Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 66
Karush-Kuhn-Tucker (KKT) conditions

Hello, welcome to another module in this massive open online course. In this module you want to start looking at KKT conditions. So, the Karush-Kuhn-Tucker conditions, which are convenient for solving any optimization problem alright.

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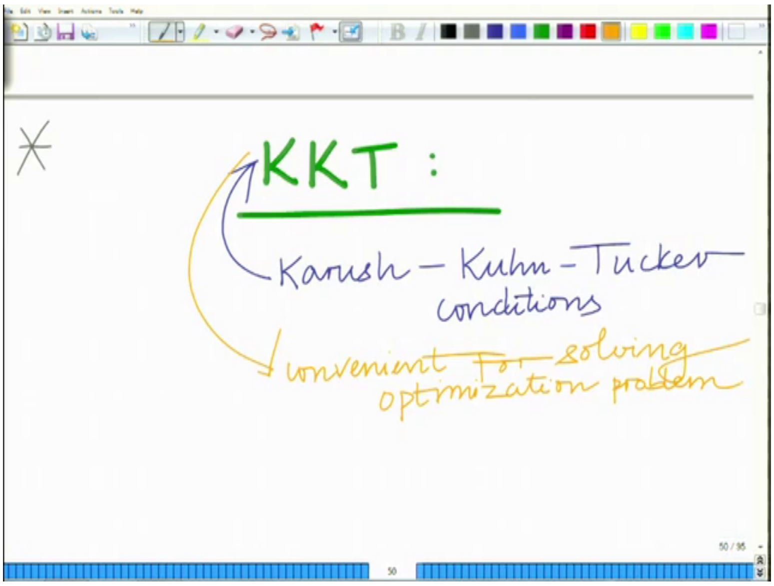

So, what you want us to start looking at is an important of play or aspect of optimization also termed as KKT conditions. Also termed as KKT conditions and these I am sorry, these are the KKT conditions which are also termed as which basically short for the Karush Kuhn or Kuhn Tucker. The Karush-Kuhn-Tucker conditions and these are convenient for solving optimization problem.

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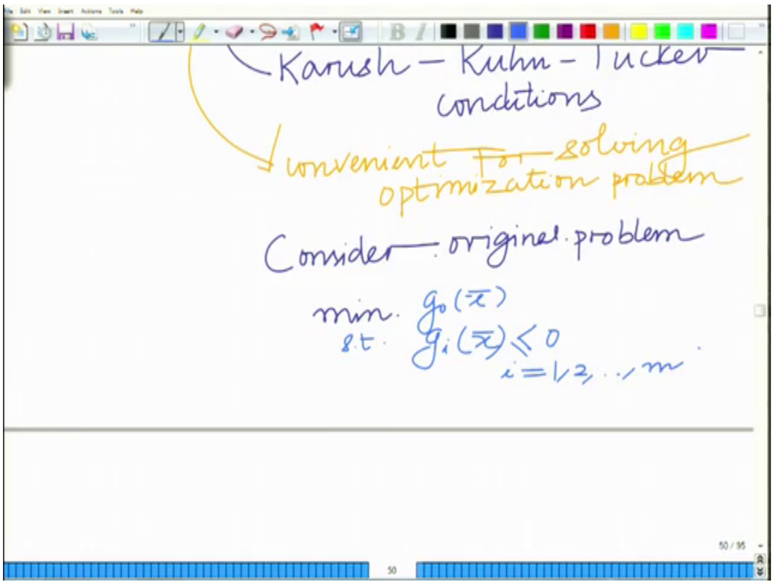

So, now consider again the original or the final optimization problem, consider the original problem what we have is remember we have minimize g naught of x bar subject to g of x bar is equals to 0, i is equals to 1, 2 up to m and g j tilde of x bar equals 0 and I am sorry, i equals to 1, 2 up to l, g j tilde x bar equal to 0, j equals to 1, 2 up to m.

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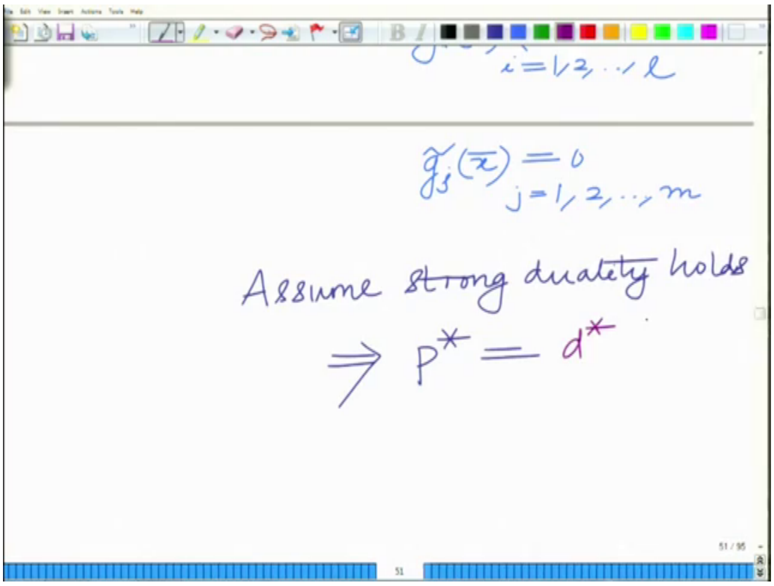

And in addition assumed that strong duality holds. Let us assume that strong duality holds, which implies at P star is equal to d star. Now we know what mean by strong duality. Strong duality employs that P star which is the optimal value of the original that is the primal optimisation problem equals d star, which is the optimal value of the dual optimization problem alright. Where the dual cost function of the dual objective function g d lambda bar nu bar maximizes subject to constrain lambda bar is greater than or equals to 0, you get the dual optimal value d star. And if strong duality holds then P star equals to d star ok.

