Applied Optimization for Wireless, Machine Learning, Big Data

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Lecture - 53
Practical Applications Probability of Error Computation for Co-operative Communication

Hello, welcome to another module in this massive open online course. So, we are looking at co-operative communication, and specifically we would like to eventually look at optimization for co-operative communication all right. So, let us continue our discussion on co-operative communication ok.

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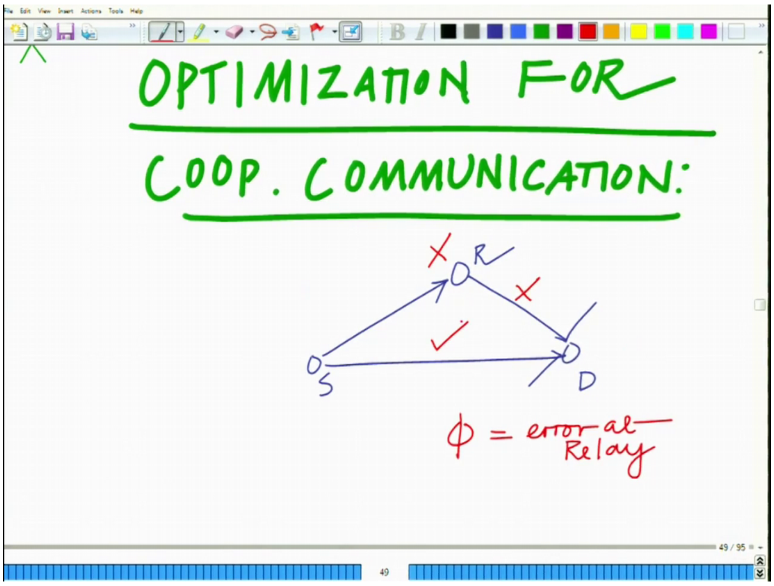

So, let us title this as optimization for optimization for co-operative communication. And what we want to look at more specifically is we want to look at optimization means minimization of something, we want to look at minimization of the error rate ok. And if we have a cooperative communication system, what we are looking as what we want to try to find is what is the error rate of this. And so this is my cooperative communication system with the your source relay and destination nodes. And if you in the event of phi that is when you have an error at the relay, which means the relay is not able to decode the symbol correctly.

Therefore, in selective decode and forward, the relay simply does not retransmit ok. So, the relative destination link that does not exist, because the relay is not if there is an when there is an error at the relay, the relay does not retransmit and selective decoded form. So, only source destination link exist that is the destination use uses only the symbol or the signal received from the source all right. So, in this case there is only the source destination symbol. So, the decoding at the destination takes place on the signal received from the source.

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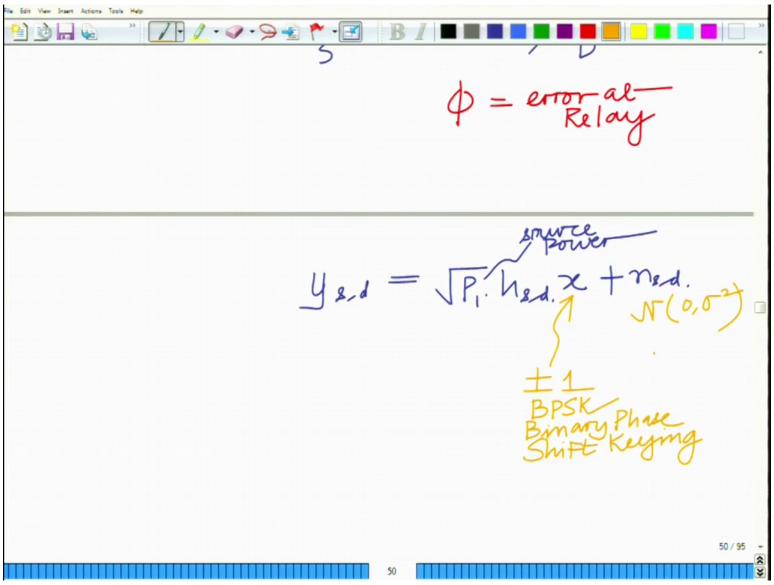

Now, let us try to model this link. This link can be modelled as the received symbol y corresponding to transmission by the y s d, because this is the source destination link, equals well square root of P 1. Let us assume that the source transmits with power P 1 square root of P 1, this is the source power times h s d channel. We have already seen this, this is a flat this is the fading channel coefficient between source and destination times x, which is the transmit symbol plus n s d.

Let us consider this x to be a BPSK symbol that is this is plus or minus 1. So, this h s d is the fading channel coefficient, P 1 is the power, and x is b plus or minus 1 which is BPSK or Binary Phase Shift Keying. And this N is Gaussian noise with mean 0, and additive white Gaussian noise with mean 0 in variance or power sigma square.

