Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 06
Properties of Norm, Gaussian Elimination, Echelon form of matrix

Hello. Welcome to another module in this massive open online course. So, you are looking at the concept of inner product and it is various properties. In particular, we have also looked at the Cauchy-Schwarz inequality all right.

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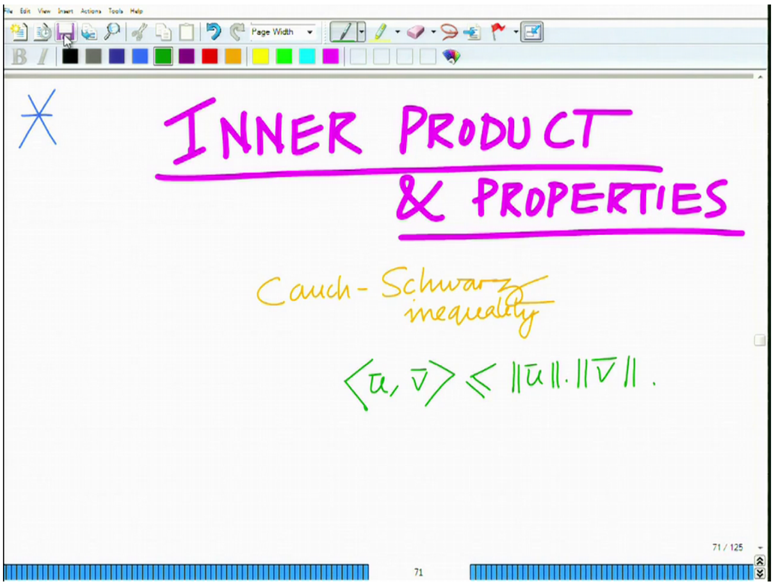

So, what we are looking at is basically this concept of the inner product, the notion of an inner product and the various properties of the inner product. And in particular, we have looked at the Cauchy-Schwarz inequality which states that you have the inner product of two vectors u bar comma v bar that is less than or equal to norm u bar times norm of v bar.

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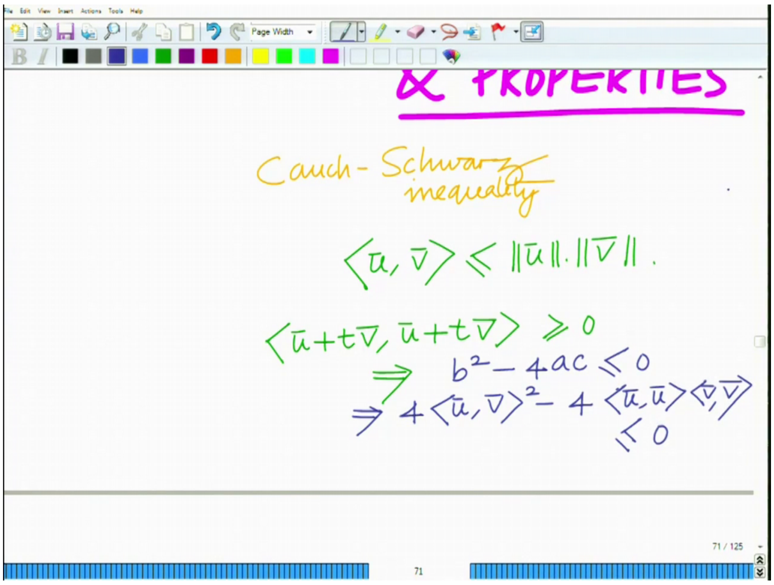

And in fact, in the derivation of the Cauchy-Schwarz inequality, what you have seen is that we have employed the property that the discriminant that is we have looked at u bar plus the inner product of u bar plus t v bar with itself and we have said that this inner product, this is always greater than or equal to 0. This implies quadratic in t is greater than or equal to 0 which implies the discriminant has to be less than or equal to 0. And which implies that the discriminant is nothing but 4 u bar comma v bar square minus 4 well u bar comma u bar times v bar comma v bar, this is less than or equal to 0.

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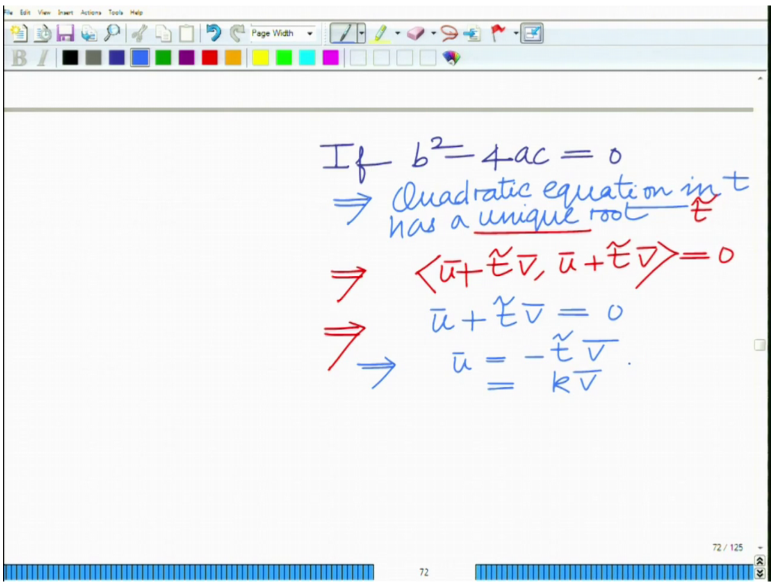

In fact, if the discriminant is 0, all right that is b square minus 4 equal to 4 a c equal to 0, this means that the quadratic equation has a unique root.