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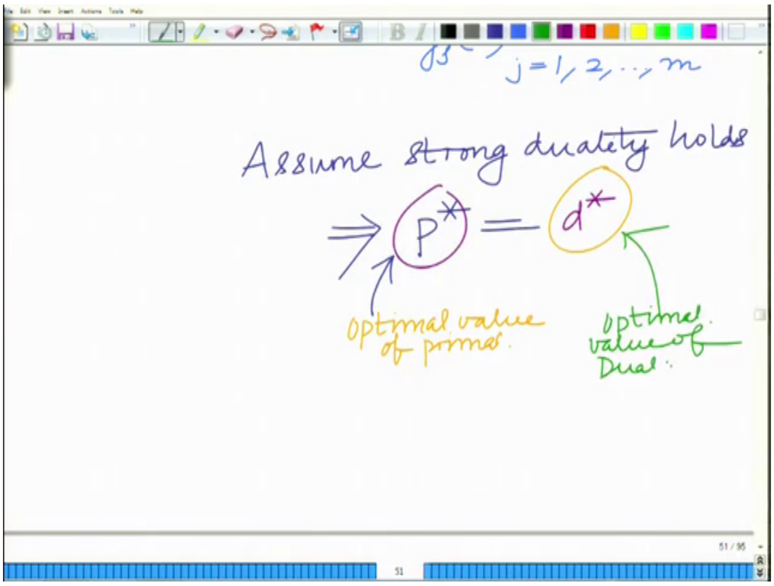

So, P star is optimal value of primal problem and d star is basically your optimal value of the d value; optimal value of the dual optimization ok.

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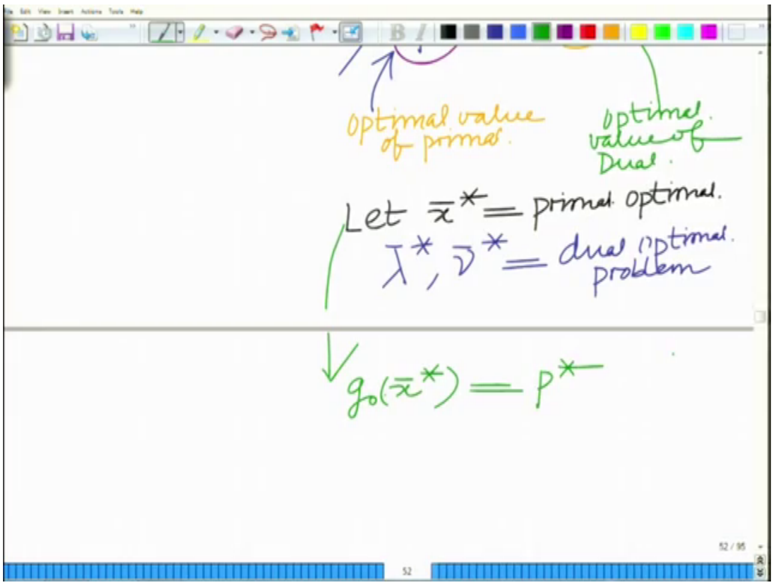

Now, let x star now in addition let x bar star equals primal optimal problem or the primal optimal solution, and that is where the optimal values achieved for the primal optimization problem. That is x bar is the solution of the primal optimization problem.

Similarly, let lambda bar star nu bar star equals the optimal value and not the optimal value that is the solution of the dual optimal problem that is where the dual optimal is achieved ok. Then by strong duality then by duality or rather strong duality, so we have so basically what this means is, that is if you look at the objective value g naught at x start that is equals to P star.

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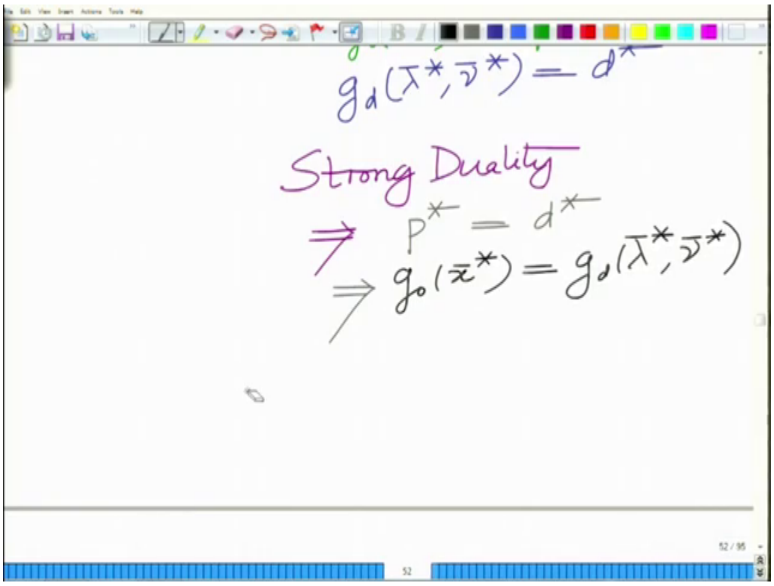

And if you look at the value of the optimization the dual objective function at lambda bar star and nu bar star, this will be the dual optimal value that is this term there is a optimal value of the dual optimization problem alright. So, they are P star and d star.

