**Applied Optimization for Wireless, Machine Learning, Big Data**
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**Lecture – 69**
**Example problem on Optimal MIMO Power allocation (Waterfilling)**

Hello. Welcome to another module in this massive open online course. So, we are looking at Optimal MIMO Power Allocation alright and we have demonstrated the solution by using the KKT conditions, alright. Now let us do an example to understand this better, alright.

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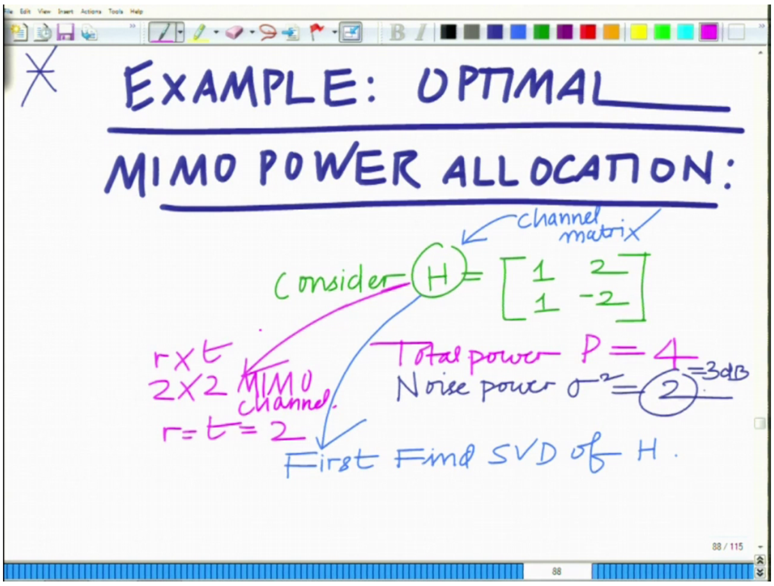


So, what we want to do is we want to do an example to understand this concept of optimal ok. And now consider the channel matrix. So, let us do an example ok. So, for this consider the MIMO channel, this is or MIMO channel in fact, this we have already seen this is the channel matrix; this is your MIMO channel matrix.

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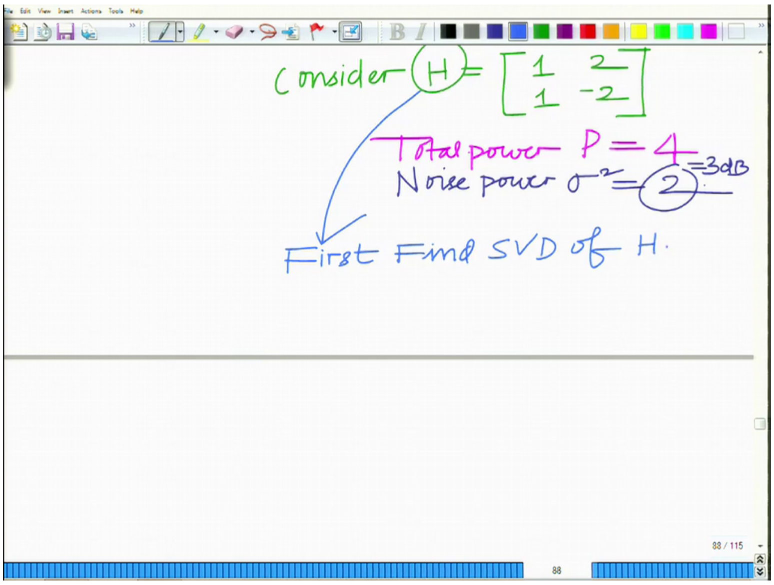


And let us say we have a total power P equal to 4 and noise power sigma square equal to 2. And we have to come up with the optimal power allocation alright; so this 2 can also be written as 3 dB ok. So, noise power is 3 dB which means sigma square equals 2, alright. And now we have to optimally allocate this power, this total power is 4 we have to optimally allocate this power. Remember once you decompose this MIMO channel as a set of parallel channels which we obtained using the singular value decomposition, alright.

So, for optimal power allocation which maximizes the sum rate alright, which achieves the capacity of the MIMO channel, we have to first start with a singular value decomposition ok. And the singular value decomposition can be obtained as follows. So, first find the singular value decomposition of H; find the singular value decomposition of H and remember we have H equals this channel matrix.

