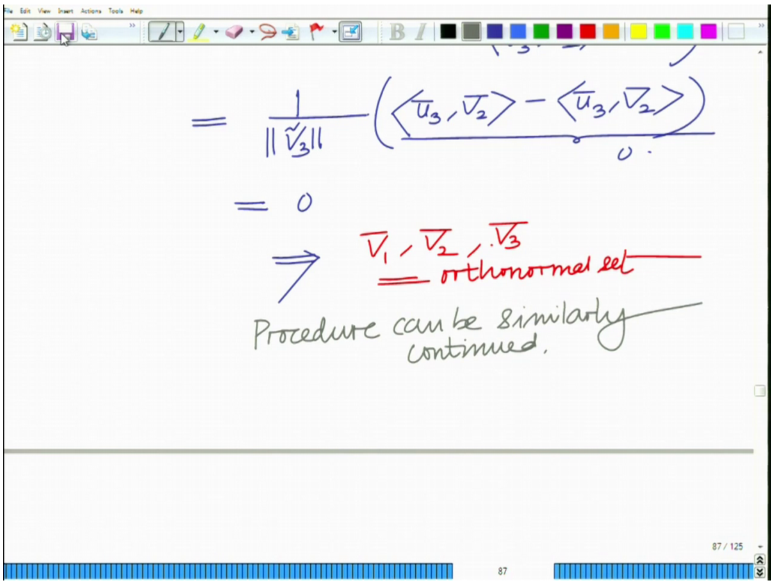
(Refer Slide Time: 12:41)



If you do V 3 bar transpose V 1 bar let us say; or V 3 bar transpose V 2 bar just to make sure it is orthogonal to the previous vector. So, I have 1 over norm V 3 tilde V 3 tilde transpose V 2 bar. This is 1 over V 3 tilde norm times u 3 bar minus u 3 bar inner product V 1 bar V 1 bar, of course this is all V transpose because we have to take the transpose minus u 3 bar comma V 2 bar into V 2 bar transpose whole times V 2 bar.

And now you can see; this will be 1 over norm V 3 tilde u 3 bar transpose V 2 bar because u 3 bar V 2 bar inner product minus u 3 bar V 1 bar inner product into V 1 bar transpose V 2 equals 0 ok. Observe the V 1 bar transpose V 2 bar equal to 0 ok, into 0 minus u 3 bar inner product V 2 bar, into V 2 bar transpose V 2 bar that is norm V 2 bar square which is once again 1.

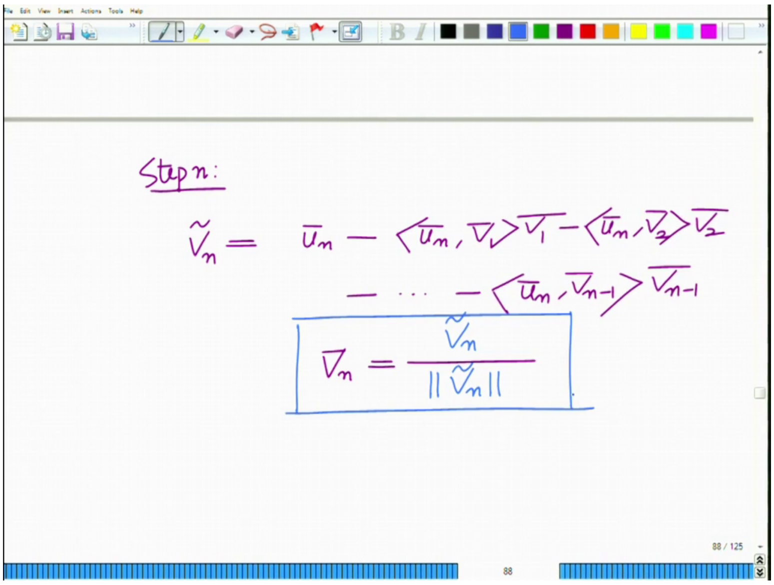
(Refer Slide Time: 14:23)



So, this is basically again norm 1 over, norm V 3 tilde u 3 bar comma V 2 bar minus inner product u 3 bar V 2 bar is equal to 0, this quantity is 0. So, these implies and even similarly verify orthogonality of V 3 bar to V 1 bar as well. So, this implies V 1 bar V 2 bar V 3 bar is an orthonormal set and this procedure can similarly be continued.