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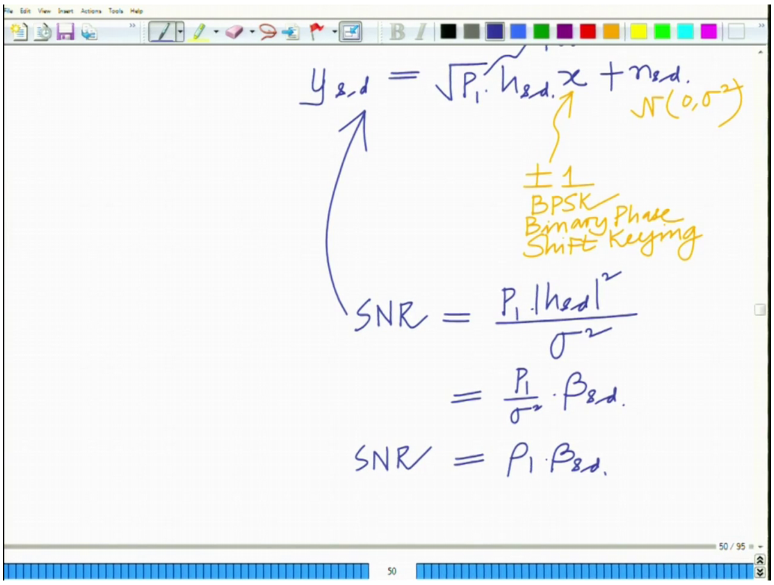

And now if you look at this the SNR at the receiver, the output SNR this is P 1 times magnitude hsd square times x is plus or minus 1, so magnitude x square is 1 divided this is the power signal power divided by the noise power, which is sigma square. I can write this as P 1 divided by sigma square times magnitude h s d square, we have already seen this is equal to beta s d all right. And I will further define P 1 over sigma square as rho 1, so I can write this as rho 1 times beta s d. So, this is your SNR all right at the output all right. When the relay is decoding in error, there is only so only the source destination link exist. There is the destination can use only the signal that has been transmitted by the source ok.

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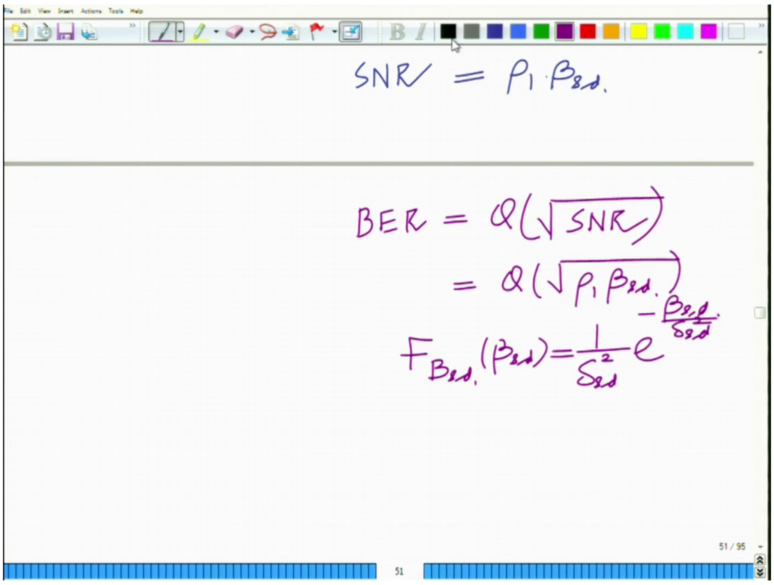

Now, what is the bit error rate? Remember for the for b p since we are considering BPSK modulation, the bit error rate has a very simple expressions. The bit error rate is the Q function of square root of SNR, which is equal to Q function of square root of rho 1 times beta s d. And now remember, we have beta s d, which is a random quantity. And we have seen in the previous module that this is exponentially distributed, it is average power delta S e d square e raised to minus B s d by delta s d square. This is a probability density function of beta s d, which is the channel gain and distributed exponential.

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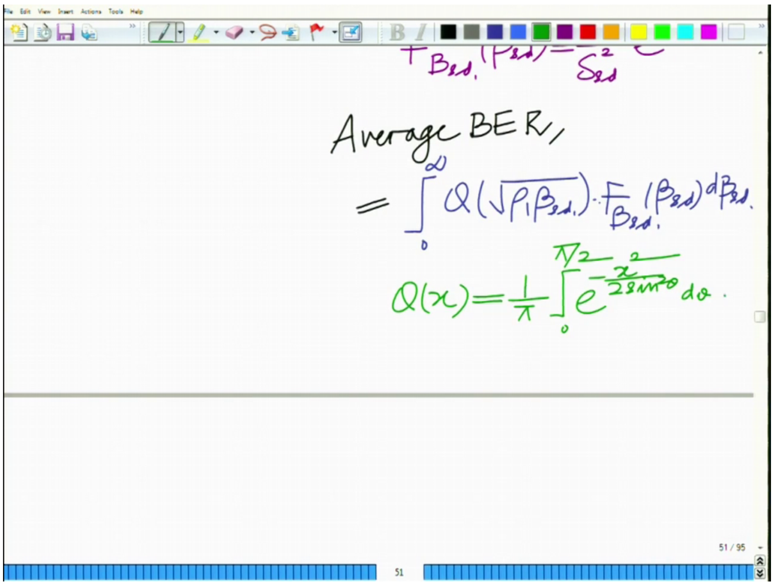

And therefore, the average bit error rate. So, to find the average bit error rate, why do we have to find the average bit error rate? Remember beta s d that is the gain of the channel coefficient the channel coefficient, the fading channel coefficient is a random quantity. So, it is changing from time to time, so varying with time. So, naturally this bit error rate, which depends on beta is going to also change from slot to slot or from time to time all right.