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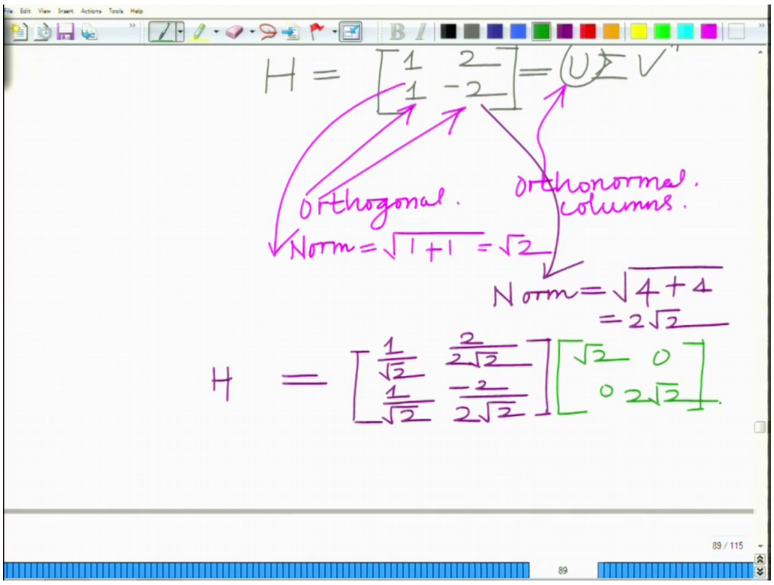


What I am going to do now remember, we have to write this as U sigma V Hermitian where U contains U orthonormal columns; and this unit norm columns: unit norm columns that are orthogonal to each other, so this contains orthonormal content.

Now we can already see that these 2 columns are orthogonal to each other. You can see these two columns 1 1 2 minus 2 these are already orthogonal, so all we have to do is we have to simply normalize them to unit norm alright which means divide them by their norm. And that I can simply do as follows remember, each column I have to divide it by its norm. So the first column, you can see norm equals square root of 1 plus 1 equals square root of 2. And second column the norm a square root of 2 square that is 4 plus 4 square root of 4 plus 4 that is square root of 8 which is 2 square root of 2.

So, the corresponding orthonormal columns will be. So, I can write H as 1 divided by square root of 2 1 divided by square root of 2. So, each column I am simply dividing by the norm 2 divided by 2 square root of 2 minus 2 divided by 2 square root of 2. Times; obviously, since I am dividing by the norm I have to also multiply each column by the norm which is basically diagonal matrix comprising of the norms: square root of 2, 2 square root of 2 0 comma 0 ok.

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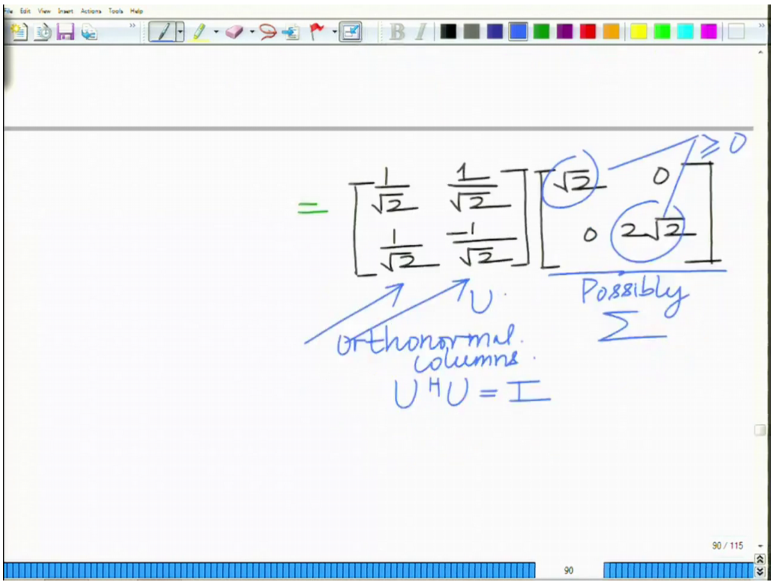


Now obviously, you can simplify further I can simply write this as 1 over square root of 2 minus 1 over square root of 2; I am sorry 1 over square root of 2, 1 over square root of 2, 1 over square root of 2 minus 1 over square root of 2 times the product square root of 2 0 ok. So, we are done to some extent, so we have orthonormal columns. Orthonormal columns means you have columns which are basically orthogonal to each other and also have unit norm.

And therefore, now you can see if you look at this matrix, you can see this satisfies the property of U, because it has orthonormal columns. Therefore U Hermitian U you can see is identity. And. In fact, if we look at this, this can be possibly sigma because this is a diagonal matrix and these are non-negative. Remember these are greater than equal to, so these are possible singular values. And now we have the V matrix; remember V is a unitary matrix. All I need is a unitary matrix and I can simply use the identity matrix in this case as unitary simply multiplied by the 2 cross 2 identity matrix; that is a unitary matrix.