So, we have V 1 bar V 2 bar V 3 bar this is a orthonormal set, and the procedure can be similarly continued.

(Refer Slide Time: 15:48)

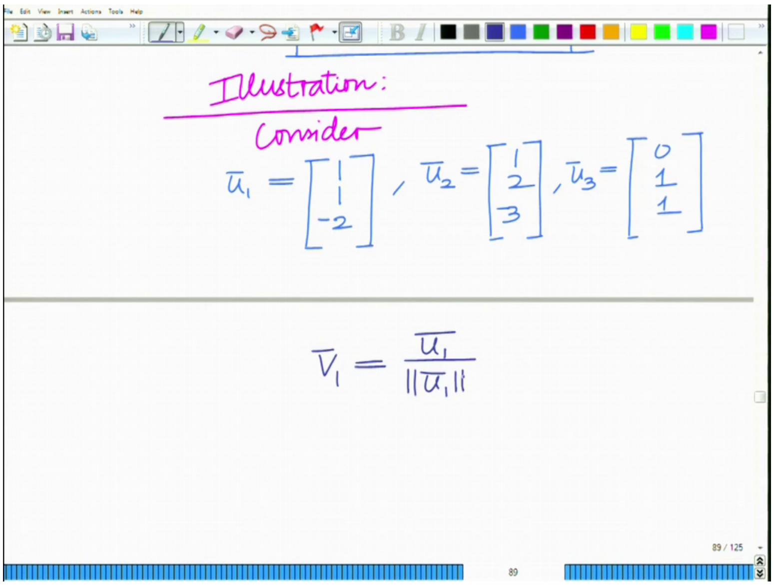


And what we do in step n; if you look at the nth step correct in step n, what we do is we have V n tilde this will be u n bar minus u n bar comma V 1 bar into V 1 bar minus projection of u 1 bar on V 2 bar remove that V 2 bar minus so on so forth.

Last one is you remove the projection along V n minus 1; I remove the projection along n minus 1 bar ok. And finally, we have V n bar V n tilde divided by norm V n tilde ok, and this generates the unit norm. Remember the orthogonality property is not affected by the norm. So, all we have to do is take the vector and simply divide by it is magnitude to get the divide by the norm to get the corresponding unit norm vector. That basically summarizes the Gram Schmidt Orthonormalization procedure.

Let us look at a specific instance of this procedure application of this procedure considering a set of vectors to demonstrate how this procedure actually works in practice. So, this is to think of this is an example inside an example.

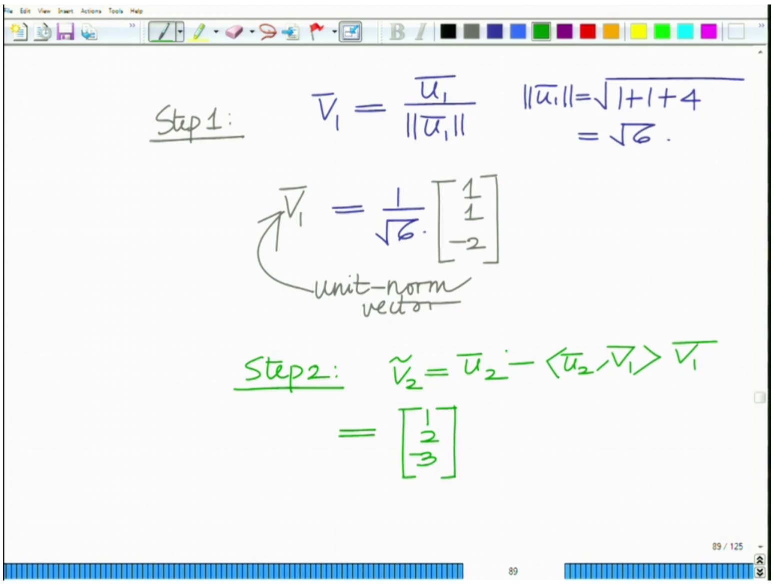
(Refer Slide Time: 17:24)



You can think of this as an illustration a practical illustration of the Gram Schmidt Orthonormalization Procedure.