Therefore, we want to find what is the average bit error rate, corresponding to these observations or these decoded symbols that are decoded symbols, decoded symbols at the destination over a long period of time ok. And that average bit error rate is given as well to compute the average of a random quantity, you multiply this a bit error rate Q of square root of rho 1 beta s d, you multiply it by its probability density function F of beta s d, and you integrate ok. So, multiply the probability by the probability density function and integrate over its domain that is from 0 to infinity ok. And here, we are going to use the formula for the Q function, Q of x equals 1 over pi integral 0 to pi by 2 e raised to minus x square by 2 sin square theta d theta, this is also known as the craigslist formula.

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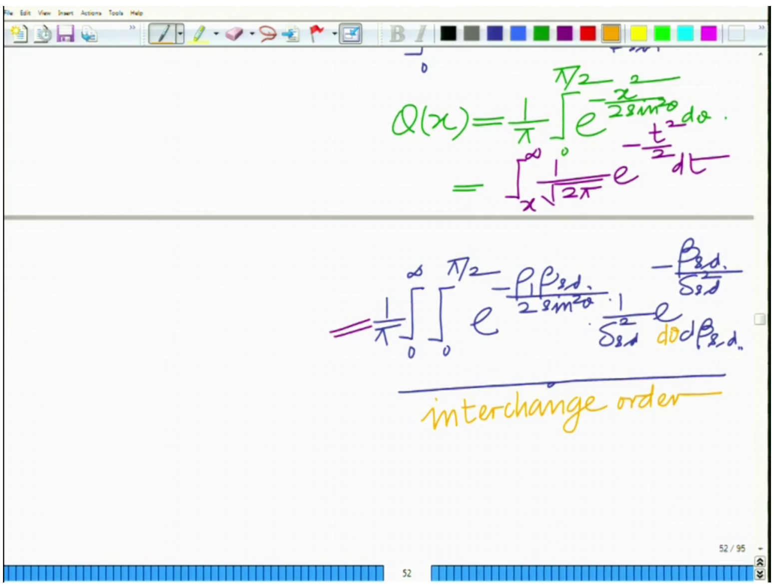

And using this, this is an alternate you can think of this as an alternative definition of the Q function normally. The Q function is defined as x to infinity 1 over square root of 2 pi e raised to minus x square by 2 d x I am sorry e raise to minus t square by 2 d t that is the tail probability of the standard normal random variable, but this is an alternative definition of the Q function, which is convenient in this scenario.

So, using this I have the average bit error rate is well, I will replace the expression for the Q function 1 by pi is a constant 1 over pi. So, I am going to take that out, so that will give me 0 to pi over 2 e raise to minus x square, which in this case is rho 1 beta s d by 2 sin square theta times 1 over delta s d square e power minus beta s d divided by delta s d square d beta s d. So, I am integrating the bit error rate over the probability density function of random variable beta s d, which is the gain of the s d channel.

Now, what I am going to do is I am going to interchange the integrals ok. This is this integral is first with respect of course this has to be theta, so there has to be d theta; this integral first with respect to theta. Next with respect to B s d, do, I am interchanging the order. So, first now I am going to make the inner integral with respect to beta s d, the outer integral with respect to theta.

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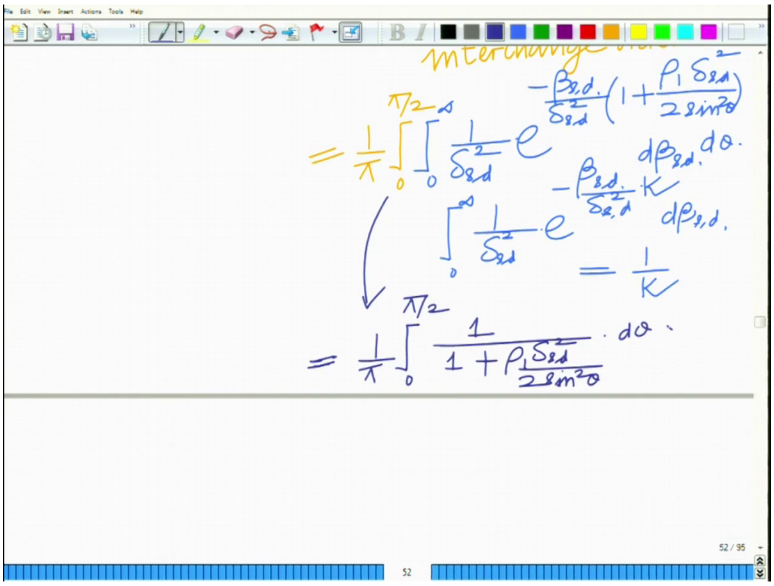

So, this is going to become 1 over pi 0 to pi by 2 0 to infinity, and combine the inner terms 1 over delta s d square e raised to minus beta s d divided by delta s d square into 1 plus rho 1 deltas s d square divided by 2 sin square theta d beta s s d into d theta. Now, if you look at the inner integral just pay attention to the inner integral, inner integral is of the form 1 over delta s d square e raised to minus B s d beta s d divided by delta s d square into some constant K times d beta s d integrated from 0 to infinity. Why is this constant K? Because remember this K it depends on theta, but theta is a constant. When you are looking at the inner integral, the quantity theta is a constant is it is K ok.