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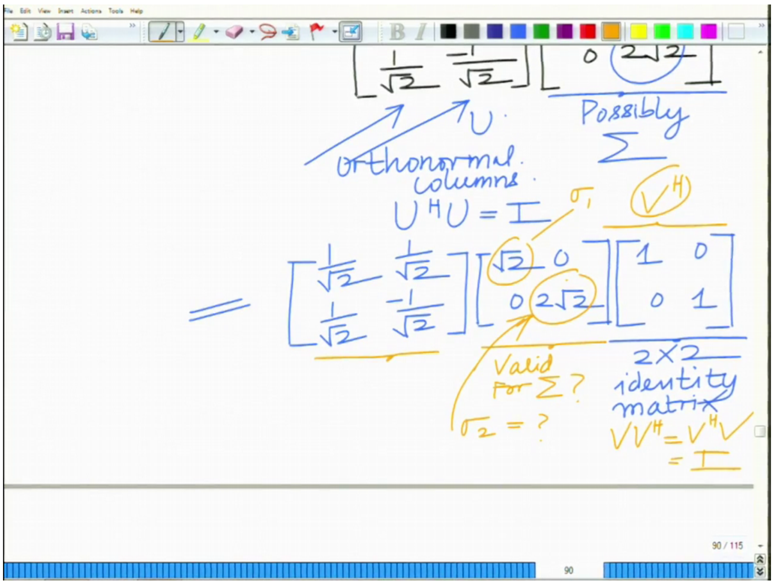


So, I can simply write this as since I already have U and sigma I can simply obtain V by multiplying by the identity matrix. So, this will be 1 minus 1 over square root of 2 correct; square root of 2. And here I can simply use a identity matrix; I can simply use the 2 cross 2 identity matrix. This is the 2 cross 2 ok. And this can be your V Hermitian that is it. So, this can be your V Hermitian.

Now the only problem is the following now if you look at this is your U, but is this valid for sigma? That is this can be sigma 1 we can say this can be singular value sigma 2. Is this valid? Now the problem is now of course, you will realize that identity matrix is V because you can have VV Hermitian this satisfies VV Hermitian is V Hermitian V is identity. So, identity matrix is a unitary matrix. Therefore it satisfies VV Hermitian equals V Hermitian V equals identity.

Now the problem with this matrix is, if you look at sigma 1 we have sigma 1 equal to square root of 2 sigma 2 is called square root of 2 square root of 2 sigma 1 is less than sigma 2.

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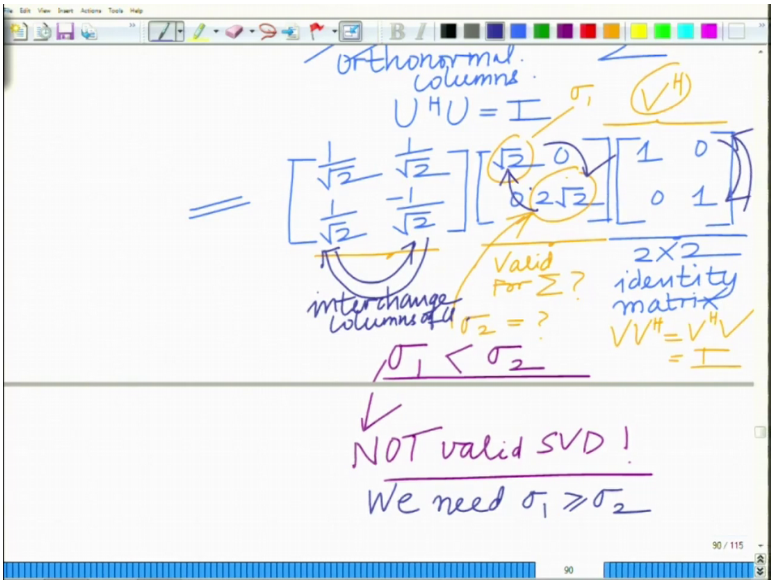


So, here the problem is sigma 1 is less than sigma 2 alright and therefore, NOT a valid SVD. Because remember for an singular value decomposition we have the signal values have to be ordered in decreasing order that is we have sigma 1 greater than equal to sigma 2 greater than equal to sigma 3 and so, on. So, we need singular values have to be ordered in decreasing order. So, we have to somehow switch these 2 singular. So, we have everything we have the basic structure of the singular value decomposition; some of you have to switch the singular values, we have to interchange the singular values.And you will realize that that is possible if I basically interchange the columns of U and here I interchange the rows of V Hermitian and then I can flip the singular values.