So, consider 3 vectors that is, u 1 bar that is, the vector 1 1 minus 2, u 2 bar equals the vector 1 2 minus 3 and u 3 bar equals the vector 0 1 1. And now we have V 1 bar equals u 1 bar divided by norm u 1 bar.

(Refer Slide Time: 18:42)

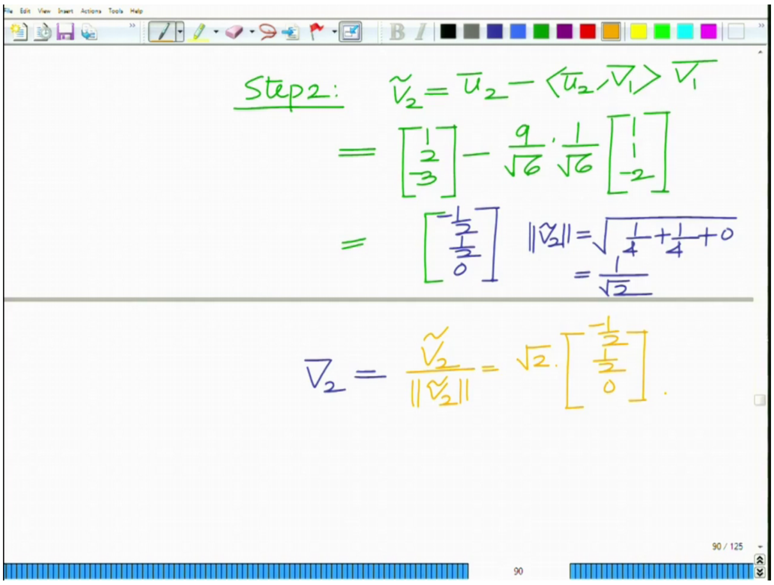


Now observe that norm u 1 bar equals square root of well one square plus 1 square plus 2 square equals square root of 6. So, this V 1 bar equals 1 over square root of 6 times 1 1 minus 2. So, this is your unit norm vector ok.

Observe that this is a unit norm vector, this is the first step. Step one we do not remove any projection because there is no vector in the set V bar yet ok. Remember we have to remove the projections on the previously chosen vectors V 1 bar V 2 bar V n minus 1 bar at step n minus 1. So, step 2 onwards we remove the projection, so that is V 2 tilde equals your u 2 bar minus the projection of u 2 bar on V 1 bar, which is given by this inner product since V 1 bar is a unit norm vector.

And this is basically 1 2 so u 2 bar is 1 2 minus 3 the vector 1 2 minus 3 minus.

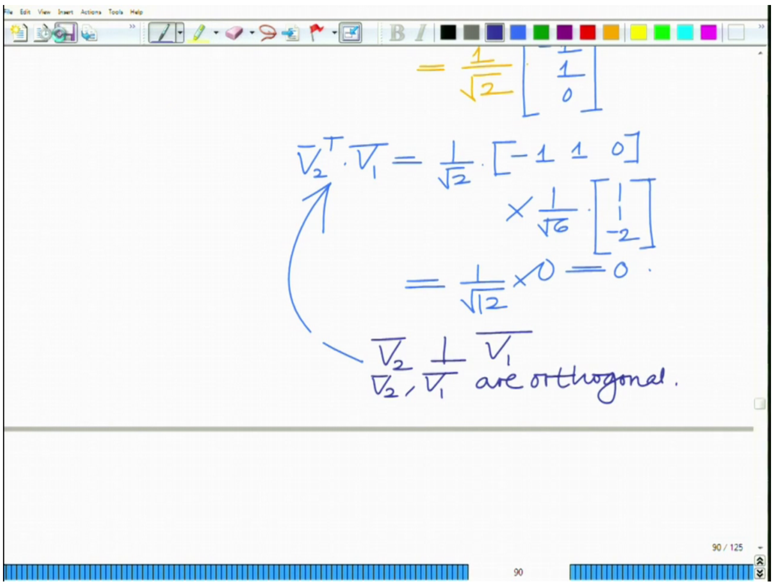
(Refer Slide Time: 20:16)