Now, if this is the discriminant, if the discriminant equals 0, this implies the quadratic equation. Remember, the quadratic equation in t has a this has a unique root which implies that at that root you will have let us call that let us denote that by let us say t tilde ok. So, at that value of t tilde, we will have u bar plus t tilde v bar, u bar plus t tilde v bar, this inner product is 0. All right at that root because the quadratic is nothing but the inner product of u bar plus t v bar with itself ok. So, at that value of t tilde this inner product vanishes and we know that the inner product u bar between of any vector with itself vanishes only if the vector is 0.

So, this implies that this is possible only if u bar plus t tilde v bar equals 0 which implies u bar equals minus t tilde v bar. For some t tilde, you can denote it as some k constant k times v bar.

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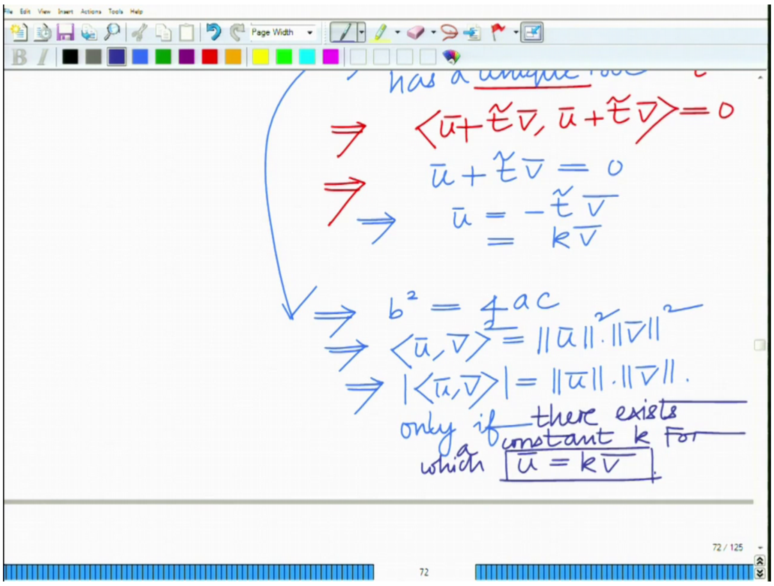

So, this shows that we have what this means is be sure that b square minus 4 a c equals 0, this implies b square equals 4 a c, this implies, inner product of u bar comma v bar equals that is u bar comma v bar square equals norm u bar into norm v bar square. Or this implies magnitude of u bar comma v bar equals norm u bar into norm v bar only if there exists a k, that is determinant is 0 only if there exists a constant k for which u bar equals k v bar.

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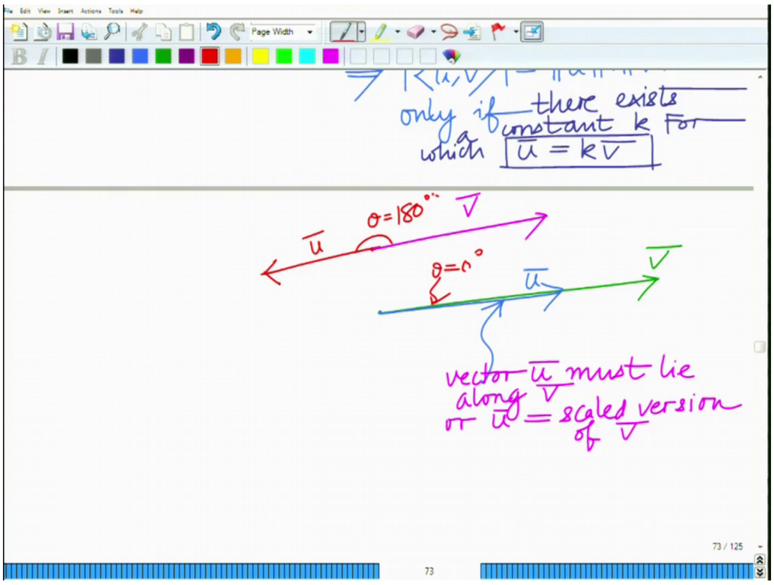

So, the inner product that is the magnitude of the inner product is equal to the product of the norms of the two vectors only if the vector u bar is a constant times bigger. What this means basically is that if we have a vector v bar, then the vector u bar must be k times v bar which means that the vector u bar must also lie along v bar. So, this implies that the vector u bar must lie along v bar or u bar equals scaled. That is simply a scaled version of v bar, ok. So, the inner product between two vectors, magnitude of the inner product between two vectors equals the product of the norms only if u bar rise along v bar or u bar can also be exactly opposite the direction of v bar. In fact, because we are looking simply at the magnitude, you can also have v bar and u bar, ok. So which means either the angle equal to theta equal to 0 degrees or theta equals 180 degrees, ok.