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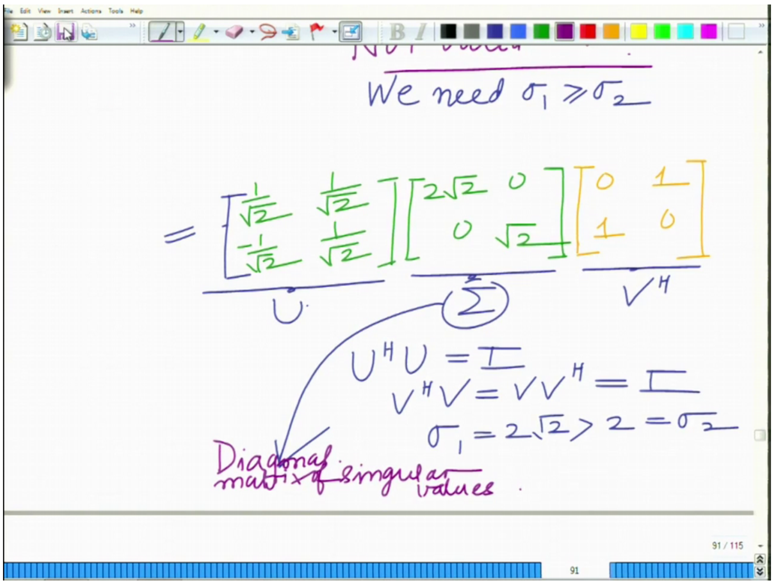


And therefore, I can equivalently write it as first I will as I told you, I will interchange the columns. So, this will become 1 over square root of 2 minus 1 over square root of 2, 1 over square root of 2, 1 over square root of 2. I can now switch the singular values 2 square root of 2 0 diagonal matrix 2, square root of 2 square root of 2.

And finally, I switch the rows of V. So, that becomes 0 1 1 0 ok. And this is now your matrix U, this is now the matrix V, this is now the matrix V Hermitian. And you can see now it satisfies all the properties for instance U Hermitian U equals identity. V Hermitian V equals VV Hermitian equals identity. And sigma 1 equals to square root 2 which is greater than 2 equal to sigma 2.

So, singular values are arranged in decreasing order and this is the diagonal matrix of remember; diagonal matrix of singular values. This is the diagonal matrix of singular values. And now we have to do optimal power allocation now perform optimal remember what this means is you can decompose this using pre coding as the combination of two parallel eternal.

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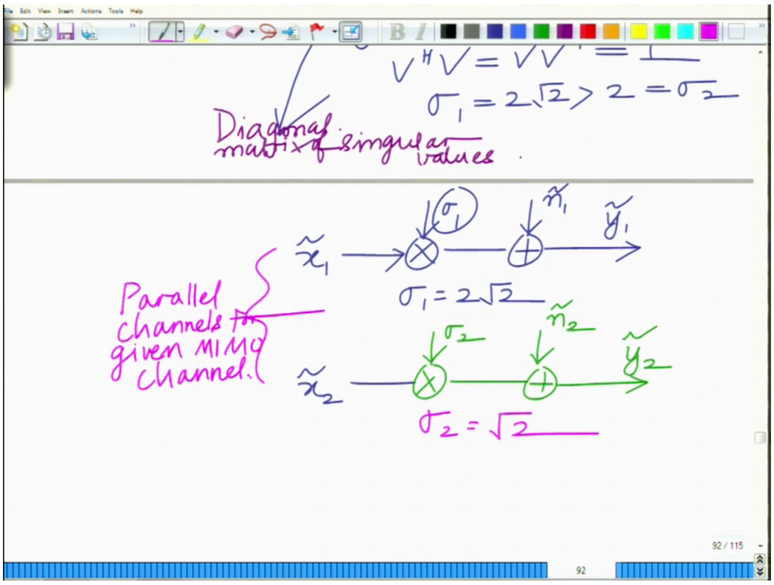


So, the first remember you have the tilde. So, you have x 1 tilde you are transmitting through the first channel that has gain sigma 1 and noise n 1 tilde to give y 1 tilde. And what is sigma 1? Sigma 1 is 2 square root of 2 and similarly through the second channel you have gain sigma 2 followed by the addition of the noise n 2 tilde followed by.

So, these are the parallel channels correct and sigma 2 equals square root of 2. So, these are the parallel channels, these are the parallel channels. What is given MIMO channel? In fact, you will realize that the given MIMO channel this is a 2 cross 2 MIMO channel ok. which we are not mentioned it 2 cross 2 and this is very easy to see this is a 2 cross 2 MIMO channel which means r equal to t equal to 2 in this scenario this is r cross t 2 cross 2 is r cross t. So, we have r equal to t equal to 2.

So, this is a 2 cross 2 MIMO channel with r equal t equal to 2 ok. So, fantastic. So, these are the 2 parallel channels.

