Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 68
Optimal MIMO Power allocation(Waterfilling)-II

Hello welcome to another module in this massive open online course. So, we are looking at KKT conditions to solve an optimization problem, we have looked at an application a specific application of KKT conditions. Now, let us look at an example that is the application in MIMO optimal MIMO power allocation, let us look at an example to better understand this paradigm alright.

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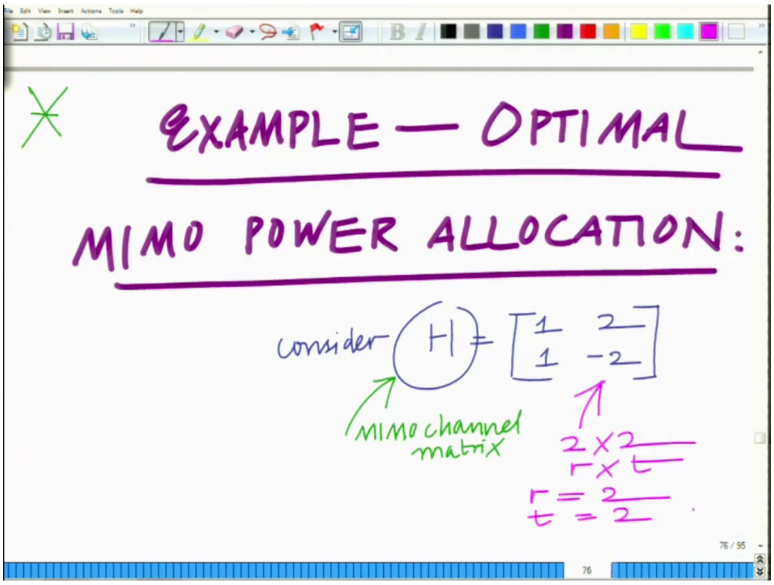

So, what you want to do today is we want to look at an example for the optimal, we have to look at an example about optimal MIMO power allocation ok. Let us consider the following, now consider the MIMO system, now remember each MIMO channel can be represented by the equivalent channel matrix. So, we have the channel 1 2 1, this is very simple MIMO channel, let us say this is our MIMO channel, this is our MIMO channel matrix ok, you can immediately see that this is a 2 cross 2 MIMO channel. So, this r cross t so, this implies r equal to 2 t equal to 2. So, basically number of receive antennas equal so, we have 2 receive antennas.

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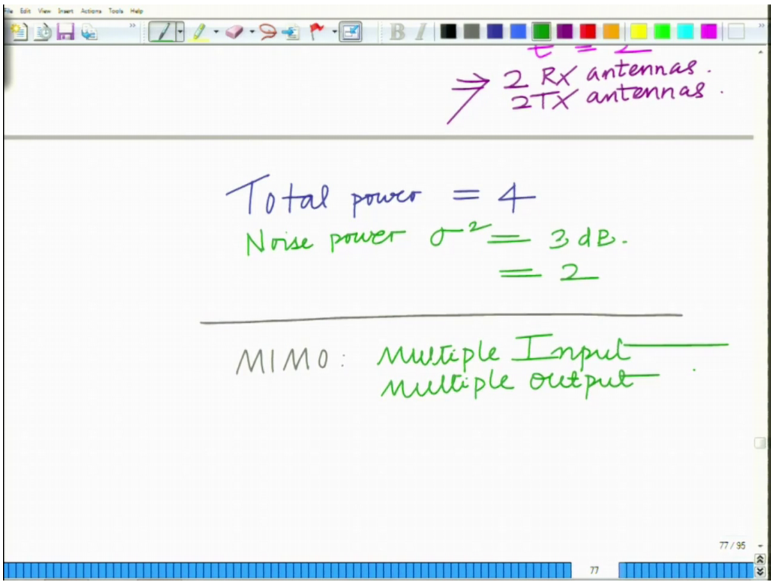

So, this implies 2 receive antennas 2 transmit antennas so on. And, what we want to do is we want to allocate power optimally to the various modes of this MIMO channel. Remember the modes are given by the singular value decomposition alright, these correspond to the channels with the gain sigma 1 sigma 2 and so on alright.

And we have to allocate power optimally to these different modes. Let us consider a total power 4 and the noise power remember, we also need knowledge of the noise power sigma square equals let us say 3 dB 3 decibels which means sigma square equals 2 ok.

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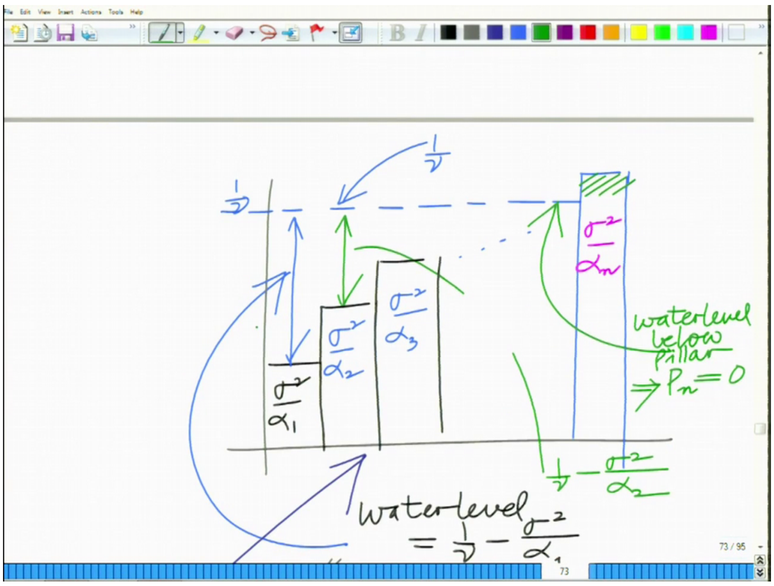

And remember we have seen optimal power allocation correct, this starts with the singular value decomposition correct.