You can check the inner product u 2 bar transpose V 1 bar is 9 over square root of 6 times 1 over square root of 6 into 1 1 minus 2. And this will basically be so V 2 tilde is minus half half comma 0. And norm of V 2 tilde is square root of 1 by 4 plus 1 by 4 plus 0 that is square root of half that is 1 over root 2.

So, V 2 bar equals V 2 tilde divided by norm of V 2 tilde. This is basically you are 1 over 1 over square root of 2, so square root of 2 times minus half half 0 taking the fact of half outside this becomes 1 over square root of 2 times minus 1 1 0.

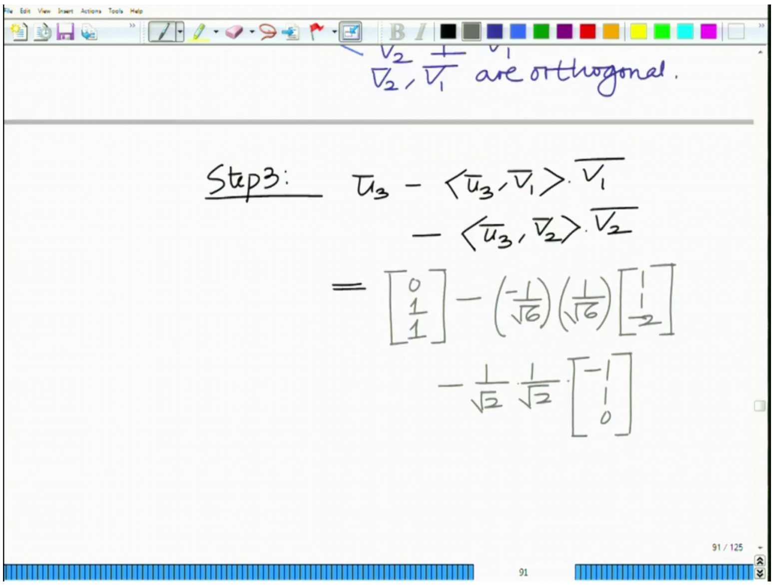
(Refer Slide Time: 21:22)



And you can clearly see this is a unit norm vector and you can also see this will be orthogonal to V 1 bar. So, V 2 bar transpose V 1 bar is well 1 over square root of 2 minus 1 1 1 0 times, 1 over square root of 6 times 1 1 minus 2. And you can clearly see this is minus 1 plus 1 which is 0 plus 0 which is 0.