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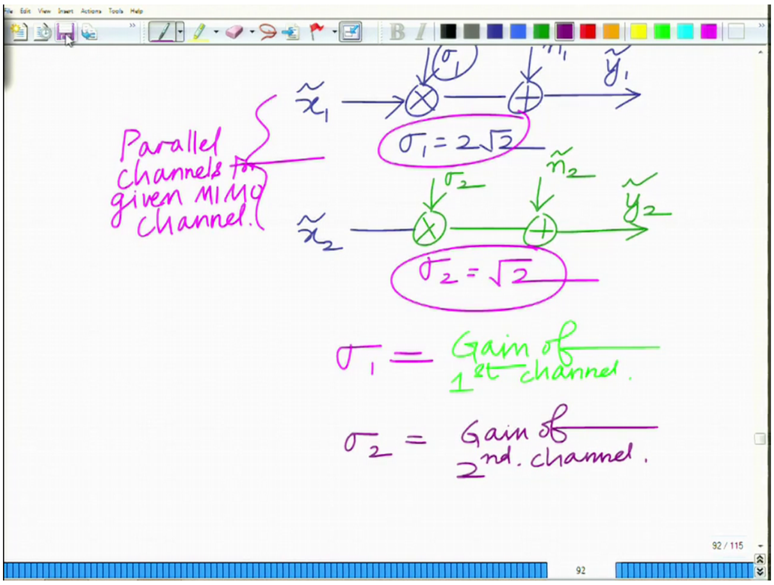


So, gain of the first channel is sigma 1 equal to 2 root 2, gain of the second channel. So, sigma 1 equals gain of first channel or you can say gain of first sub channel, and sigma 2 this is the gain of the second channel ok.

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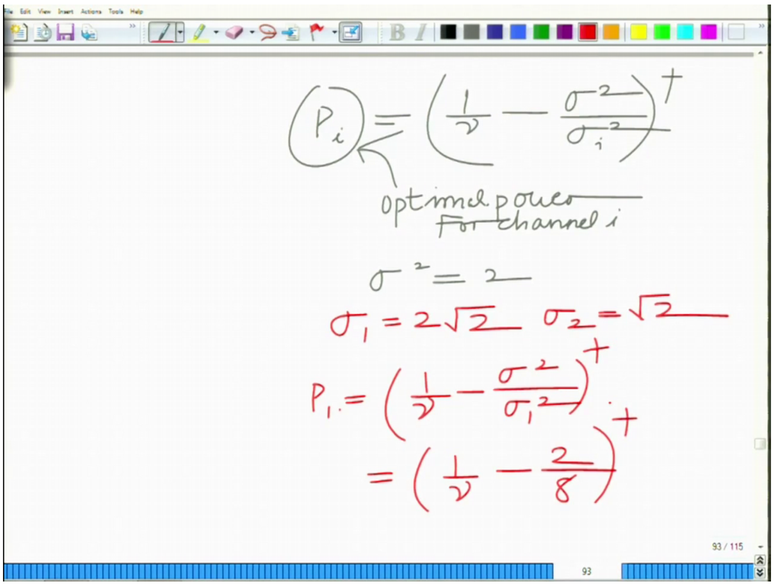


Now, what is the optimal power allocation? We have seen optimal power allocation Pi for the MIMO channel equals well what is it? 1 over nu minus sigma square sigma i square plus this is the optimal power. This is the optimal power for channel i and now we have well what we have we have sigma square remember we have seen sigma square equal to 2. And we have sigma 1 equals 2 square root of 2 sigma 2 equal square root of 2.

So, P 1 equals 1 over nu minus sigma square by sigma 1 square equals 1 over nu minus 2 divided by 2 square root 2 whole square which is 8 plus of course, which means it is this quantity if it is greater than equal to 0 or it is 0.

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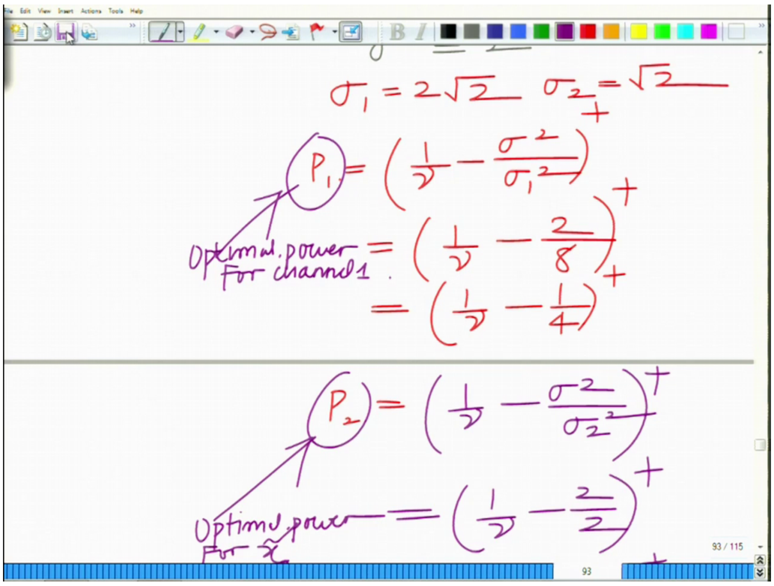


Otherwise which is basically 1 over nu minus 1 over 4 similarly P 2 equals 1 over nu minus sigma square by sigma 2 square equals which equals 1 over nu minus 2 divided by 2. Of course, there is always this function plus which is 1 over nu minus 1 ok.