So, if you can denote this by K, and therefore this integral is simply if you evaluate this inner integral, this is simply equal to 1 over K, where K is what is K 1 plus beta 1 plus I am sorry this is not 1 plus beta 1, this 1 plus rho 1 ok. K is 1 plus rho 1 delta s d square divided by 2 sin square theta. And therefore, this now reduces to 1 over pi integral 0 to pi over 2 1 over 1 plus rho 1 delta s d square divided by 2 sin square theta d theta. There is a convenient way to evaluate this.

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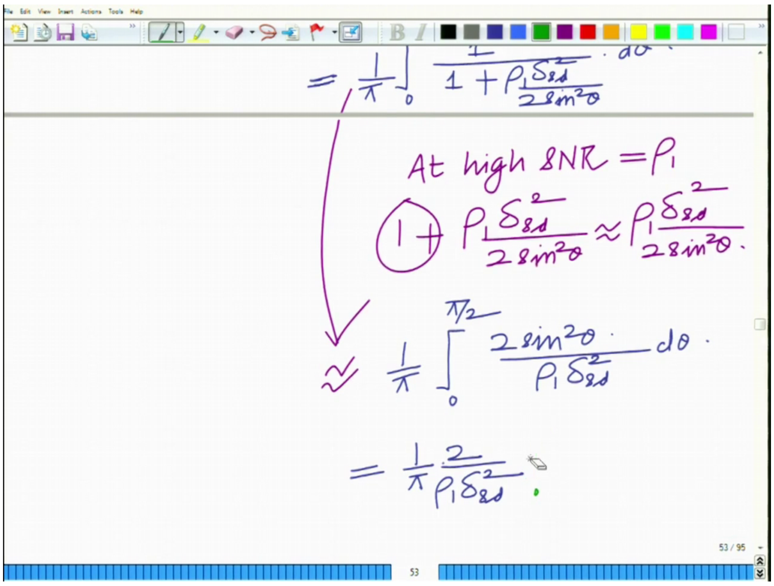

Now, observe that if you at high SNR at high SNR equal to rho 1 that is when rho 1 is very high. 1 plus rho 1 delta s d square divided by this is approximately equal to rho 1 delta s d square divided by. So, I can neglect this one all right, because when rho 1 becomes high, the second term dominates in the sum. So, I can simply approximate this by rho 1 delta s d square over 2 sin square theta, which means this integral now approximately becomes 1 over pi 0 to pi by 2, so the 1 goes, so this simply becomes 2 sin square theta divided by rho 1 delta s d square d theta, which is well 1 over pi 2 over rho 1 1 over pi 1 over pi 2 over rho 1 delta s d square.

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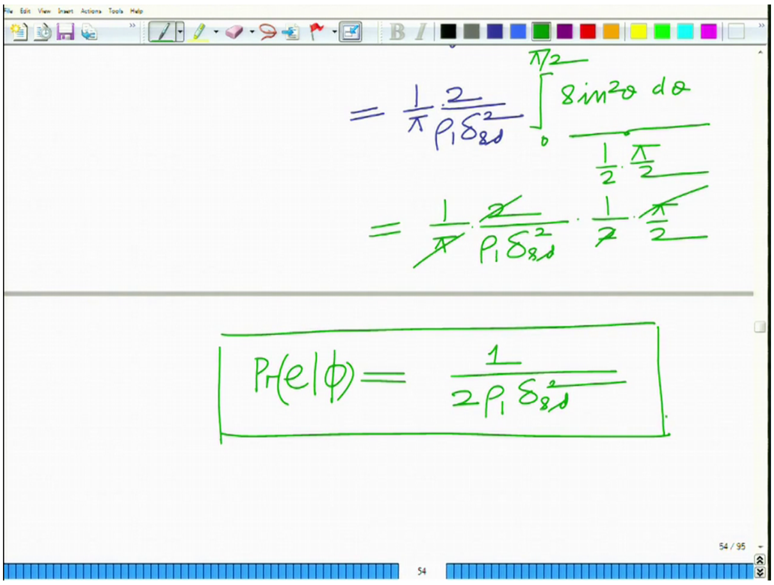

Times integral and integral 0 to pi by 2 sin square theta is easily evaluated that is 1 minus cosine 2 theta divided by 2, so that is integral of cosine 2 theta, obviously between 0 to root of pi by 2 is 0. So, this is 1 over 2 times pi by 2. And if you (Refer Time: 13:49), if you multiply 1 over pi 2 over rho 1 delta s d square into 1 over 2 pi by 2 the pi terms cancel, the 2's cancel. And what you have is basically at that point you have 1 over 2 rho 1 delta s d square, which is your probability of error given phi remember. This is the quantity that we are talking we were talking about earlier.