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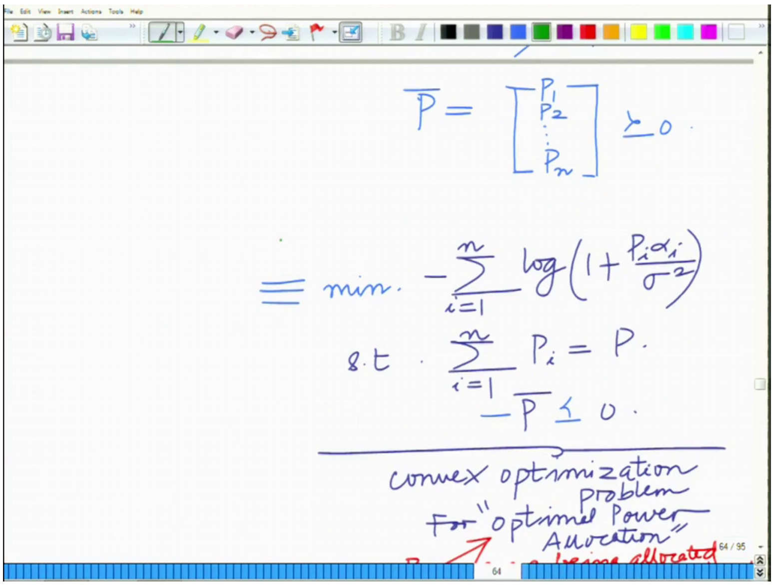

So, we have log 1 plus P i alpha i divided by sigma log of 1 plus here P i alpha i divided by actually let us start with the brief description of the MIMO system.

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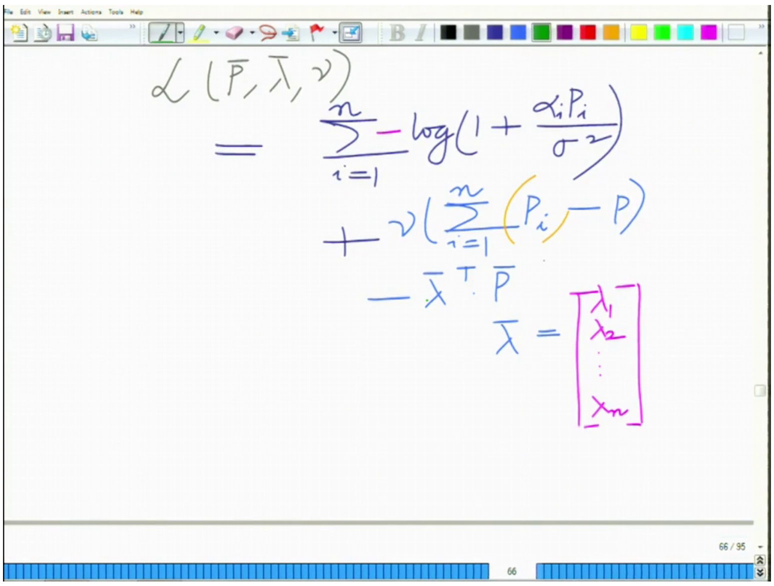

So, what we have is let us consider first the MIMO channel, remember MIMO and MIMO stands for this stands for Multiple Input Multiple Output system, which means that basically you have a wireless communication system with multiple transmit antennas.

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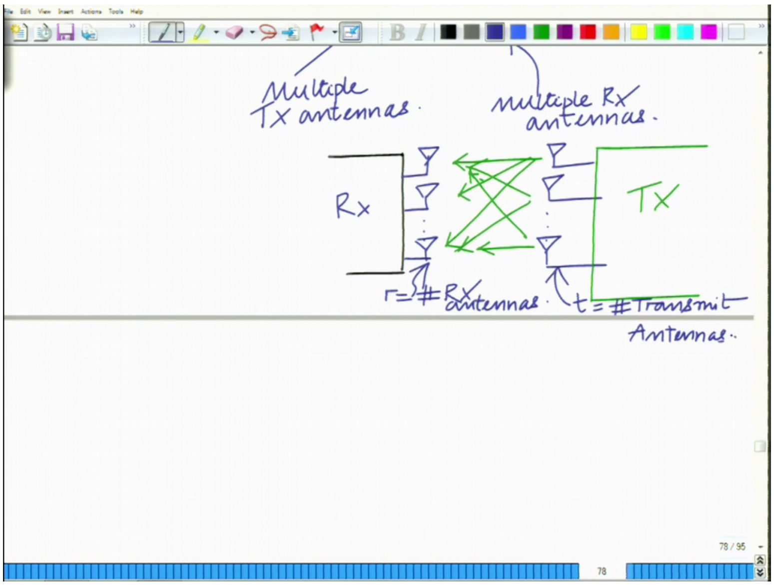

And you have multiple transmit antennas and multiple receive antennas ok. And so, you have schematically you can present, then we have seen this before you have the receiver with multiple receive antennas. So, this is your receiver you have the transmitter with multiple transmit antennas and this is the MIMO channel and so, let us say we have T or R receive antennas and this quantity t equals number of transmit antenna.

Now, what we want to do is we want to come up with a framework for optimal power allocation. So, remember we have seen the optimal power allocation problem, that is which is as follows that is we have a total power P total transmit power P. And we have a set of parallel channels each given by alpha i and we want to allocate the power optimally amongst this parallel channel so, as to maximize the total bit rate that can be transmitted across this channel.

That is the sum, that is the sum rate of this wireless channel that is the power of problem of optimal power allocation. Now, for that first we have to see how this MIMO channel can be decomposed into a set of parallel channels, because only then one can talk about optimal power allocation correct. So, let us proceed in that direction first starting first developing a framework to decompose, this MIMO channel, this seeming I mean it is not obvious what the set of parallel channels is in the context of this MIMO channel or multiple input multiple output wireless communication channel ok.