So, this is optimal power for channel 2 or you can say optimum power for x tilde. This is the optimal power for channel 1 optimal power for the parallel channel component parallel channel.

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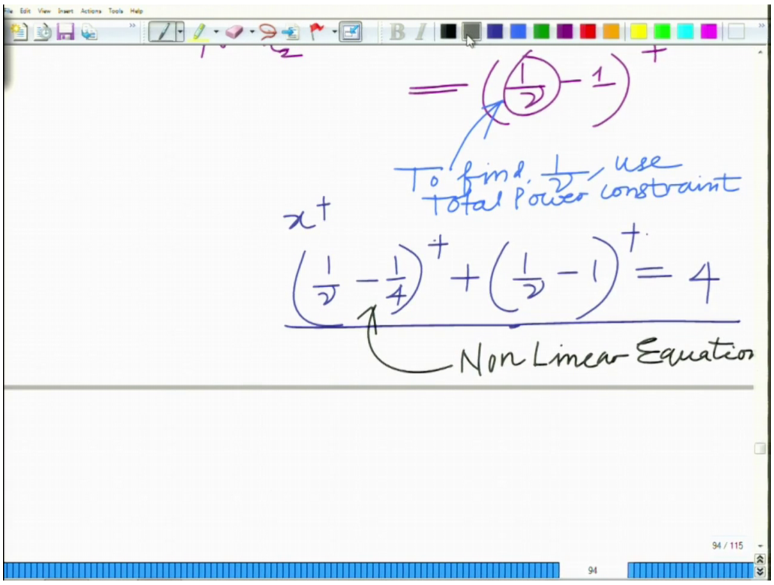


So, optimal power for channel 1 and now how to find nu 1 over nu or nu, we have to use that total power rate to find nu the LaGrange multiplier remember we have to use the total power constraint.

So, to find nu ok; so, that becomes what does that become that becomes well 1 over nu minus 1 plus plus 1 over nu minus I am sorry 1 over nu minus 1 by 4 plus 1 over nu minus 1 plus equals 4. Now observe we do not know if each of these quantities what each of this quantity is because remember we said if 1 over nu depends if 1 over nu is that is each quantity is power is equal to this quantity that is this plus indicates that this quantity plus indicates that it is equal to this quantity which is greater than equal to 0 and 0 otherwise alright.

So, we do not know now that depends again on the value of 1 over nu which we are again trying to find. So, this is basically a non-linear equation although it might appear as it because of this plus they because of this function x plus, this makes it a non-linear equation.

Now solve this we start with an assumption therefore, we have to start with the an assumption. Assume that 1 over nu is greater than or equal to 1 ok.

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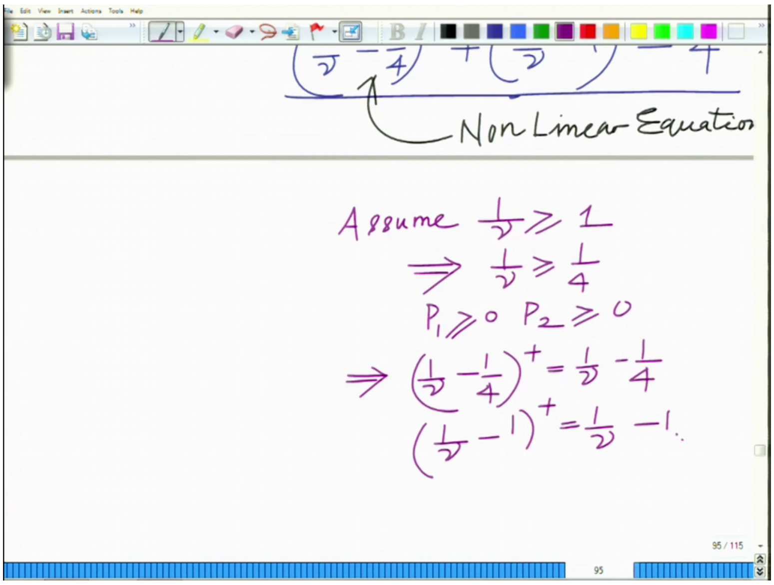


So, start with the assumption. This implies 1 over nu is greater than equal to 1 this implies that 1 over nu is greater than equal to 1 by 4; so, P 1 greater than 0 P 2 greater than or greater than equal to 0. So, this implies 1 over nu minus 1 over 4 plus equals 1 over nu minus 1 over 4 1 over nu minus 1 plus equals 1 over nu minus 1.