What is this? this is the probability of error. The probability is e given phi remember phi is the error at the relay. In the event of which there is no relay to destination transmission, so there is only source destination transmission. And this is the probability of error at the destination for decoding the BPSK symbol transmitted by the source given the error at the relay.

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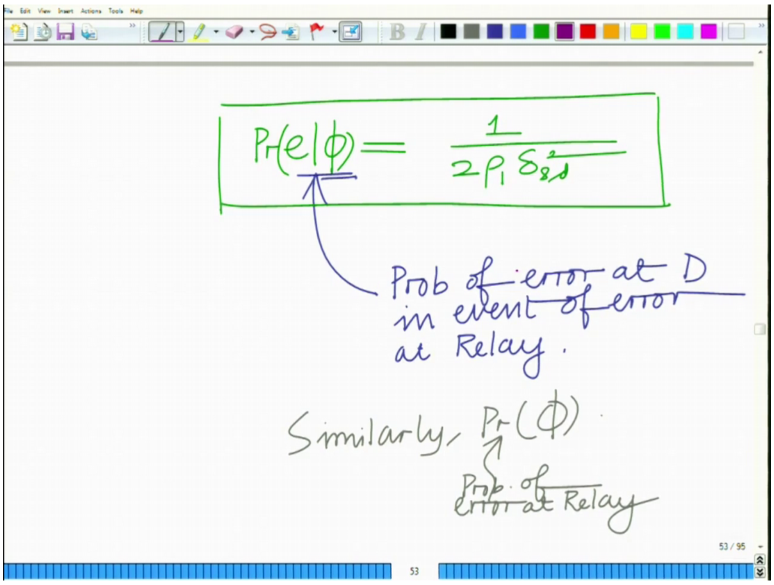

So, this is probability of so describe it in detail, probability of error at destination in event of ok. So, this is your probability of e given phi error at destination given phi that is error at relay. Now, the other thing that we need is the probability of phi, similarly probability of phi. Now, what is probability of phi? And similarly probability of; Now, what is probability of phi, this is error at relay or rather probability of and what is the probability of an error at relay.

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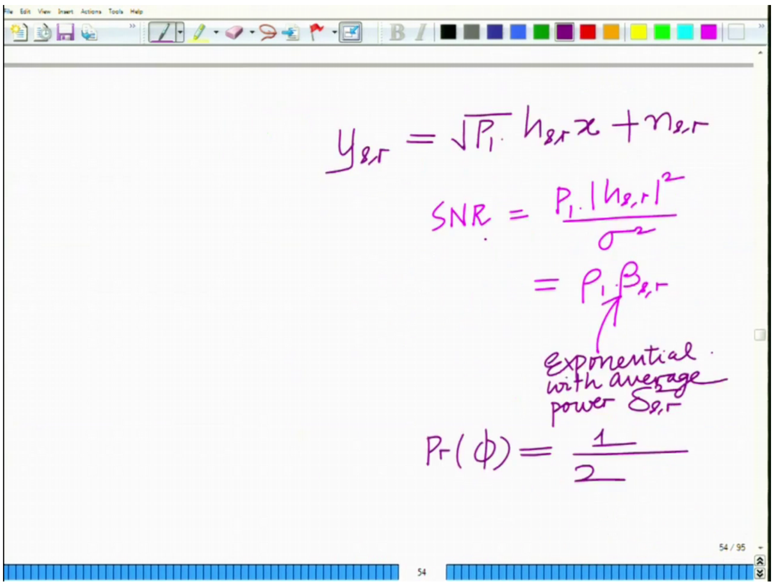

Remember the source to relay link also a fading link. So, I can model this is as y of s comma r, source power is P 1 square root of P 1 h of s comma r times x, which is the same symbol n of s comma r. Now, what is the SNR, SNR is P 1 magnitude h s, r square divided by rho sigma square, which is nothing but rho 1 beta s, r. So, SNR is basically exactly same, as that of source destination link with beta s d replaced by beta s, r.

Remember beta s, r is also exponential random variable with average power delta s, r square ok. So, this recall that this is also exponential with average power. This is also exponential average power delta s, r square. So, therefore the error rate probability of error is simply 1 over if you look at the earlier expression, 1 over 2 rho 1 delta s, d square I simply have to replace delta s, d square by delta s, r square corresponding to the source relay link.