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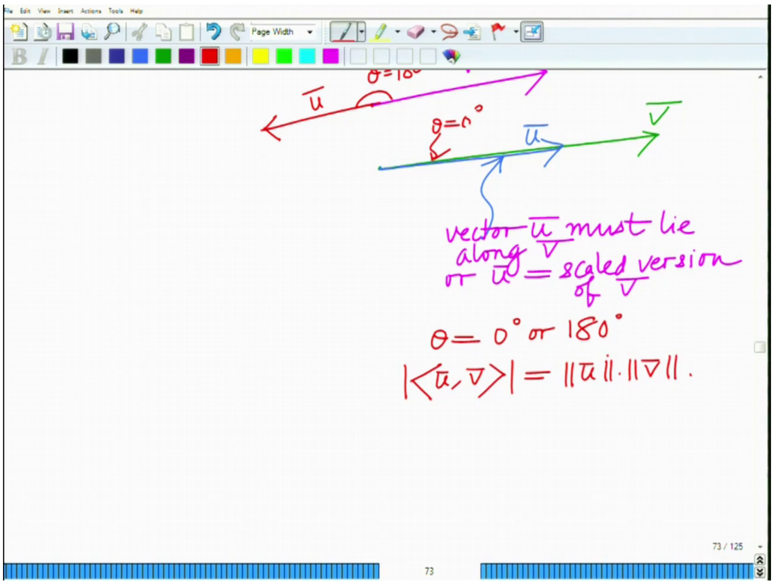

So, in both the cases that is theta equal to 0 degrees or 180 degrees, we will have magnitude u bar comma v bar equals norm u bar into norm v bar. Of course, if theta equals 180 degrees the inner product between u bar v bar is negative, so u bar, v bar is minus norm u bar times v bar that is a inner product between u bar v bar is minus norm u bar times norm v bar and if theta equals 0 degrees, then the inner product is norm u bar simply norm u bar times norm v bar ok. So, they have to be for the inner; so, magnitude of the inner product to equal to the product of the norms, these vectors have to be either aligned or exactly anti aligned or that is an angle between them is 180 degrees ok.

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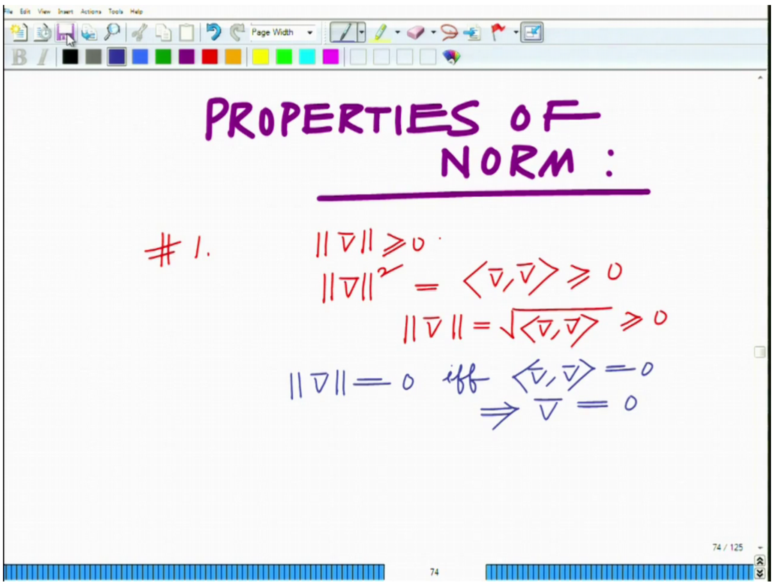

And this is an interesting property that is related to the inner product which we will again frequently invoke at instances during our discussion on optimization. Let us now look at another important concept that is a various properties. So, what we want to look at is the properties of this norm, the properties of the norm and these are as follows. The first property will be that norm v bar is greater than or equal to 0. In fact, norm v bar this is in fact, norm v bar square equals v bar the inner product of v bar comma v bar which is greater than or equal to 0.

And norm v bar equals square root of the inner product between v bar and v bar which is also greater than equal to 0. In fact, norm v bar equal to 0 if and only if, the inner product v bar between v bar and itself equal to 0 which implies we have seen inner product is 0 only in only the inner product of v bar with itself is 0 only when v bar is 0.