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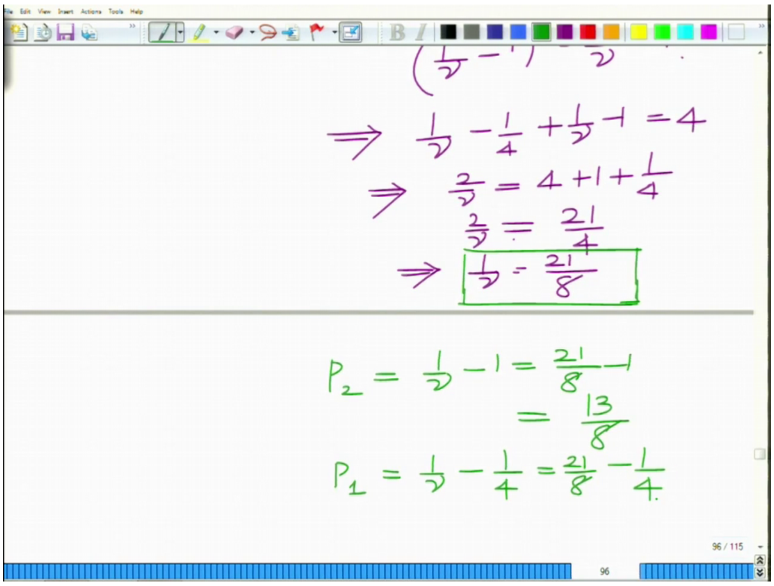


So, therefore, this implies we must have; now substitute this in the total power 1 over nu minus 1 by 4 plus 1 over nu minus 1 equals 4. Now remember this is all based on the assumption that 1 over nu is greater than equal to 1. If that does not assumption does not hold, then the entire procedure here alright. So, the values derived will not hold alright.

So, this is conditioned on the assumption that 1 over nu is greater than equal to 1. So, this implies that 2 over nu is 4 plus 1 plus 1 over 4 which is 5 plus 1 over 4 which is 21 over 4. So, that is your 2 over 2 over nu is 21 over 4 which implies 1 over nu is 21 over 8 ok.

So, this is the value of 1 over nu we do not need to really compute nu because we only need 1 over nu alright. So, that is what we need which implies if you look at P 2. This is equal to 1 over nu minus 1 which is 21 by 8 minus 1. So, that will be 13 by 8 and P 1 is 1 over nu minus 1 over 4 which is 21 by 8 minus 1 over 4 which is 19 over 8 ok.

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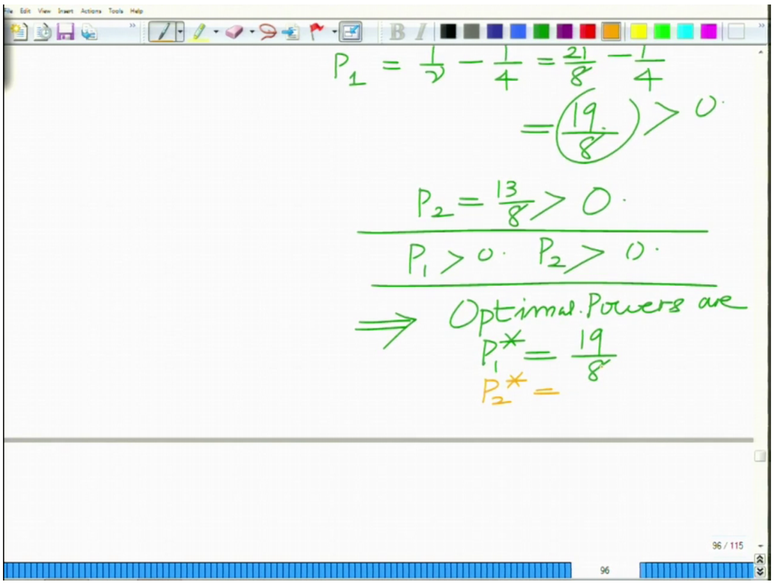


And now you can see that both P2 is greater than 0 P1 greater than 0 which implies and you can see P2 equals 13 by 8 greater than 1 greater than 0 ok. So, both powers P2 greater than 0 and P1 greater than 0 so, and you can also see that 21 over 8; this is greater than your 1 ok. So, the original assumption holds 1 over nu is greater than 1 greater than 1. So, original assumption holds P2 greater than 0, P1 greater than 0 implies that optimal powers are indeed P1 star equals well P1 star equals 19 by 8 and P2 star equals P2 star equals 13 by 8.