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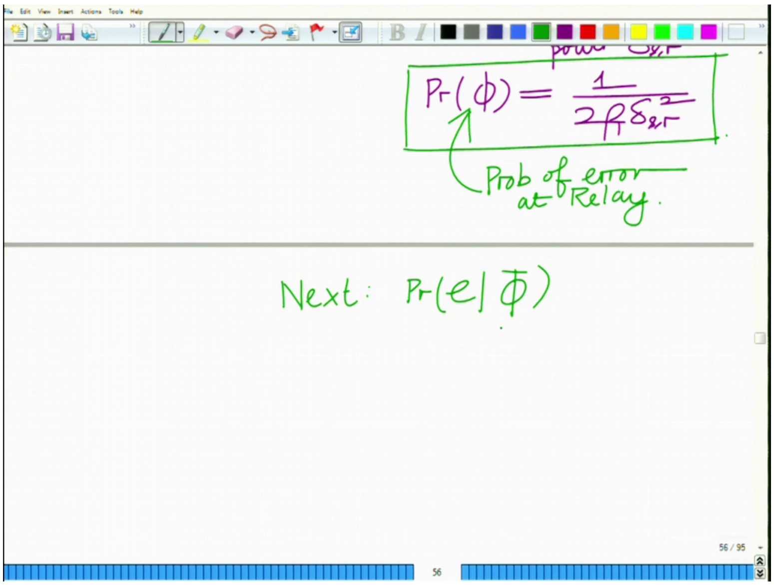

So, this is simply going to be 1 over 2 rho 1 delta s, r square; I hope this is clear. What we are saying is we are simply replacing in the source destination SNR beta s, d with beta s, r, which is exponential with average power delta s, r square all right. So, in the bit error rate expression, one can simply replace delta s, d square by delta s, r square. And you will get the appropriate bit error rate expression for the source relay link ok. And therefore, this is your probability of phi; this is the probability of error. This is the probability of error at the relay ok, so that is two components remember, we have probability of e, we want to find out which we have written in terms of probability of e given phi and probability of phi. The next thing that we want to find is the key component, which is probability of e given phi bar.

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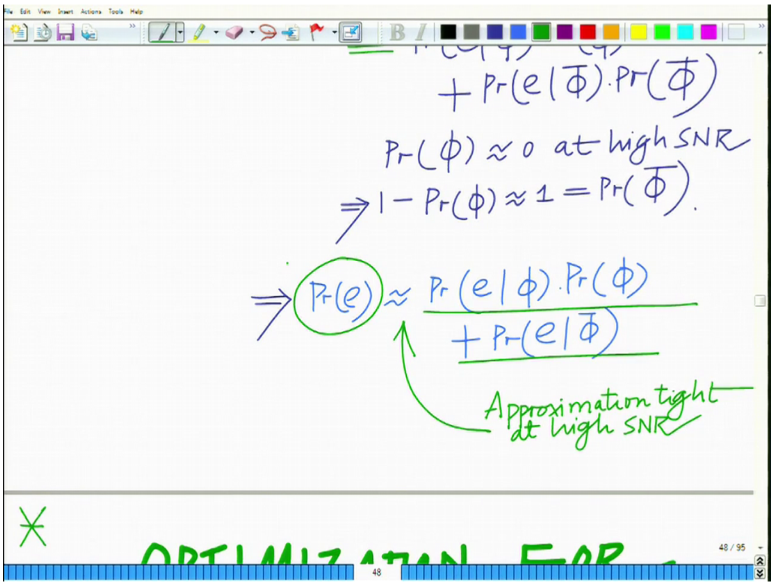

Remember we are talking about if you go all the way back, go back to the previous module we have this probability of e error at the destination, which you want to find eventually. Now, that depends on probability of e given phi, it is approximately probability of e given phi into probability of phi plus probability of e given phi bar. So, we have found out probability of e given phi probability of phi, what needs to be what remains to be found is probability e given phi bar. What is probability of e given phi bar that is the probability of error at destination given that given phi bar that is no error at relay that is the really decodes accurately, in which case relay also retransmit.

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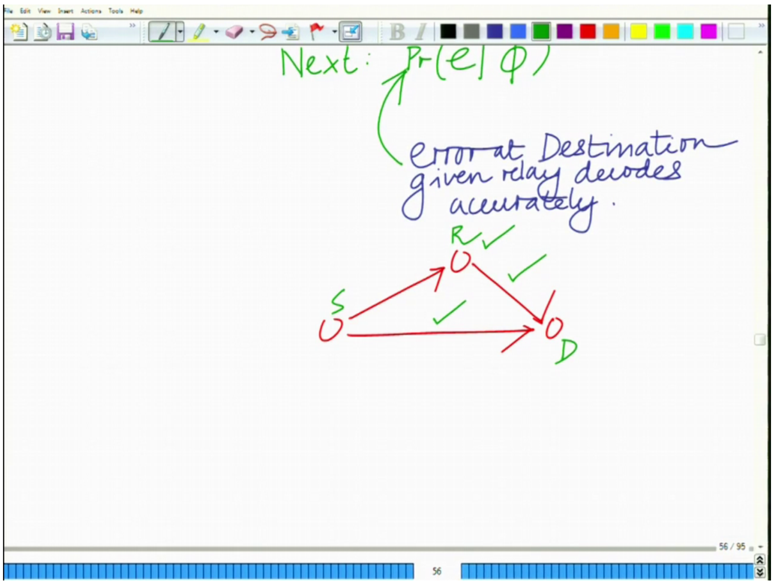

So, probability of e given phi bar; what is this? This is the error at destination given that the relay decodes accurately. So, in this case what happens is if you look at the diagram, you have the source. In this case, what happens is you have the source, relay, destination. Relay is decoding accurately, so source transmits relay also transmits. So, destination you have two signals, signal received by the source, signal received from this relay.