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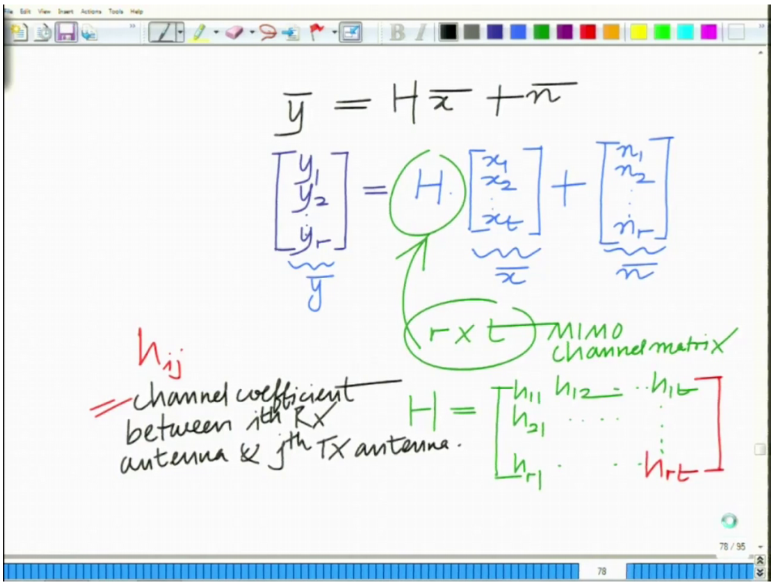

For that, now first realize that and we have probably seen this before that is the received symbol vector y is given as y equals H x bar for this MIMO system, where y bar equals y 1 y 2 y r these are the r received symbols across the r receive antennas. H which is now a matrix times the transmit vector remember you have t transmit antennas so, you have t t transmit symbols plus n bar which is the thermal noise samples additive white Gaussian noise samples at the r receive antennas.

So, this is your vector y bar which comprises of your r receive symbols, this is your vector x bar this is your vector n bar. And this is the MIMO channel matrix H, which is now you can see it relates r receive symbol vector r to a transmit symbol vector t or transmit symbol vector x bar of dimension t. So, this has to be naturally of dimension r cross t, where r is a number of receive antennas t is the number of transmit antennas.

So, this is an r cross t MIMO channel matrix. And for this MIMO channel matrix you have the coefficients h 1 1 h 1 2 h 2 1 so on. This one last row first column will be h r 1 last column first row will be h 1 t and finally, last row last column will be h r comma t. So, h ij is the channel coefficient between ith receive antennas. So, let me just write that between ith RX antenna and jth ith RX antenna and jth transmit ith receive antenna and jth transmit antenna. And now the key to understand this decomposition of MIMO channel into a set of parallel channels is what is known as a singular value decomposition alright.

So, we will use this technique of singular value decomposition, which we have probably briefly referred to once earlier in the context of optimal beam forming. We will use this construct of singular value decomposition and subsequently demonstrate optimal power allocation, for this MIMO that is multiple input multiple output wireless channel ok.

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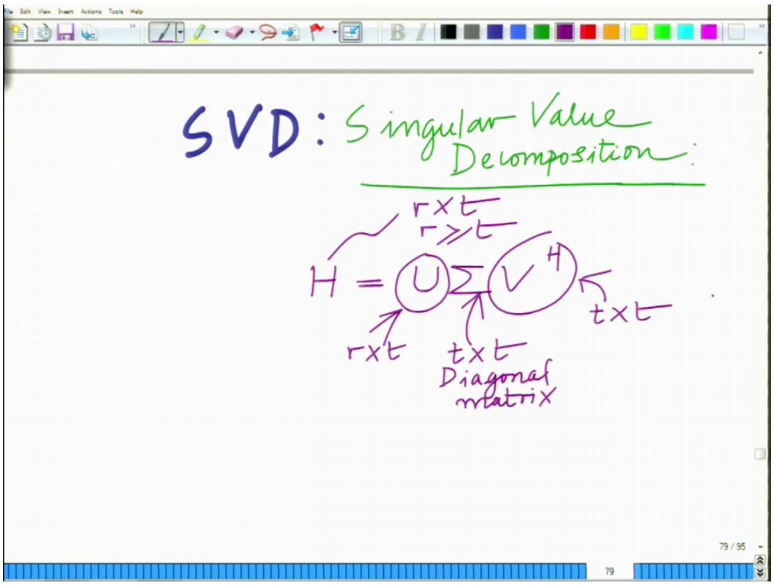

And so, this is an important concept in fact, this is an important concept that arises or this is an important technique or we can think of an important decomposition, that has several applications in the context of signal processing and wireless communication. The theory is general so, this is termed as singular value, this is termed as singular value decomposition ok.

Now, in a singular value decomposition what we do is given this channel matrix H, you decompose this as a product of 3 matrices U sigma V hermitian, consider now for the sake of simplicity we considered this to be an r cross t matrix with r greater than equal to t just for the sake of simplicity all those singular value decomposition itself is very general. And can be applied for any general channel matrix, we considering for simplicity we are considering a scenario with r greater than or equal to t.

Now, this matrix U has the following property U can this is an r cross t matrix sigma is a t cross t diagonal matrix. And V Hermitian is again a t cross t and this is a unitary matrix, this is a unitary matrix.

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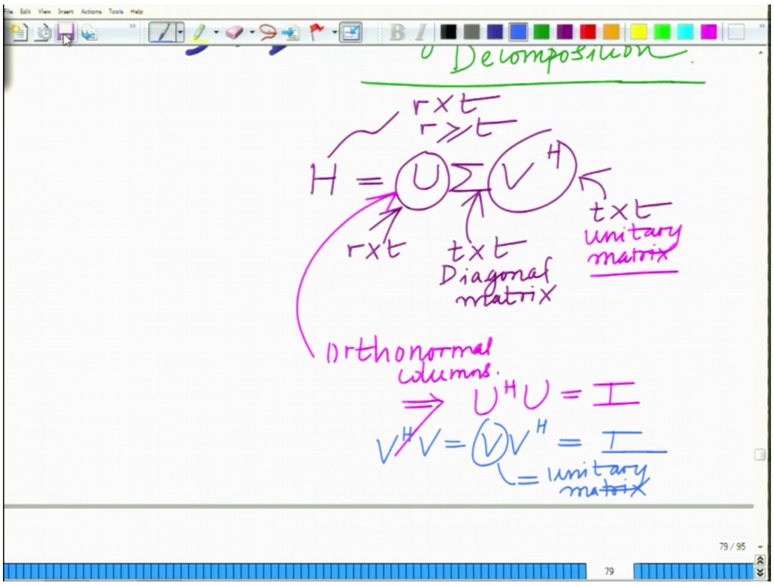

Now, U has orthonormal columns, orthonormal columns which implies that each column the columns are orthogonalities are the unit norm and therefore, if you perform U Hermitian U you will get identity. Now, V is a unitary matrix that implies that which implies that V Hermitian V equals V V Hermitian equals identity so, V equals a unitary matrix.