And now from strong duality, we have already said this implies P star equals to d star, which implies that g naught of x bar star equals the dual function g d of lambda bar star nu bar star.

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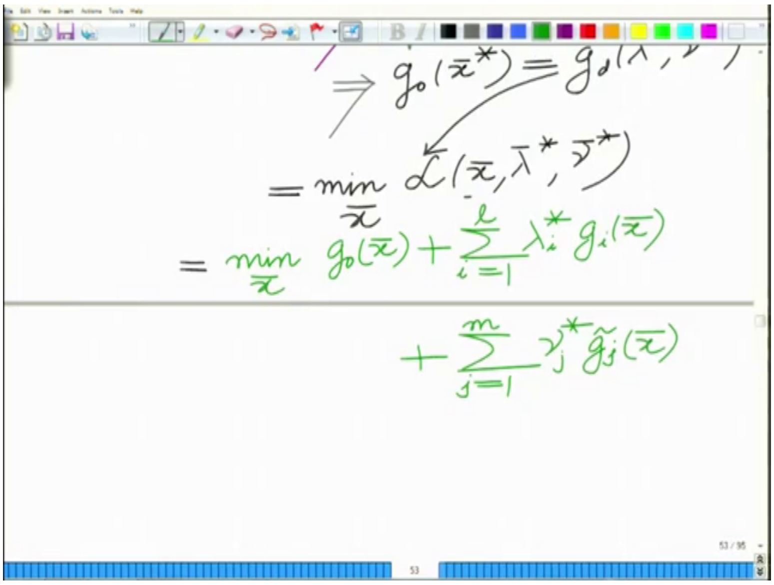

Now g naught; now, if you look at this g d of lambda bar now dual function remember, the dual function is the infimum of the Lagrangian, alright.

So, the dual function g d is nothing but the infimum with the Lagrangian over the set off all feasible points x bar, alright. So, this is basically or infimum or basically is the minimum ok. And this is not the minimum over x bar of g naught x bar plus summation i equal to 1 to l lambda i star g i of x bar plus summation g equals 1 to m nu j star g j tilde of x bar ok.

So, g d of lambda bar star nu bar star is nothing but the infimum of the Lagrangian over all x bar at lambda bar star and nu bar star ok. So, that is the key argument here this transaction ok. Now once you realize that the rest of the argument is very simple.

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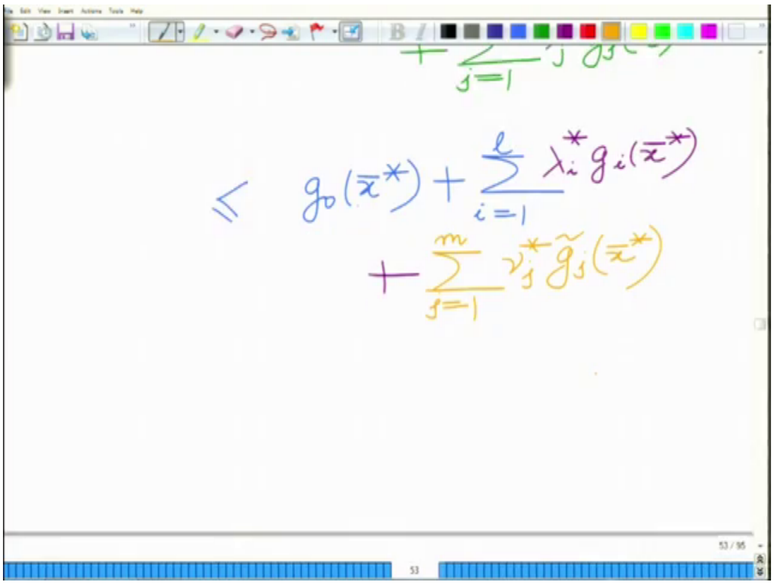

Now, if you look at this, this is the minimum over all x, bar which means this is less than or equal to; so remember the optimal value of this is the minimum or all x bar which means this is less than or equal to the its value corresponding to x bar star ok. So, this is less than because, x bar star is one particular x bar that belongs to feasible set ok. So, therefore, this is less than or equal to g naught of x bar star plus summation i equals 1 to l, summation i equal to 1 to l lambda i star g i of x bar star plus summation g equals 1 to m nu j star g j tilde x bar star, which is less than or equal to g naught.