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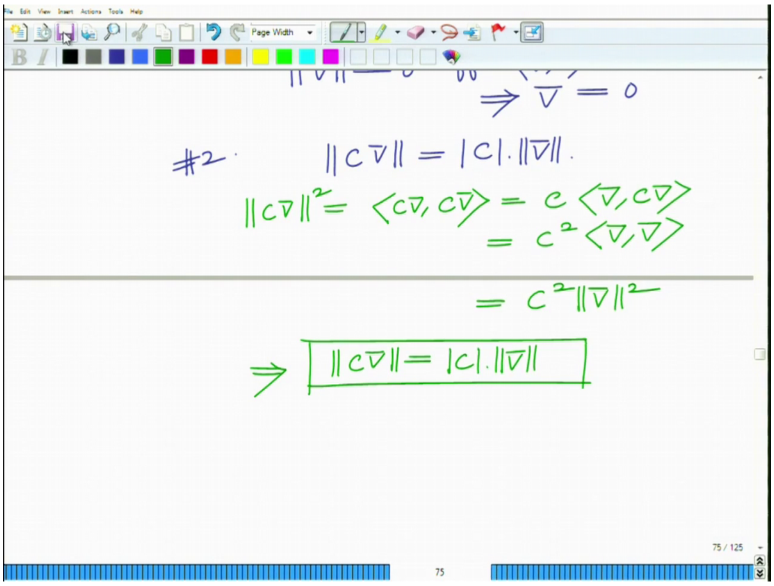

Now, the second aspect is that the norm of c v bar, the constant times v bar equals the magnitude of the constant times the norm of v bar. In fact, you can see the norm of c v bar square equals c v bar, c v bar which is c times inner product of v bar with c v bar which is c square times the inner product of v bar with itself which is c square times the norm of v bar square which implies the norm of c v bar equals taking the square root of on the right, you have magnitude of c times norm of v bar, it was norm of v bar ok.

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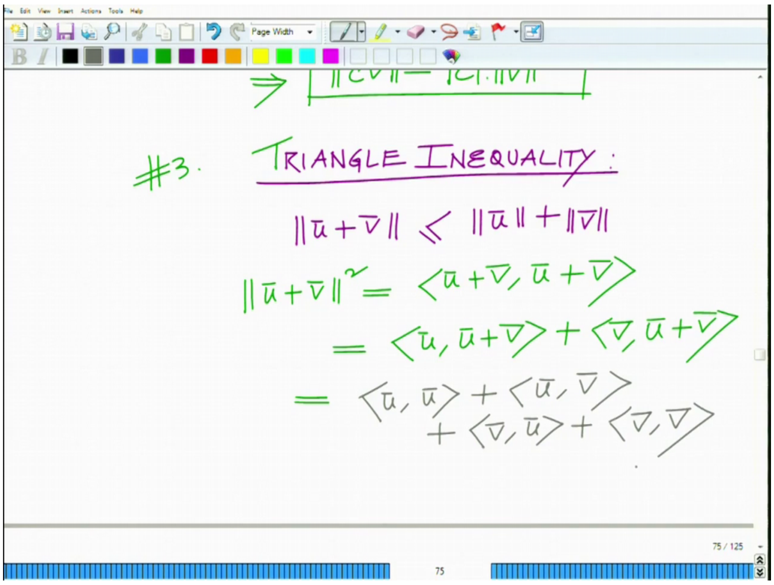

And the third property is the important property. This is known as the triangle inequality. And the triangle inequality, this states that norm u bar plus v bar less than or equal to norm u bar plus norm v bar and this follows as follow this follows, this can be seen as follows if you look at norm u bar plus v bar square that is the inner product of u bar plus v bar with itself which is basically again u bar, inner product with u bar plus v bar plus v bar, inner product with u bar plus v bar which is equal to u bar inner product with u bar plus u bar inner product with v bar plus v bar inner product with u bar plus v bar inner product with v bar equals, well norm u bar square plus twice inner product between u bar and v bar because inner product v bar u bar is the same as inner product u bar v bar plus norm v bar square.

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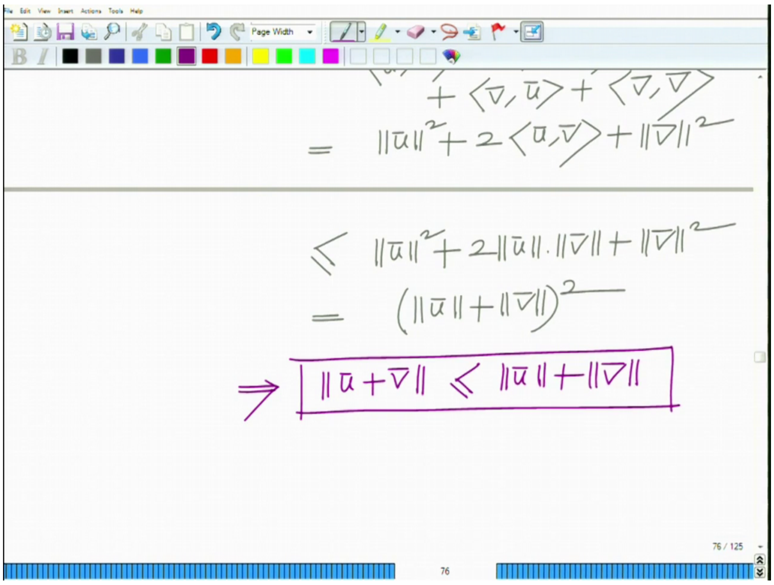

Now, applying the Caushys-Shwardz inequality, we know that the inner product of u bar v bar is less than the product of the norms u bar into v bar. So, this is less than or equal to norm u bar square plus twice norm u bar into norm v bar plus norm v bar square which is equal to norm u bar plus norm v bar whole square, norm v bar whole square. And therefore, we have this implies taking the square root since all the quantities are positive, this implies norm of u bar plus v bar is less than or equal to norm of u bar plus norm of v bar. This is basically the triangle inequality of the norm. This is valid for any norm that is induced by the, this is valid for any norm in particular for the norm induced by the inner product, ok.