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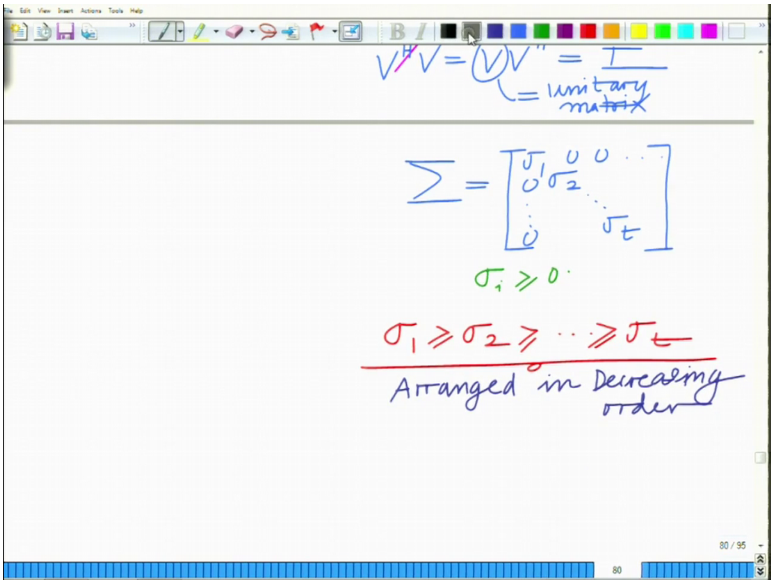

And further this quantity sigma is a diagonal matrix of what are known as singular value. So, sigma 1 sigma 2 up to sigma t each of this quantity sigma i is non negative that is it greater than equal to 0. And these are arranged in decreasing order that is sigma 1 is greater than or equal to sigma 2 is greater than or equal to sigma t. So, these are arranged in its important to realize that these are arranged in, these are arranged in decreasing order ok.

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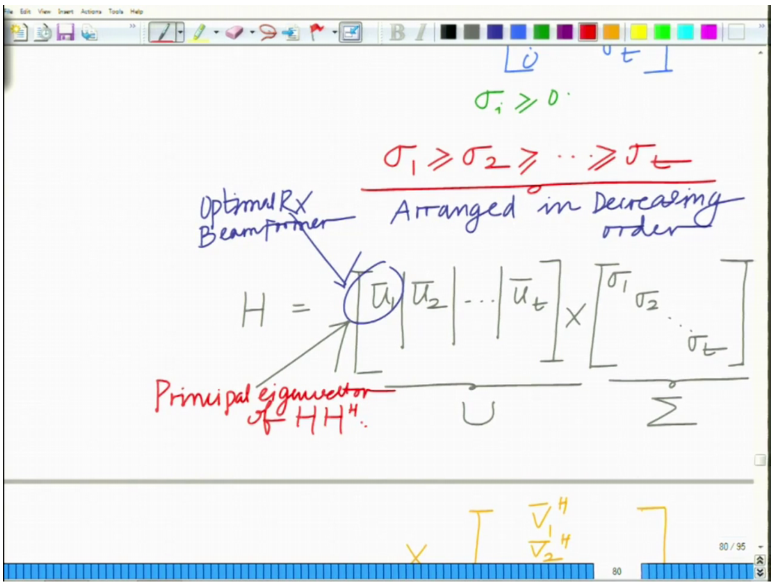

And therefore now you can write it has the matrix H equals u which contains r cross t which means it contains t columns each of size r u 1 bar u 2 bar so, on up to u t bar which are orthonormal columns.

So, this is your matrix U times the diagonal matrix sigma, which contains the non-negative singular values in decreasing order into the matrix V Hermitian, which contains the rows t rows and naturally these are also orthonormal, since V is a unitary matrix so, these rows are also that is.

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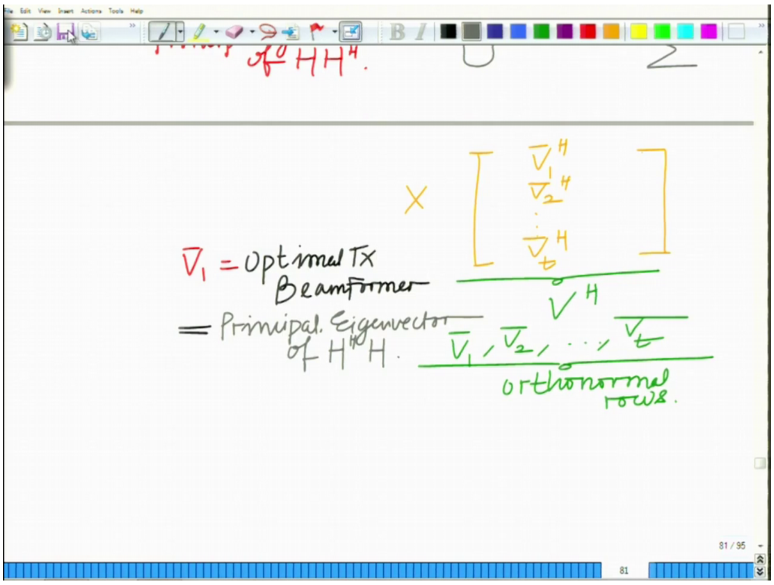

So, this is your V Hermitian V 1 bar V 2 bar V t bar these are orthonormal rows, these are orthonormal rows ok. Now, in fact we have seen this earlier in fact, what we have seen is precisely if you remember, we have seen that u one bar this is the optimal receive beam former for the MIMO system. When you talk about MIMO, we have mentioned briefly u 1 bar is the optimal receive beam former in fact, u 1 bar is the principle that is principle eigenvector, there is a eigenvector corresponding to the largest singular value of H H Hermitian ok.