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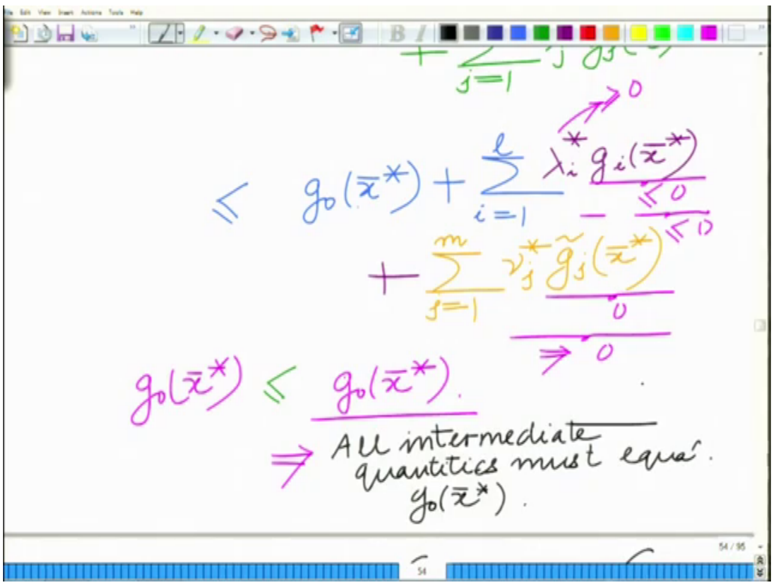

And now if you look at this quantity; now look at this again x bar star is the optimal value belongs to the feasible set satisfy the constraints implies this is equal to 0 which implies this is equals to 0. So, this is equal to 0. Now this again lambda i star remember is greater than or equal to 0, g i x bar star x bar belongs to feasible set x bar star belongs to feasible set. So, this is less than equal to 0. So, this is less than or equal to 0.

So, this is less than or equal to g naught x bar star. And therefore, now you have very interesting observation. You have started from g naught x bar star, you have ended with g naught x bar star, which are indeed equal this implies all the intermediate quantities. So, you have left g naught x bar star less than equal to right g naught x bar which are nothing but one and the same; which means all the intermediate quantities which are sandwiched in between must be also equal to g naught x bar star ok.

So, which means that all intermediate quantities must equal, this is very simple for instance, we are the in equality 5 less than or equal to x less than equal to y less than equal to 5. So, remember the end we have 5, 5 and x 5 less than equal to x, x less than equal to 5. This is only one possibility that is x equals 5, y equals 5.

So, all the intermediate quantities; so similarly if you follow this chain of arguments towards both an g and g naught x bar star, which means all the intermediate qualities must be g naught x bar star.

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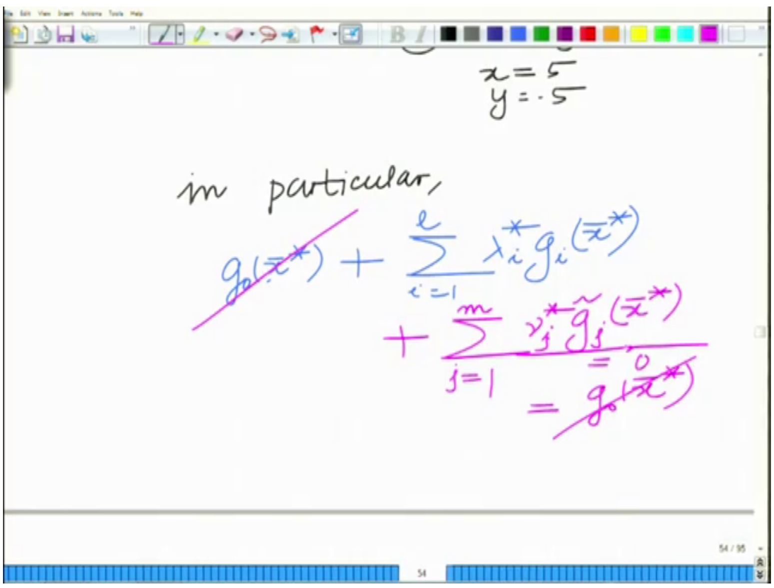

And therefore, this implies in particular we must have summation i equals 1 to l lambda i star gi x bar star plus equals 1 to m nu is nu j star gj tilde x bar star hum this is equals to g naught x bar star, which implies that. Now if you cancel these things g naught x bar star g naught x bar star we know this is 0 anyway.

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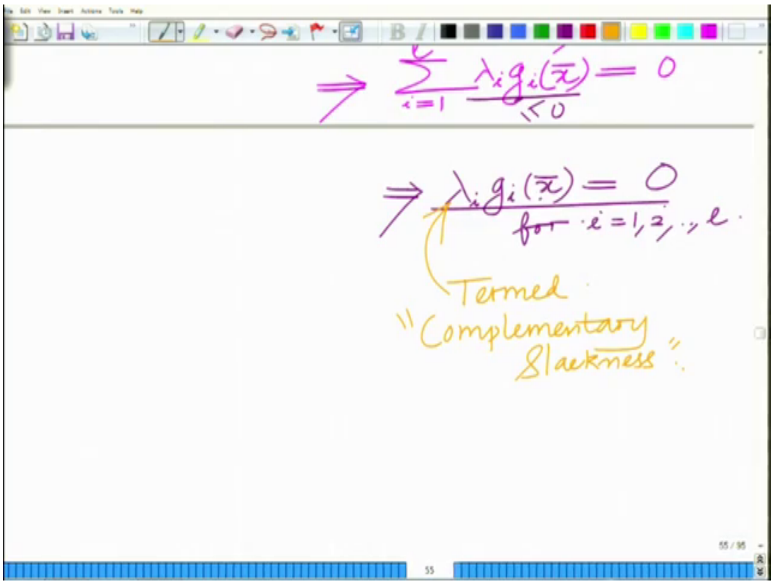