Now, how does a destination decoded, naturally the destination has to employee some kind of combining. And we already know, what is the optimal combining structure so. And this is something that I am going to something that is very interesting shows the broad applicability of the optimization principles that we have seen so far. You can treat this in fact as a beam forming problem, and that is very interested. It is a beam forming with the multiple nodes rather than multiple and so.

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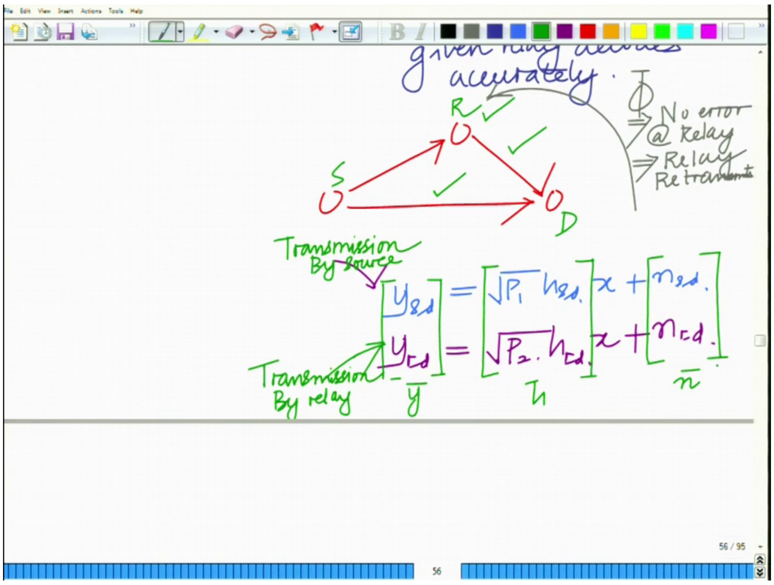

In this case, the relay retransmits. Phi bar implies, no implies, no error at relay. Remember phi implies error at relay, so phi bar implies no error at relay, implies relay retransmits, because relay remember selective decode and forward, relay retransmits only fits able to decode. So, relay retransmits, so now what happens? You have two symbols y s, d, which you already had square root of P 1 h s, d x plus n s, d. Now, you will have transmission by the relay. So, y r d equals square root of P 2 that is the relay power h r d fading channel coefficient between relay and destination plus n r d ok.

And so this is the original transmission by source, and this is the transmission by the relay and this is the transmission by the relay. And now you have these two symbols received at the destination; one from the source, one from the relay. Now, what are you suppose to do, obviously you have to combine them in some kind of an optimal fashion. And what is the optimal combiner; we know what is the optimal combiner. The optimal combiner, the optimal beamforming this is similar to the beam forming problem with multiple receive antennas. And we know that the optimal combiner is the maximal ratio combiner all right. And therefore, if you treat this two received symbols as your receive vector y bar, and this as your channel vector h bar, and this as your noise vector n bar.

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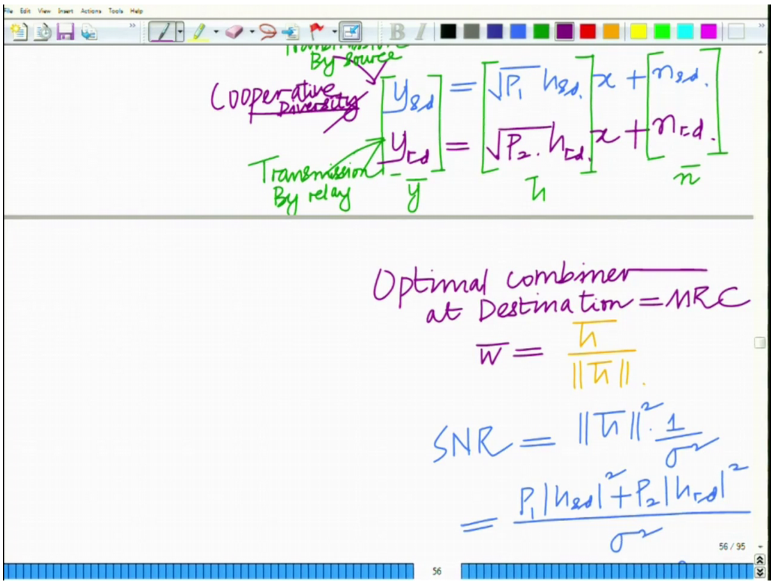

We already know, what is the optimal combiner. Optimal combiner for this optimal combiner at destination equals the MRC that is the Maximal Ratio Combiner. What is the maximal ratio of combiner? That is you have your beam former W bar equals sorry h bar divided by norm h bar, we know this. And what is the SNR, SNR is norm h bar square P divided by but P is 1 that is power of x is 1 into 1 divided by sigma square. And what is norm h bar square remember, norm h bar square is P 1 magnitude h s, d square plus P 2 magnitude h r, d square ok.