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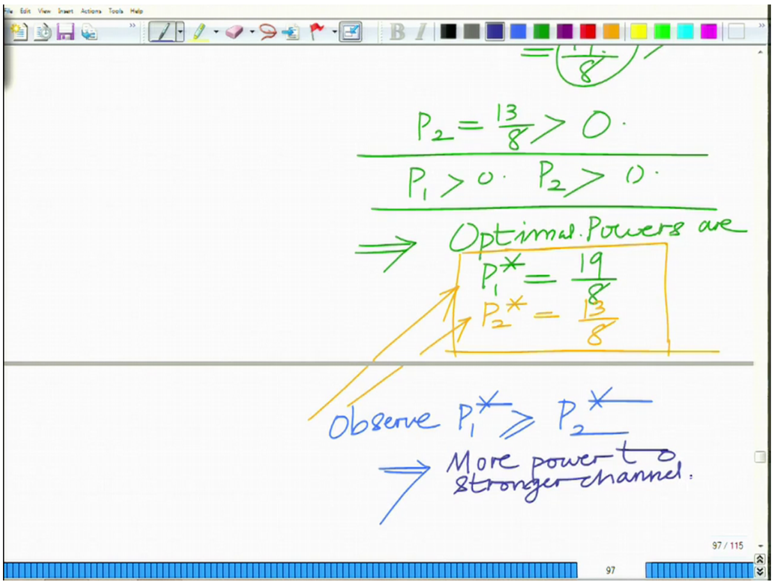


Now, if 1 of the powers would have been negative for instance let's say P2 star would have been negative. And that implies our original assumption is incorrect. So, therefore, we have to assume something else and then redo it alright. So, here we are able to show that the powers P1 star and P2 star are indeed non negative indeed they are positive alright. So, therefore, the original assumption is correct non negative powers are indeed allocated to both the channels otherwise if 1 of the power is negative it implies that that power is that is less than remember there is this concept of notion of water level. So, the power is negative it implies that that corresponding channel is above the water level.So, power is not allocated.

So, power; so, the corresponding channel the power has to be set to. And the problem has to be repeated with the total power (Refer Time: 23:06) ok. So, this is the procedure alright and now you also observe that observe P1 star greater than equal to P2 star implies there is more power more power to the stronger channel.

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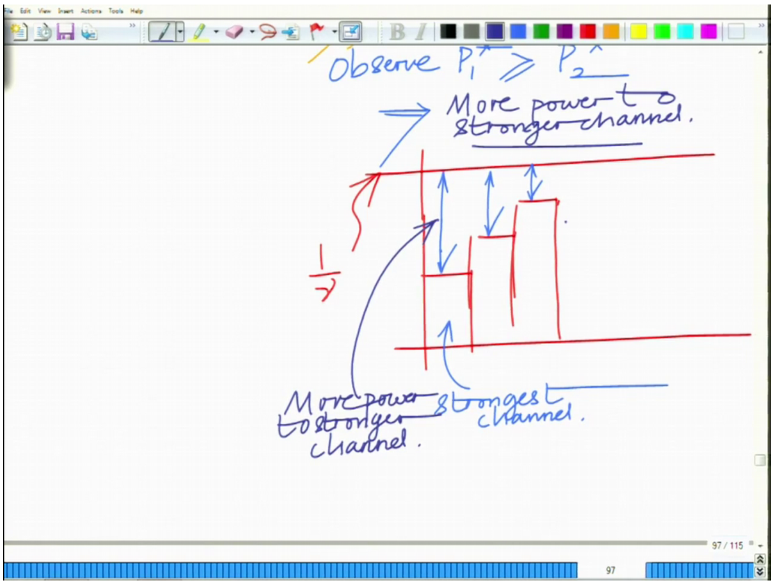


Remember this is what our water filling algorithm tells us for instance if you remember our water filling algorithm, you have these different bars correct and water level indicates the power.

So, if this is the level this is 1 over nu the power. So, this is the power and this is the strongest channel and what you see is that which is indeed true is that more power because there is a bar corresponding bar is smaller more power. So, what this shows is to maximize capacity because of the nature of water in filling power allocation more power is allocated to the stronger channel in this case the stronger channel will be the 1 with the larger singular value.

So, sigma 1 is greater than equal to sigma 2 is greater than equal to sigma 3; if there is 1 and so on. So, which means more power will be allocated to the channel with gain sigma 1 in comparison to the channel with gain sigma 2 and so on and so forth and that is what indeed we observe in this exercise for example, of optimal power allocation alright. So, we will stop here and continue in the subsequent modules.

Thank you very much.