So, this implies that the only remaining term that is summation i equals 1 to l lambda i g i x bar this is equals to 0. Now remember each of this quantity as lambda i g i x bar is less than equals to 0, the sum of several non positive quantities is equal to 0 which means each of this quantities is equal.

You remember each of this quantity is less than equal to 0, but there some is equals to 0 which means each of these quantities equals to 0. This implies lambda i into gi x bar equal to 0 for each i, for i equals to 1, 2 up to l. And this is the very interesting property this is termed as complimentary slackness and I am going to explain this in a movement, this is termed interesting name. The meaning of this is following this implies lambda i into g i x bar is equals 0 which implies either.

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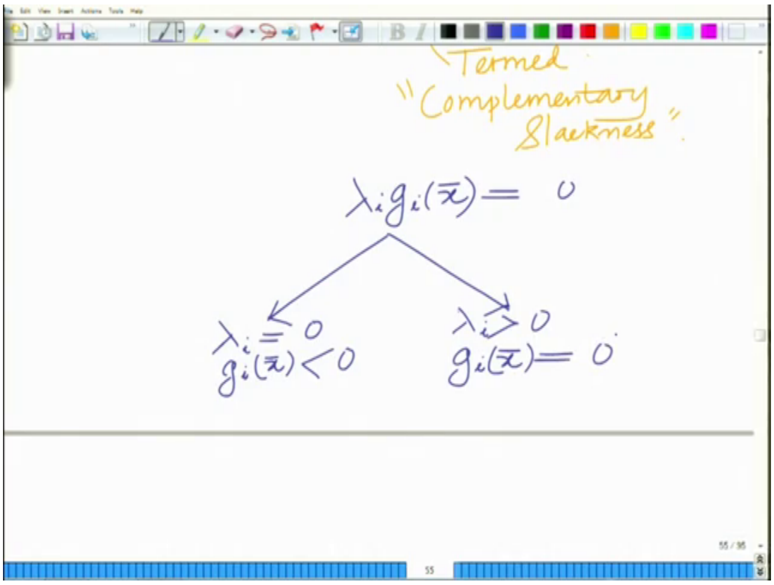

So, we have lambda i g i x bar equal to 0. So, this implies 2 things either lambda i equals 0 correct, which means g i of x bar can be less than 0 or lambda i is greater than 0 and g i of x bar equal to 0.

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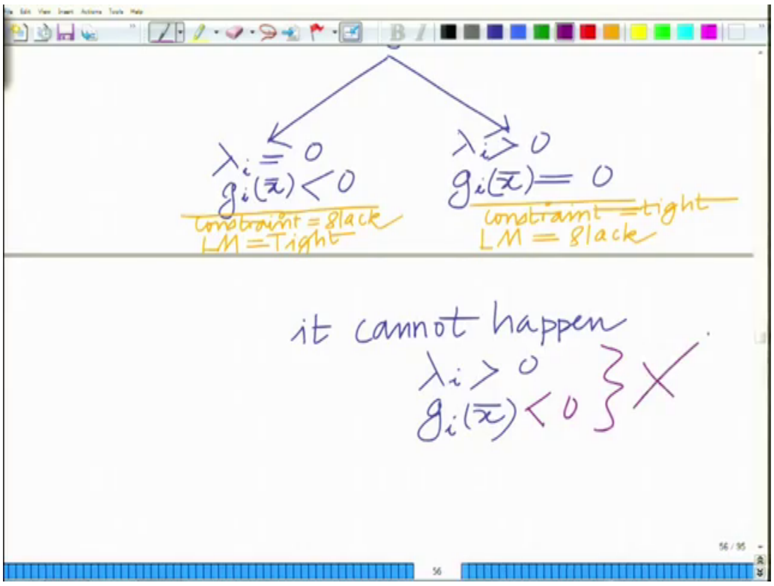

So, in this case now let us look at this in this case, the constraint is tie, it is met with equality. And Lagrange multiplier LM equals slack that is greater than. Now in this case the constraint is slack g i x bar is less than 0, so there is some slack ok, g i x bar is less than 0, which means whatever is gi x bar minus 0 that is the slack, so the constraint is slack.

And LM Lagrange multiplier equals tight because, the Lagrange multiplier equals 0. So, this is the meaning of the complimentary slack that is either the Lagrange multiplier is slack or the constraint is slack. It cannot happen that both the Lagrange multiplier and the constraint are slack; that is it cannot happen that lambda i is greater strictly greater than 0 and g i of x bar is strictly less than 0, this cannot happen because lambda i into g f x bar must equals ok.