So, u 1 bar if you look at it principle eigenvector and V 1 bar equals optimal transmit beam former, that is it maximizes again optimal receive beam former optimal TX beam former, that is also equal to principle eigenvector of H Hermitian H H Hermitian H ok. Now, for the parallel decomposition into parallel channels, the MIMO transmission scheme is as follows.

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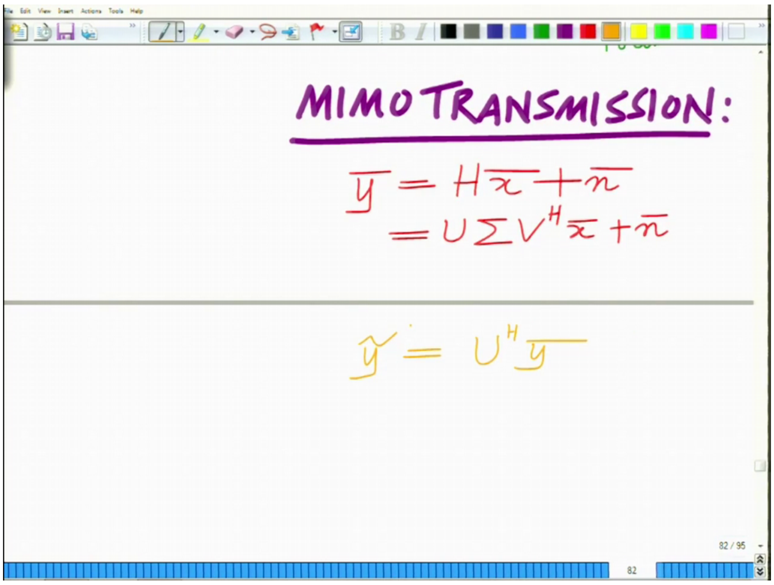

Now let us look at the following MIMO transmission scheme. What is the MIMO transmission scheme? Now, remember you have y bar received signal vector equals H x bar plus n bar and we have the singular value decomposition.

So, I can write H as U sigma V Hermitian x bar plus n bar, now what I am going to do is the step one first step at the receiver, I am going to process with U Hermitian ok. So, I am going to multiply y bar with U Hermitian to get y tilde. So, this is my receive processing.

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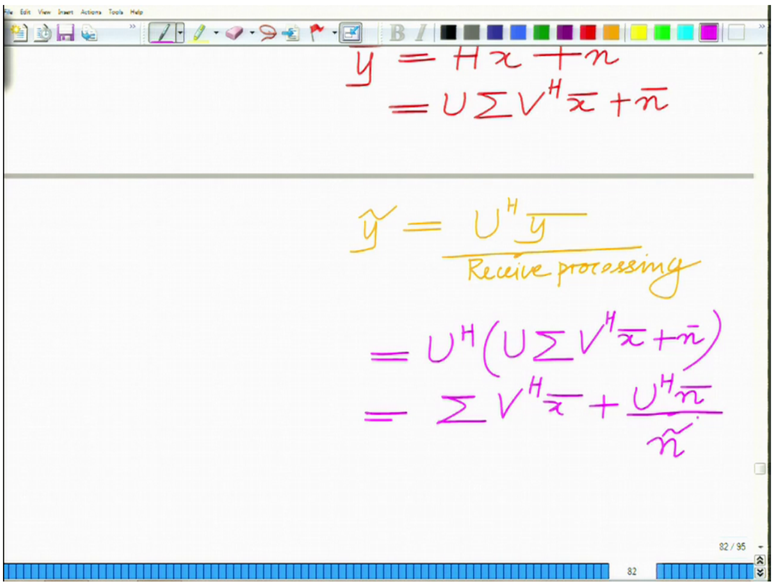

This is my receive processing so, y tilde equals U Hermitian let us substitute the expression for y bar which is U sigma V hermitian, that is H nothing, but H times x bar. Now, if you look at this you have U Hermitian U which is identity. So, what remains is sigma V Hermitian X bar plus U Hermitian n bar, which we will call as n tilde ok, because remember U has orthonormal column. So, it satisfies the property U Hermitian U is identity. So, we have U Hermitian U is identity so, implying that we have sigma V Hermitian x bar which is sigma V Hermitian x bar plus n tilde, that is after receive processing by U Hermitian or multiplying by U Hermitian at the receiver.

Now, what we are going to do is similarly at the transmitter even before transmission of x bar, we are going to employ a pre processing operation or what is also known as a pre coding operation and that is key. So, MIMO reception processing can be done at both ends one is at the transmitter and the receiver the receiver what operation at receiver is receive combining or post processing. The operation at the transmitter is either preprocessing or pre coding that is your pre coding the symbols prior to transmission alright.

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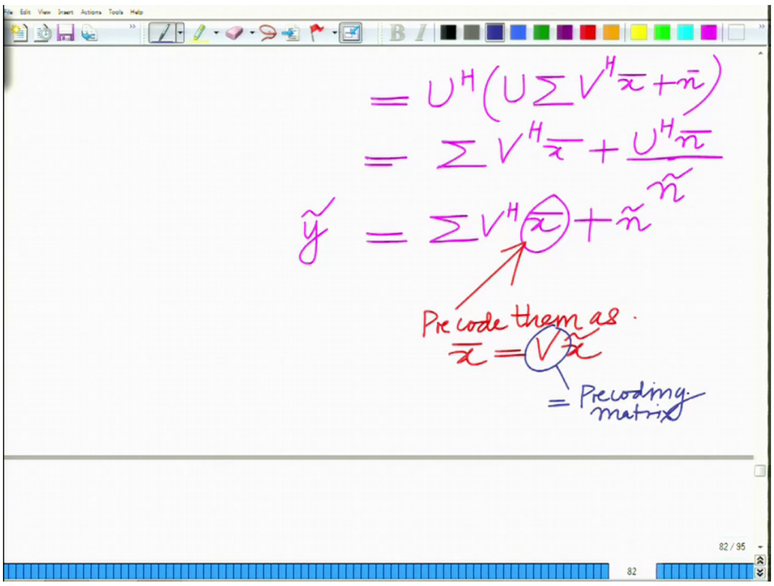

So, x bar you pre code them x bar equals V times x tilde, this is a pre coding operation this V equals the pre coding matrix.