So, that basically concludes our discussion on the various aspects of the norm and the properties of the norm that is induced by the inner product. So, now, let us start doing some examples on this mathematical preliminaries. So, what we want to do is, we want to do some examples on this mathematical preliminaries.

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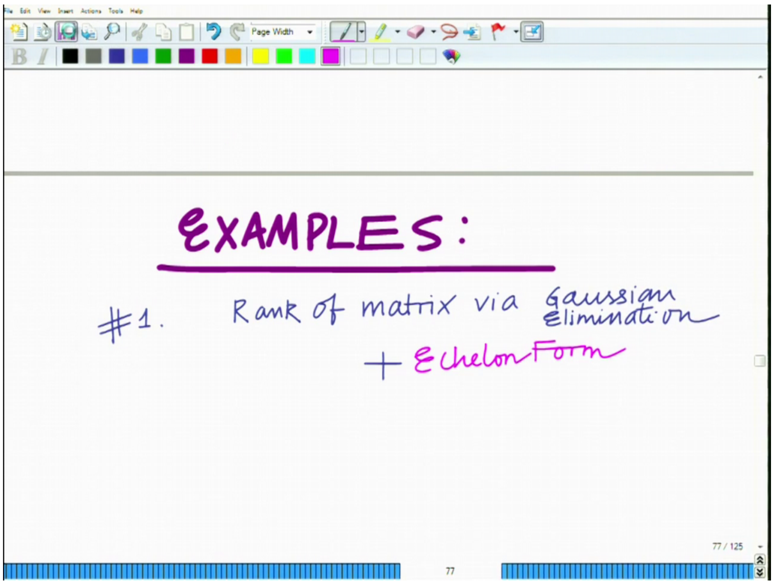

For instance, let us start by considering a simple example. It is an important aspect of matrices that we have not discussed previously which is the row elimination that is a reduction to the row, what is known as the row echelon form or simply the echelon form

So, in this example, what we will want to do is the rank of matrix via Gaussian elimination. So, we have seen the concept of a rank of a matrix and the echelon form and we want to also discuss this notion of an of an echelon form.

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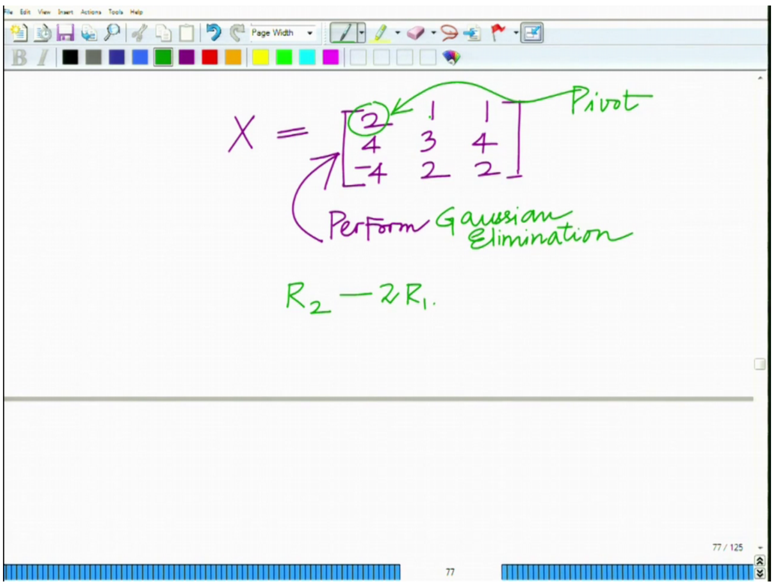

So, we have x let us say, we have a 3 cross 3 matrix, 2, 4, minus 4, 1, 3, 2 and 1, 4, 2. So, what we are going to do is we are going to perform row operations on this. So, what first thing we are going to, we are going to perform Gaussian elimination.

So, first perform R 2 that is row 2 minus twice row 1. So, what we are doing is basically, we are using this element as a pivot as what is known as a pivot and we are subtracting this to reduce a corresponding elements in the second row, we are multiplying the first row all right the pivot all right, we are multiplying it, we are multiplying the first row, scaling it appropriately, subtracting it from the second row so as to reduce the corresponding element in the second row to the element below the pivot in the second row to 0.