So, it cannot happen that lambda i is greater than 0 g j or g i of x bar is strictly less than 0 this is not possible.

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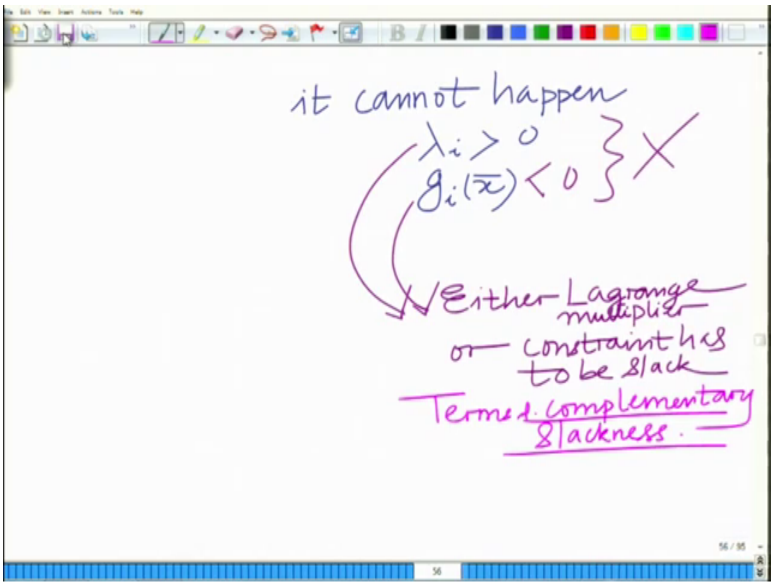

Which means either constraint or Lagrange multiplier has to be slack this termed as to be complimentary slack. So, these complement each other; either Lagrange multiplier or constraint ok.

This is termed this termed as complimentary. This property is termed as complimentary slackness. And this is a unique aspect of the KKT conditions that is that is the both cannot be and both cannot be that is cannot happen that Lagrange multiplier is strictly greater than 0. The constraint inequality constraint is less than 0, it can only happen only that only one of them is slack, let the other is 0.

So, the Lagrange multiplier is 0, the constraint is slack, is the constraint is 0, the Lagrange multiplier is slack and this is known as complimentary slack. So, only one or the can be slack ok, termed only either Lagrange multiplier or constraint can be slack, not has to be slack, can be slack, only one or other can be slack.

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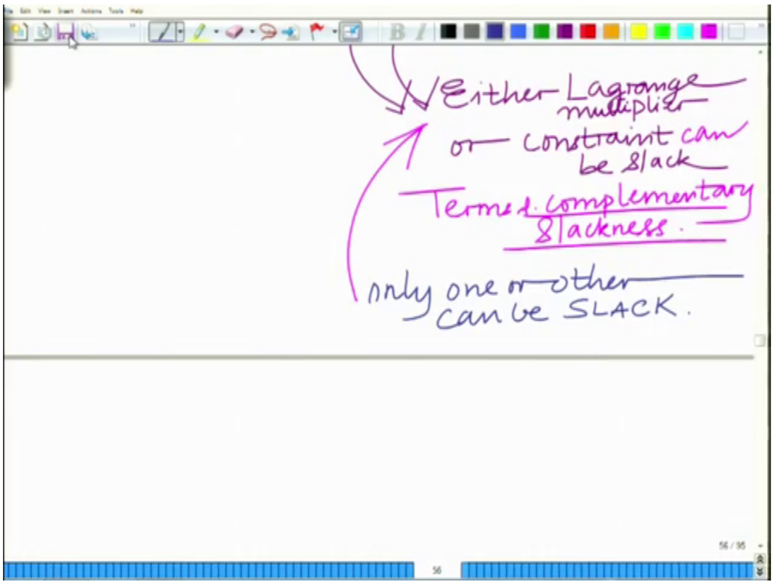

Only one or the other can be slack. That is basically your complimentary slackers. And the KKT conditions can be finally, stated as follows; the Karush-Kuhn-Tucker conditions.

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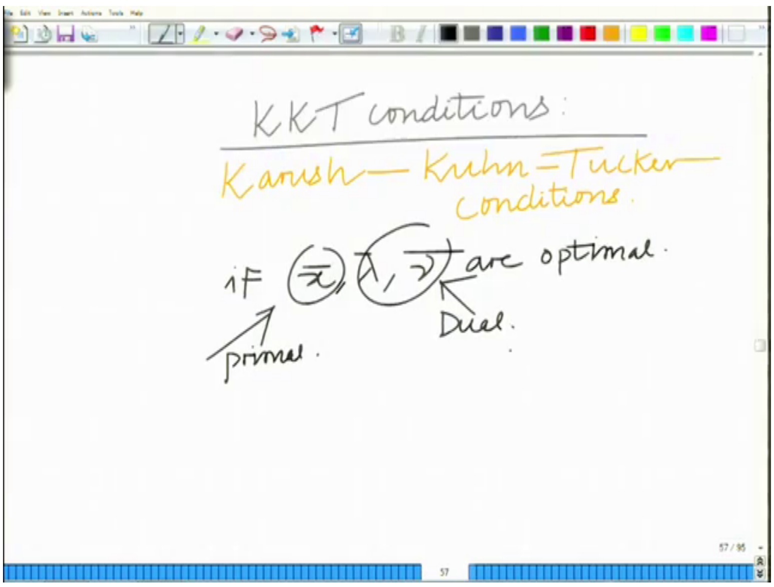

These are your Karush the Karush-Kuhn-Tucker. And the KKT conditions are basically if x bar lambda bar nu bar are optimal, that is a x bar for obviously the primal problem, this is for the dual and strong duality holds.