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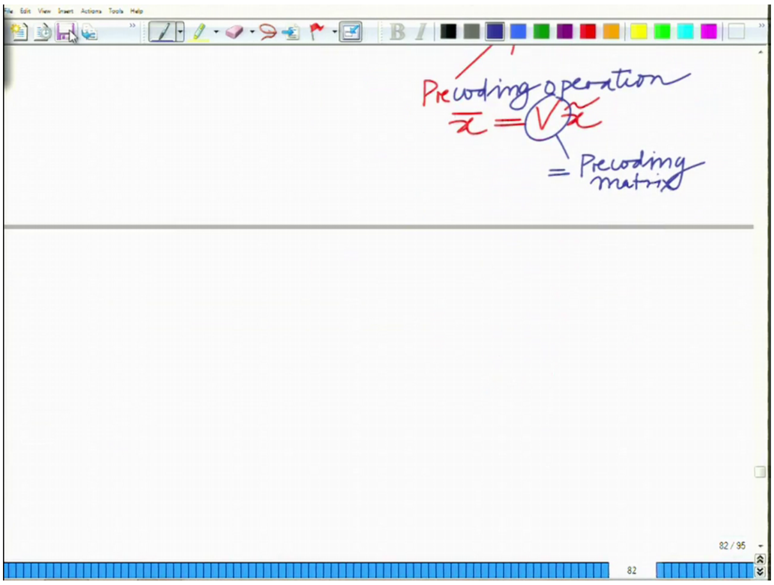

Just write this as pre coding operation ok.

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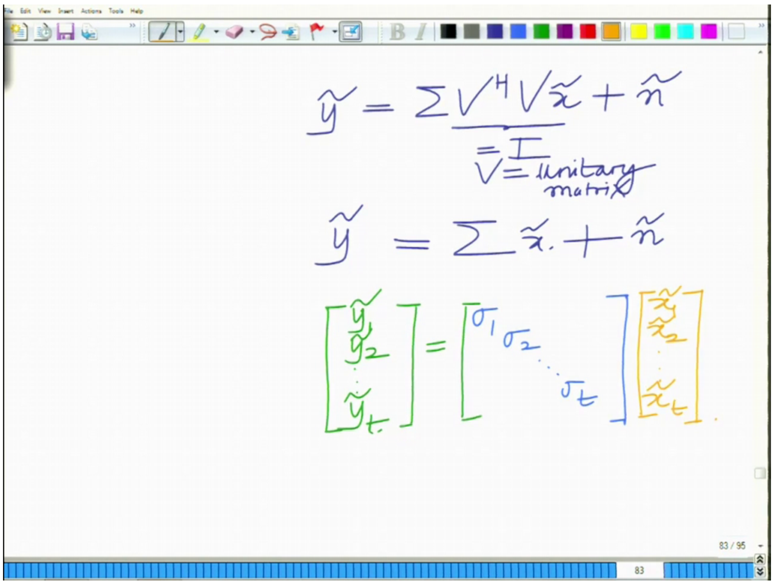

And now if you substitute x bar equal to V x tilde so, what you have y tilde equals U or V is gone sigma V Hermitian x bar is now V x tilde. So, substitute V x tilde plus n tilde. Now, we use the other property V Hermitian V equals identity since V is a unitary matrix. So, this will become sigma V Hermitian V sigma times x tilde plus n tilde ok. So, this will be y 1 tilde y 2 tilde y t tilde in fact, just write it out explicitly. So, you have y tilde equals sigma times x 1 tilde x 2 tilde that is the original symbols before pre coding which is what we are interested in plus you have your n tilde vector.

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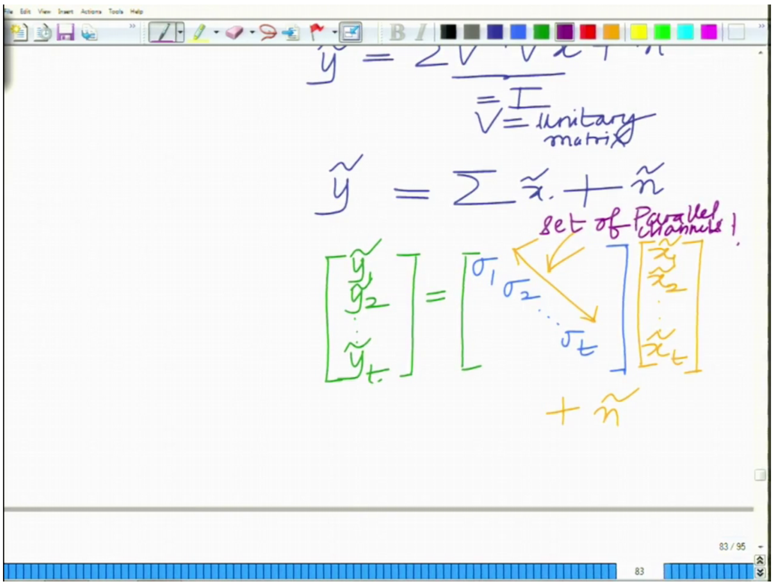

Now, if you can look now we have a set of parallel channels now this is our set of now if we observe closely, you have y 1 tilde equals sigma one times x 1 tilde y 2 tilde equals sigma 2 times. So, this is our set this diagonal matrix in fact, is nothing, but our set of parallel channels that is so, singular value decomposition is what gives us our set of parallel channels.

In fact, you will observe something interesting the gain of the first channel is sigma 1 square, that is alpha a alpha 1 gain of the second channel is sigma 2 square alpha 2 gain of the last channel is sigma t square alpha t and in fact, since sigma's are arranged in decreasing order this implies the alpha 1 this channel 1 is stronger than channel 2 is stronger than channel 3 and so on. Similar to the paradigm, that we have seen in the original, optimal power optimal power allocation module ok so, the framework for optimal power allocation.