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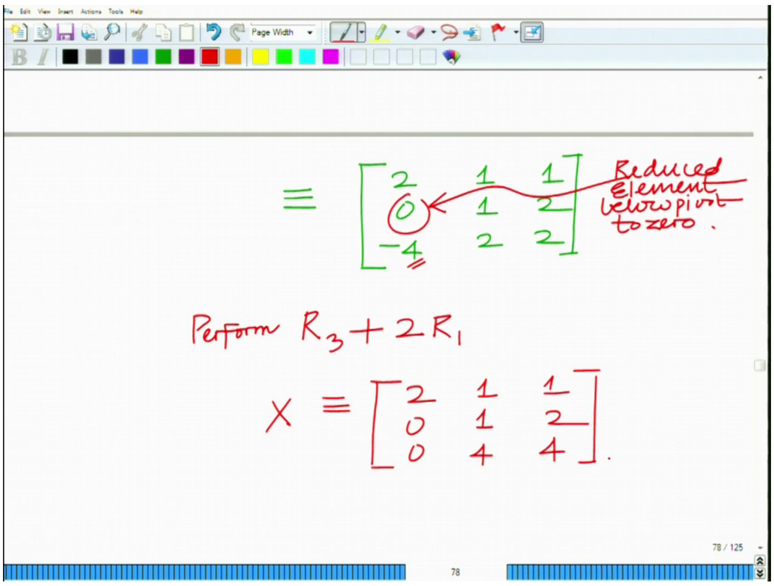

So, this will give you an equivalent representation that is when you multiply 4. So, the first row will remain as it is so, that is 2, 1, 1; the second row will be 0, 1, 2; third row will be minus 4, 2, 2. So, we have reduced this element below the pivot; reduce the element below the pivot to 0 ok. Now, similarly you can also reduce this to 0 by performing R 3 plus 2 R 1. So, we want to perform R 3 plus 2 R 1 and that will give you x equals the first row will remain as it is. Second row will give 0, 1, 2 and the third row will be minus 4, plus 4 that is 0, 2 plus 2, that is 4, 2 plus 2, that is 4.

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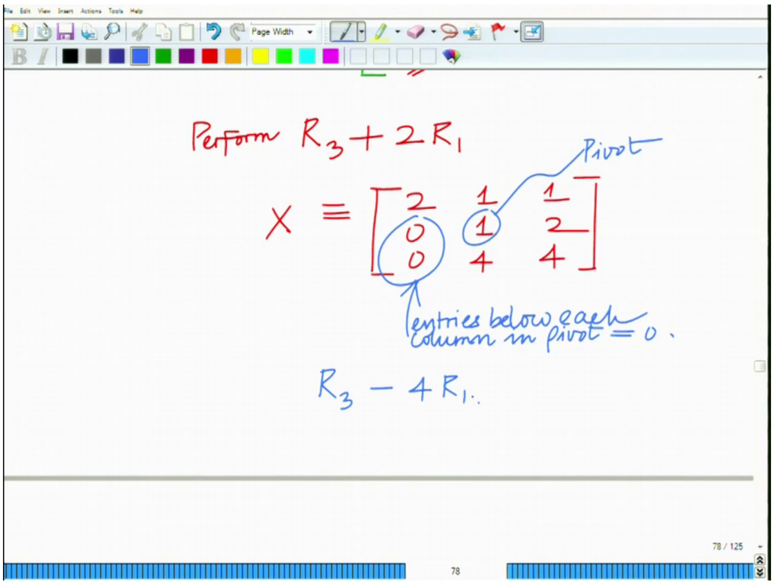

So, what we have done is we have chosen the pivot and we have reduced all the corresponding entries in each column below the pivot to 0 ok. And now, similarly we can use this as a pivot and we can again perform R 3 minus 4 R 1, now.

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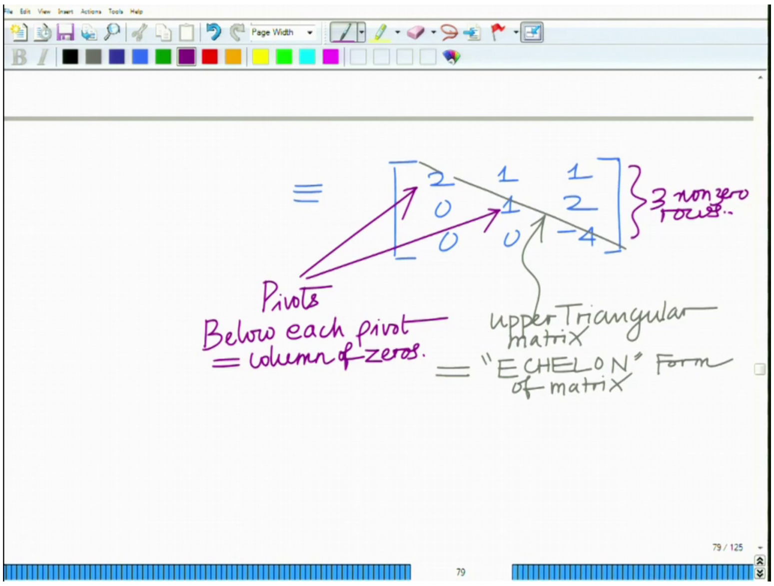

And that equivalently yields the first rows as it is. Second row is also as it is and third will become 0, 0 and 4 minus 8, that is minus 4 and if you look at this now, what you will observe is that this is an upper triangular matrix. If you look at this, this is an upper triangular matrix.