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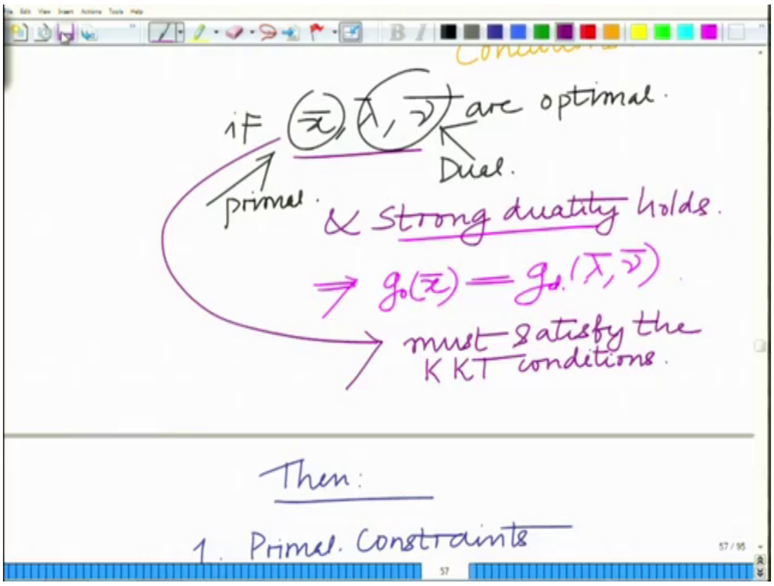

And strong duality holds, this implies that your g naught of x bar equals g d of lambda bar nu bar.

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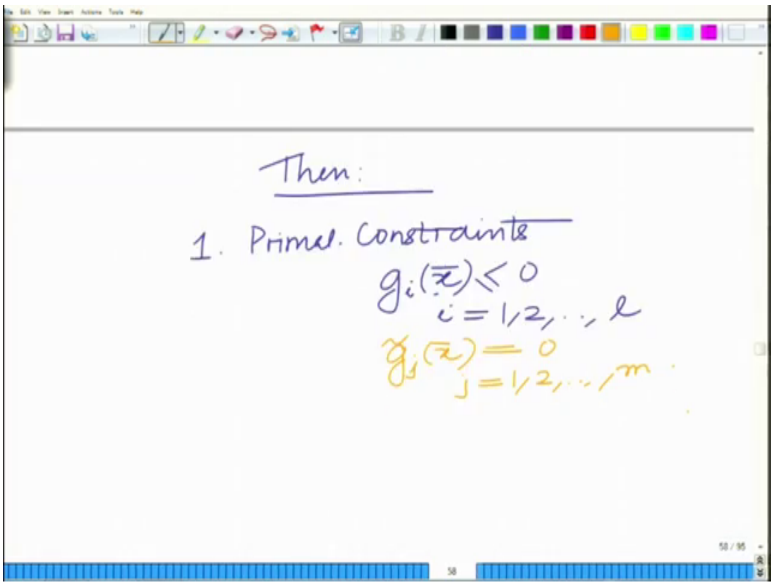

Then it must be that; then first we have the primal constraints, these must hold. The primal constraints are basically g i of x bar less than equal to 0 for i equals 1, 2 upto l, g j tilde of x bar equals 0 g equals 1, 2 up to m.

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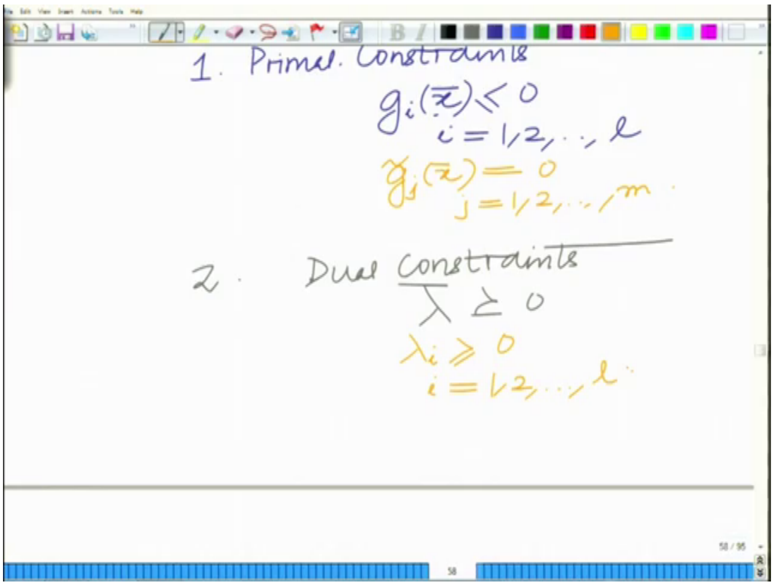

The dual constraints must hold. What are the dual constraints? Remember lambda bar is component wise greater than equals to 0 which implies each lambda i is greater than equal to 0, for i equals 1, 2 up to l that is Lagrange multipliers associated with the inequality constraints are greater than or equals to 0, alright.