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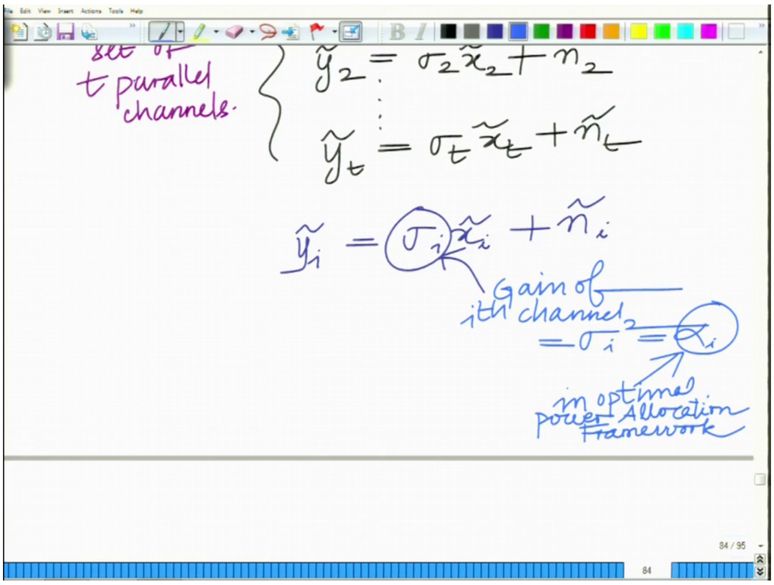

So, what we have is we have a set of parallel channels y 1 equals or y 1 tilde equals sigma 1 x 1 tilde plus n 1s tilde y 2 tilde equals sigma 2 x 2 tilde y t tilde equals sigma t x t tilde plus n t tilde and this is your set of t parallel channels.

This is the set of t parallel channels correct, we have well if you look at the ith channel this is y i tilde equals sigma i xi tilde plus n i tilde gain of ith channel sigma is the amplitude gain. So, the power gain will be sigma i square, equals sigma i square which is alpha i in your optimal power allocation framework, for the optimal power allocation framework alpha i is sigma i square.

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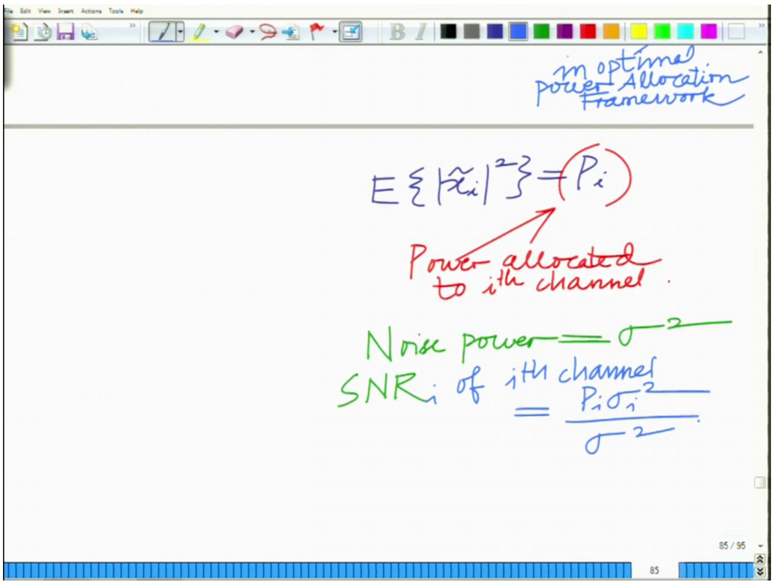

And therefore, now if you have expected that is power allocated to ith channel, that is expected magnitude x i tilde square equals P i, what is P i this is power allocated to ith stream or ith channel, this is the power at which you are transmitting symbols over channel i that is again sigma i square which is alpha i. And noise power we have already seen that is sigma square. Now, therefore SNR of ith channel equals when P i alpha i which is sigma i square divided by sigma square.

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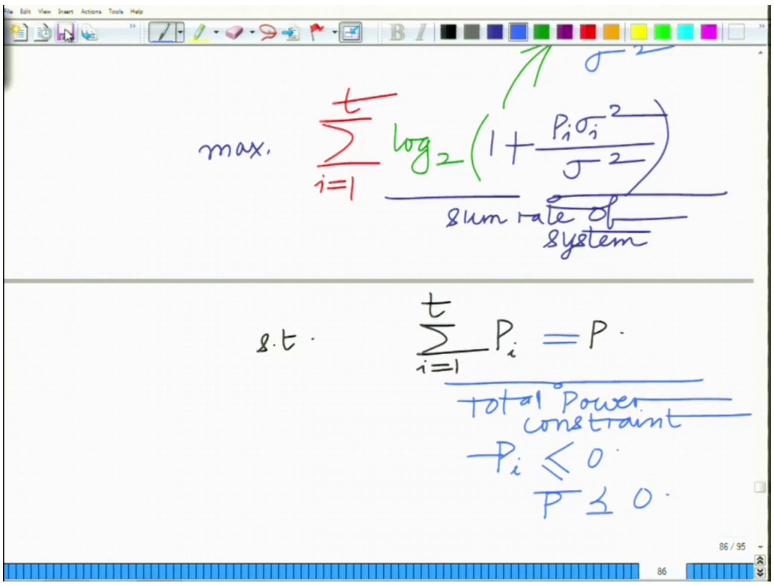

The rate maximum rate of ith channel is log to the base 2 right, log to the base 2 1 plus P i sigma i square divided by sigma square. So, optimization and sum rate will be sum across all the t parallel channels, this is the sum rate of the system. The maximum sum rate will be when you maximize, this when you maximize this subject to the power constraint.

That summation of i equal to 1 to i t P i less than equal to P, this is your total power or this you can talk of this as total power constraint or you can say summation of P i equals P. This is your total power constraint P i greater than or equal to 0, or rather minus P i less than or equal to 0, that is vector P bar is component wise less than equal to 0. This is the non-negativity power constraint.