And this is termed as the echelon form of the matrix. This is termed as Echelon form of the matrix and these are the pivots, these are your pivots which are used to scale and subtract from the rows below. These are the pivots and below each pivot, you have a column of zeros below each pivot equals, below each pivot you have a column of zeros. And the interesting property here is if you look at the number of nonzero rows, the number of nonzero, in this case you have 3 nonzero rows and the number of nonzero rows is equal to the rank of the matrix.

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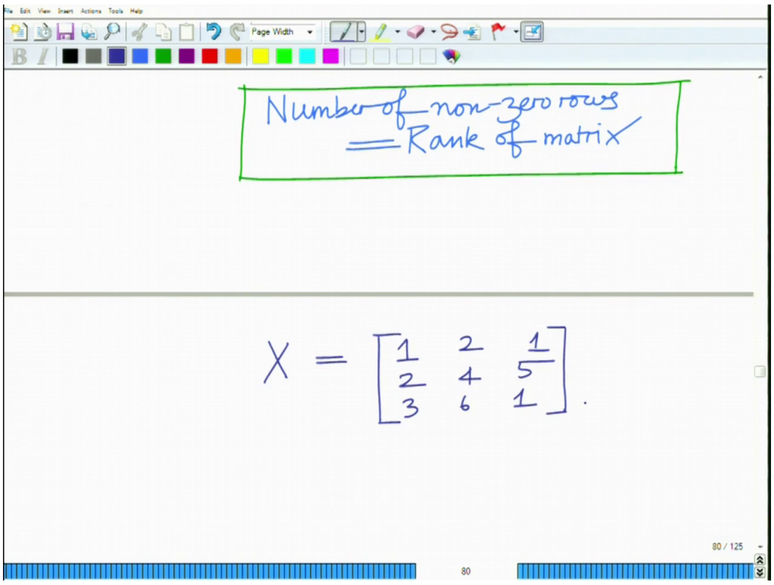

This is an interesting number the number of nonzero rows equals the rank of the matrix and this is basically the interesting. So, one this is how one can see the rank by the, this is how one can deduce the rank of the matrix why are this pivoting and this process of Gaussian elimination during using the pivots and subtracting the scaled versions, the this scaled versions of the rows from the rows beneath ok. Let us now consider another matrix. Consider another example let us consider again a matrix X equals 1, 2, 1 and 2, 4, 5, 3, 6, 1.

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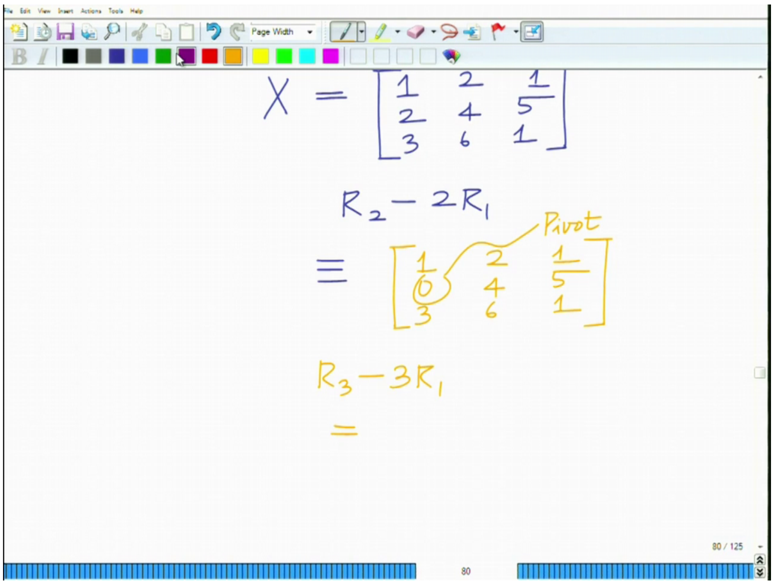

And if you look at this, you will have now perform R 2 minus 2 R 1, that will give you the matrix 1, 2, 1, 0, 4, 5, 3, 6, 1. So, you have this as your pivot. Now perform, R 3 minus 3 R 1 and that gives you well, you can see what that gives you.

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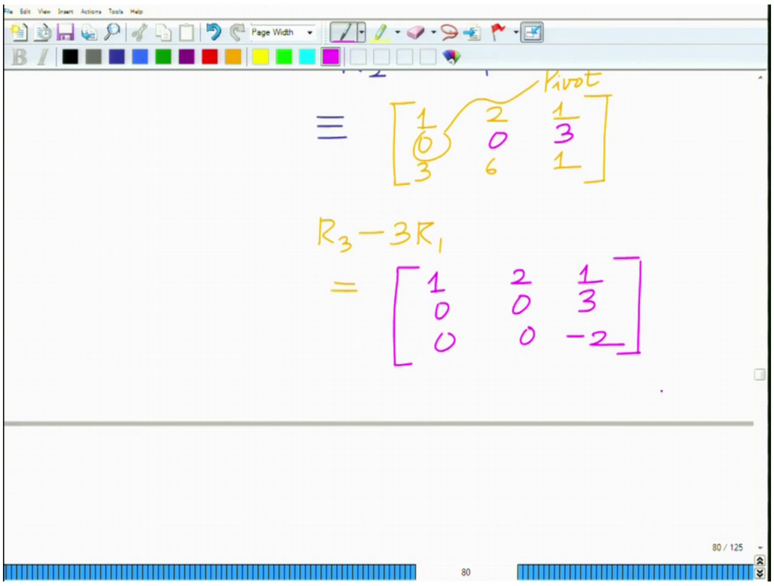