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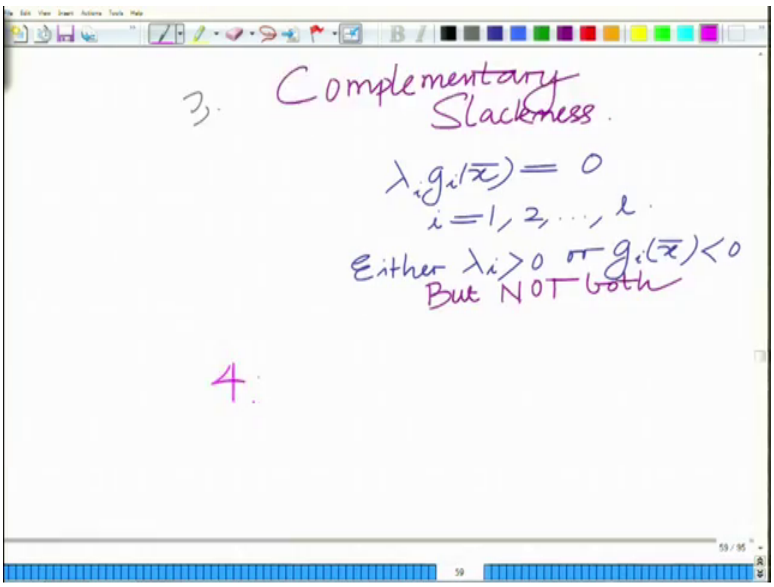

Complimentary slackness, this is the third condition. Complimentary slackness what happens in complimentary slackness, we have with respect Lagrangian, with respect to the gradient, with respect to the Lagrangian equals 0.

This implies that, the gradient of the Lagrangian vanishes that this implies basically the gradient with respect to x because, remember at x bar this is the infimum of the Lagrangian correct, this optimal value x bar we have the infimum of the Lagrangian; which means the gradient we have the minimum of the Lagrangian function. Therefore, the gradient with respect to x bar of the Lagrangian has to vanish at x bar ok. Plus I am sorry, this is not complimentary slackness, complimentary slackness let me just mention the complimentary slackness aspect this will come later.

So, the complimentary slackness we have already seen that. That is lambda i into g i of x bar equals 0, i equals 1, 2 up to l, that is either lambda i equals 0 or g i x bar equals either lambda either lambda i greater than 0 or g i x bar less than 0. That is either a slack, but not both either a slack but not both.

And finally, since x bar is the infimum or at x bar you have the minimum of the Lagrangian, of the Lagrangian function Lagrangian of x bar lambda bar nu bar.

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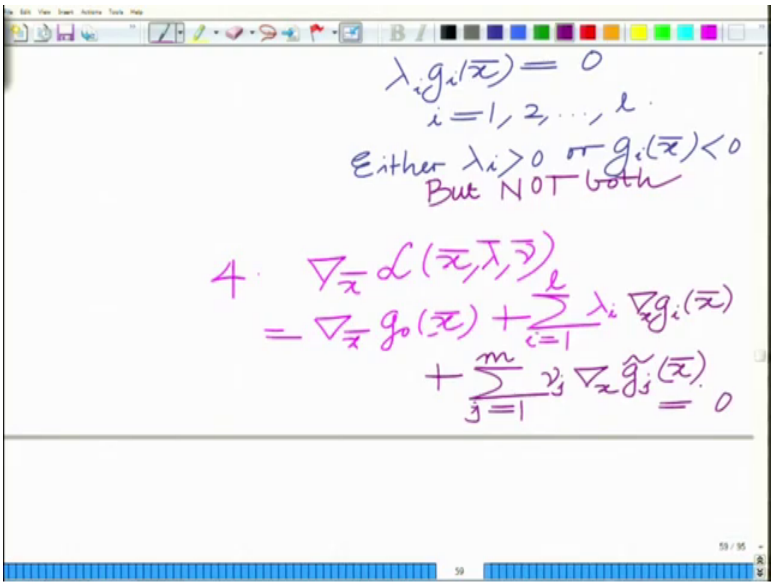

The gradient with respect to x bar of the Lagrangian must vanish at this point x bar. That is we must have delta the gradient of the respective x bar g naught x bar plus summation i equal to 1 to l, gradient with respect to i, g of x bar plus summation g equals 1 to m nu j, the gradient with respect to x of g j tilde x bar, this must be equal to 0, alright. And these are the KKT conditions and if these are x bar lambda bar nu bar are the optimal solutions of the primal, and the dual optimization problems x bar lambda, and strong duality holds x bar lambda bar nu bar must satisfy must satisfy the KKT conditions ok. So, these are the 4 KKT conditions that must be satisfied by the solution solutions x bar of the x bar of the primal optimization problem and lambda bar nu bar of the dual optimization alright.

So, let us stop here and continue in the subsequent modules.

Thank you very much.