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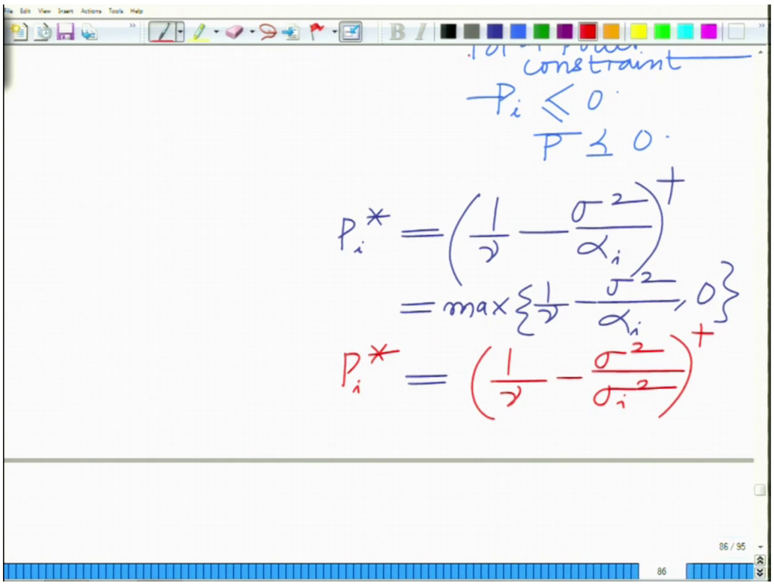

And we have already solved this problem P i or P i star, this is given by the water filling power allocation that is one over nu minus sigma square by alpha i plus, where plus indicate that it is the same quantity if it is greater than or equal to 0 if it is less than that is this is equal to your maximum of 1 over nu minus sigma square by alpha right comma 0 which is nothing. But, if you substitute for alpha i as sigma i square, that is the sigma i is the singular value ith singular value this is 1 by nu minus sigma i sigma square divided by sigma i square plus, this is P i star. This optimal power allocation allocated to the ith mode.

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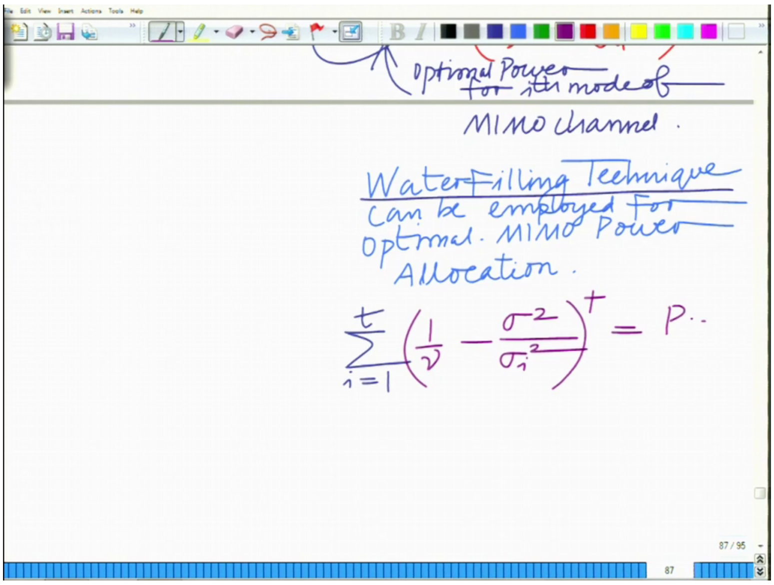

These are known as the modes of the MIMO channel optimal power or ith mode of MIMO channel ok. So, thus what we have shown is that the same water filling paradigm and now this is remember this is nothing, but the water filling power allocation. So, the same water filling paradigm the water filling optimal, water filling water filling water filling technique can be used for optimal MIMO power allocation.

So, thus water filling thus the water filling technique can be employed for optimal MIMO power allocation and, if you observe that these channels the parallel channels are already arranged in the decreasing order of k. So, sigma one square is larger than or equal to greater than or equal to sigma 2 square greater than or equal to so, on until sigma t square. So, naturally first channel is allocated a larger fraction of the power, compared to the second channel compared to the last channel. And there might be some channels remember which are below the water level 1 over nu which are not allocated any power.

And in fact, the technique to find the nu the Lagrange multiplier nu is through the total power constraint correct, 1 equal to 1 to t sigma square y sigma i square plus equal to P ok.

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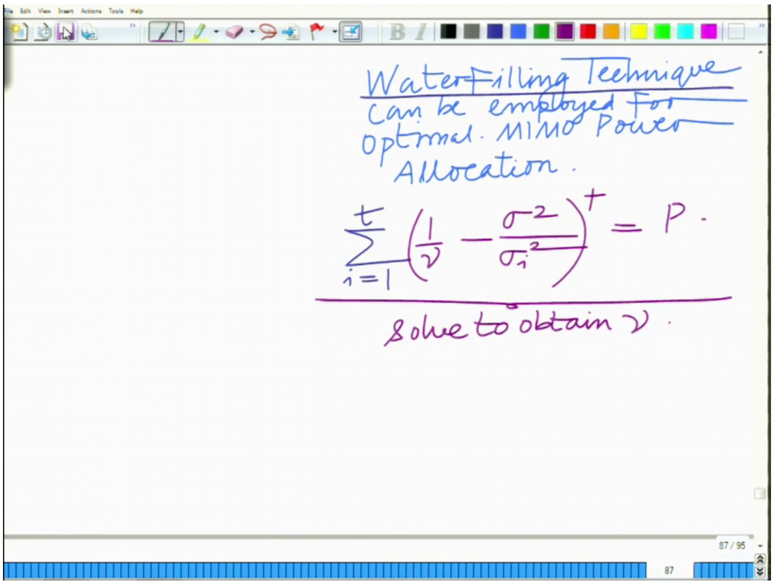

Solving this so, you solve this. So, solve this equation solve this to obtain. So, this is the framework for optimal power allocation. So, the key idea here is that you use the singular value decomposition to decompose, the MIMO channel it was set of parallel channels with the gains given by the squares of the singular values. And naturally once you get these alpha is 1 can use the water filling technique for optimal power allocation, given total power P and noise power sigma square alright. So, we will stop here and continue in the next lecture.

Thank you very much.