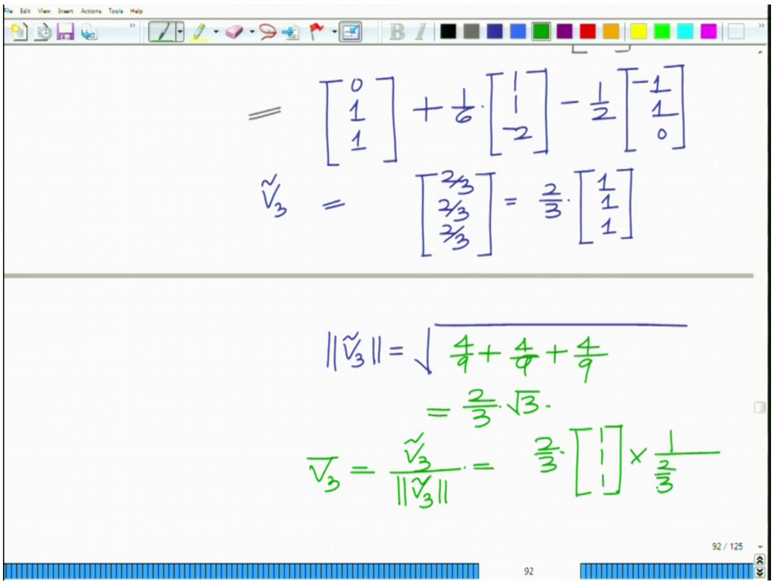
So, 1 over square root of 12 times 0 so this is 0. So, implies these are orthogonal that is, V 2 bar perpendicular to V 1 bar or the same thing as saying V 2 bar V 1 bar or orthogonal. So, that completes your step 2, plus remove the projection of V 1 bar from u 2 bar and then divide by it is norm that is, you obtain V 2 tilde divide by the norm of V 2 tilde to obtain the orth normal vector alright. And V 2 bar you can see is also orthogonal to V 1 bar therefore, it makes it an orthonormal sector.

(Refer Slide Time: 22:59)



Now, step 3 is again similar very similar. So, u 3 bar minus u 3 bar projection along V 1 bar into V 1 bar minus u 3 bar V 2 bar inner product V 2 bar, and this will be; you can check this is u 3 bar is you have seen this is given 0 1 1, this is the vector 0 1 1 minus the project the inner product is minus 1 over square root of 6 times 1 over square root of 6 times 1 1 minus 2, that is your V 1 bar minus 1 over square root of 2, that is a projection 1 over square root of 2 times minus 1 1 0 all right.

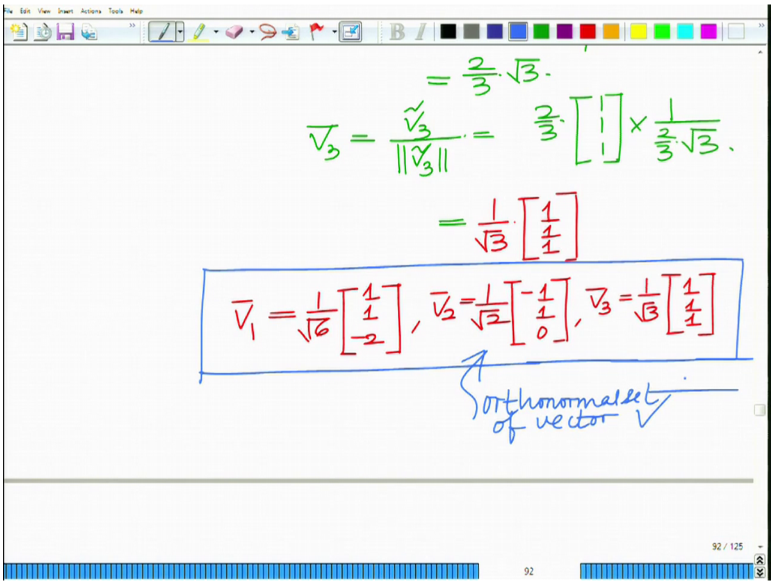
(Refer Slide Time: 24:11)



So, and that will be basically if you can simplify this this will be 0 1 1 plus 1 over 6 1 1 minus 2 minus 1 over 2 minus 1 1 0. So, this will be 0 plus 1 by 6 so if you look at this 0 plus 1 by 6 that is half, so half plus 1 by 6 half plus 1 by 6 that is 4 by 6. So, this will 2 by 3, you can also check second entry will also be 2 by 3 and third entry will also be 2 by 3, so that is 2 by 3 into 1 1 1.