That gives you this is 0 and this is 5 minus 2, this is 3. So, this is 0, 0, 3 and now you will have R 3 minus 3 R 1, this is 0, 0 and this is 1 minus 3, this is minus 2. And now, you can interestingly see that all the pivots possible pivots in the second column are also 0.

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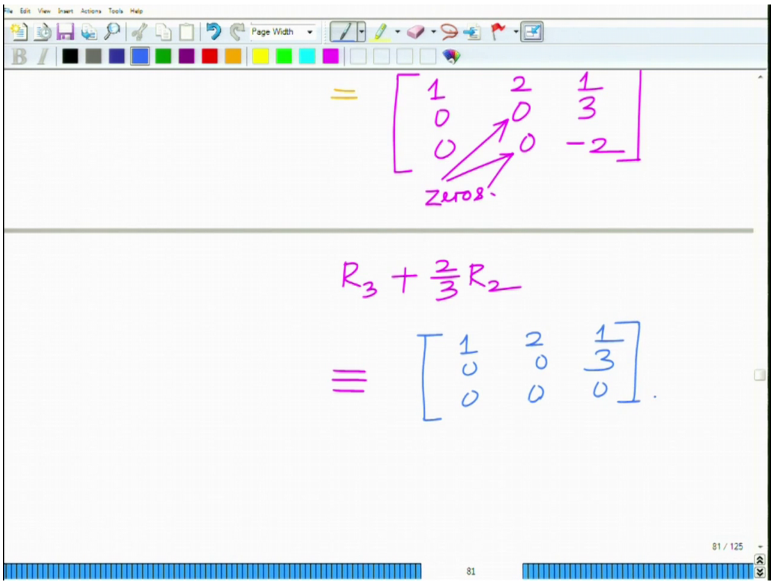

So, these entries which can be used as possible pivots, these are zeros. And now what we do is we perform, R 3 plus 2 by 3 R 2 and that gives us something interesting what you will see is you have 1, 2, 1, 0, 0, 3 and you will finally have a row of all 0s sSo, this is a row

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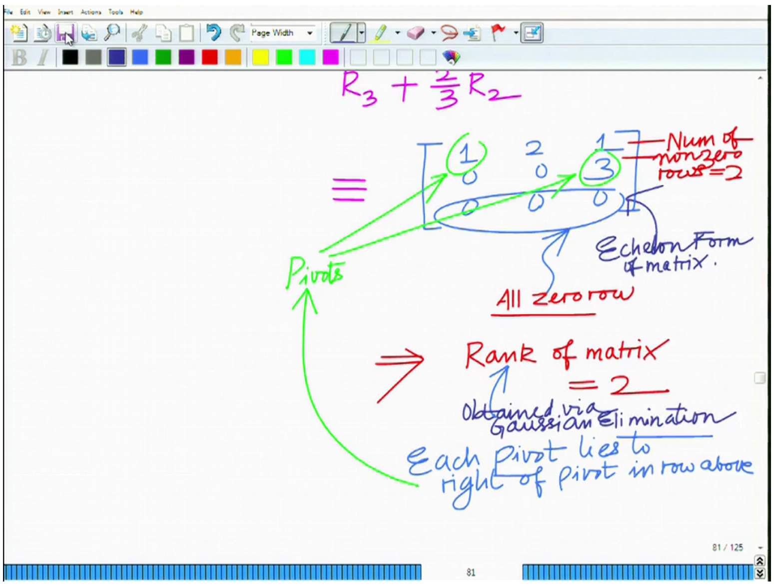

So this, in this Echelon form, you will see you have a row, all 0 row. So, this is an all zero row. So, number of nonzero rows, nonzero equal to 2 and implies the rank, rank of matrix equal to 2 ok.

And in this case, the pivots you can clearly see the pivots that we have used these are the pivots. So, 1 and 3, these are the pivots and note this interesting property of the pivots that each pivot has to lie of the right of the pivot in the row above. So, each pivot each pivot lies to the right of pivot in the row, in the row above. And therefore, the rank of the matrix is 2 and this is obtained via Gaussian elimination, obtained as obtained via.

And this is the Echelon form of this matrix. This is a Echelon form of this matrix and this is how the simple procedure of Gaussian elimination using pivoting can be used to determine the rank of the matrix and reduce it to the Echelon form and this is also very helpful makes it much easier to solve a system of linear equations. We will stop here.

Thank you very much.