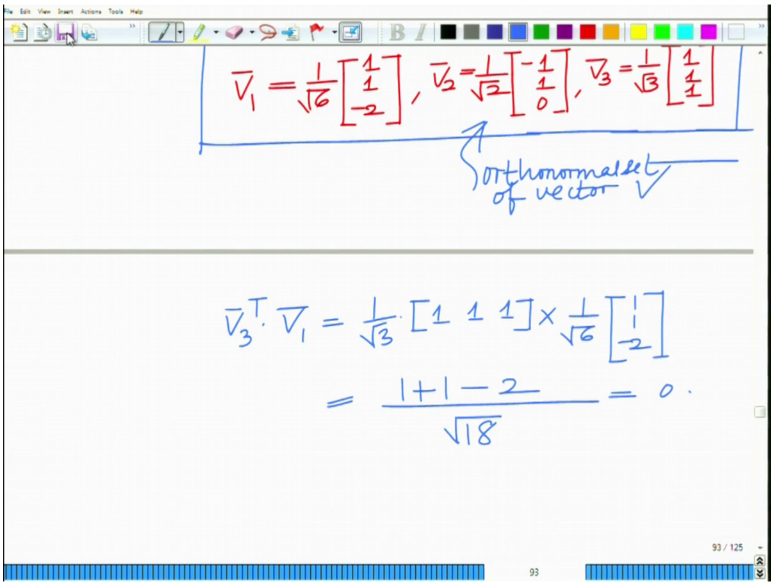
And norm and this is basically your V 3 tilde ok. And norm of V 3 tilde you can also compute that very easily, that is square root of well that will be 4 by 9 into 3 4 by 9 plus 4 by 9 plus 4 by 9; that will be 2 by 3 square root of 3. And therefore, finally, V 3 bar equals V 3 tilde divided by norm V 3 tilde equals well 2 by 3 1 1 1 divided by the norm into 1 over 2 by 3 square root of 3 and that will be equal to well 1 over square root of 3 times 1 1 1.

(Refer Slide Time: 25:59)



So, finally, we have the orthonormal set of vectors, which has the same span space as u 1 bar u 2 bar u 3 bar remind you. So, this is 1 over square root of 6, 1 1 minus 2 V 2 bar equals 1 over square root of 2, apologize 1 over square root of 2 minus 1 1 0 and V 3 bar equals 1 over square root of 3, and this is 1 1 1. This is 0 orthonormal set of vectors V ok.

(Refer Slide Time: 27:19)



And you can once again check if you do V 3 transpose V 1 bar, let us do a quick check that would be; V 3 bar transpose V 1 bar. So, that would be well 1 over square root of 3 times 1 1 1 the rho vectors times 1 over square root of 6 times 1 1 minus 2, which is basically one plus 1 minus 2 divided by square root of 18 and you can see this is equal to 0.

And therefore, V 1 bar V 2 bar and you can similarly check for the inner product or V 2 or V 3 bar transpose V 2 bar that in fact, you can readily see that they are also orthogonal therefore, V 1 bar V 2 bar V 3 bar is an orthonormal set. And remember the important thing about this is the span the same subspace as the original vector u 1 bar u 2 bar u 3 bar.

So, that is the connection between this set, the new set V and the old set u bar. And in several times it is very convenient to represent to find out an orthonormal span. So, although both the given sets has the same subspace, it is very convenient to deal with V rather than u because V is an orthonormal set of vectors that spans the same subspace. And in fact, this can be used to not only find the orthonormal span for the vector subspace remember this can be used for any inner product space all right and we have already said that the set of functions, so and so continuous functions forms an inner product space continuous functions on the interval a and b.

So, given a set of basis functions linearly dependent functions on that right? Which spans subspace on that one can similarly determine an orthonormal set of functions, that is functions with an orthogonal to each other and have unit norm and that span the same subspace of continuous functions on the interval ab.

So, this Gram Schmidt Orthonormalization procedure is something that is very convenient, very handy and it is very popular or it is highly applicable a practice because of it is first, because it is a low complexity procedure and 2, it has immense utility in terms of simplifying, either be it either deriving the span of a subspace or the representation of a set of or representation of a new vector, to represent it in a this subspace can be much readily derived or much more easily derived using the orthonormal span for the same subspace. So, we will stop here and we will continue with other aspects in the subsequent modules.

Thank you very much.