You can see h bar, h bar is the vector square root of P 1 h s, d square root of P 2 h r, d ok, so that is norm h bar square equals P 1 magnitude h s, d square plus P 2 magnitude h r, d square divided by sigma square. Now, we know P 1 divided by sigma square is rho 1. So, this is rho 1 magnitude h s, d square is beta s, d plus P 2 divided by sigma square is rho 2, magnitude h r, d square is beta r d ok. So, this is your SNR.

You can see, this is the coherent combining combines the SNRs corresponding to both the source destination transmission and the relay destination. And thereby you are enhancing the reliability. This is where you get the gain from co-operative communication, because you have the signals that are transmitted by two different sources. One is the source original source and other is the relay, which is acting as the replica of the source, so that in hand and now you see the co-operative diversity aspect emerging, because there is transmission by the source, there is transmission by the relay.

So, they are co-operating you have two signal copies, and that gives diversity in a wireless communication system, which leads to a significant decrease in the bit error rate that is what we are going to see all right.

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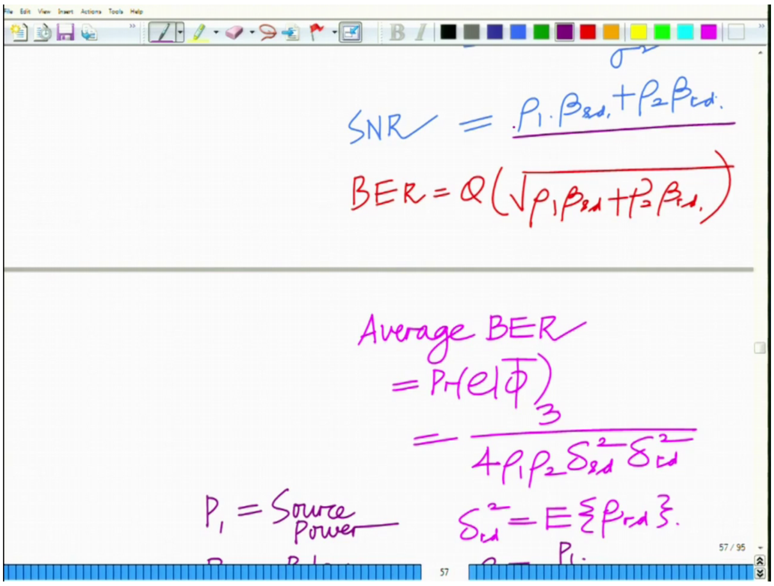

And therefore, now again you can follow the same procedure. The bit error rate for QPSK will be Q of square root of rho 1 beta s d plus rho 2 beta r, d. You can average this over the probability density function. And it can be shown that the average bit error rate for this, which is nothing but your probability of e given phi bar. It can be shown I am not going to explicitly show it, maybe in a different module separate module, because it is not necessary for a preliminary discussion this is given as 3 or 4 rho 1 rho 2 delta s, d square delta r, d square. We know what delta r d square is delta r d square is the expected value of average gain of the relay destination link.

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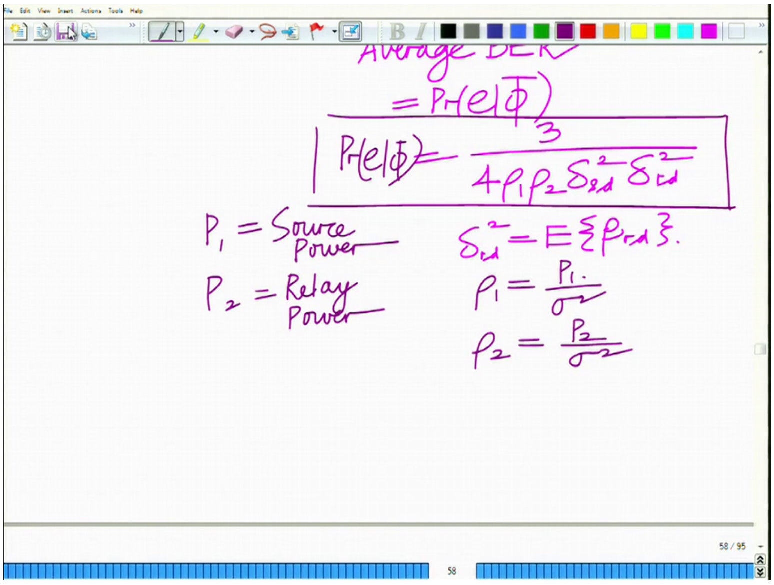

And rho 1, we know already find rho 1. Rho 2 is also similar rho 1 equals P 1 that is source power by sigma square root 2 equals P 2 by sigma square. Well what is P 1, P 1 equals source. What is P 2, P 2 equals relay power, because the source and relay need not transmit with equal power. Source power can be very different from the relay power. Relay can be have very low power depending on the relay can have high power or low power and so on.

And you know, and this is where you can see this is where you have this is coherent combining, you have this signal copies, and you have this two signal copies. And this is where you have this co-operative diversity really coming actually if you realize this. And diversity is an important principle in wireless communication system, which results in a significant decrease in the error rate ok. And therefore, this is your P r of e given phi bar. So, now we have this elegant expression for the probability of error given phi bar that is correct decoding at the relay or no error at the relay all right.

So, now putting all these components together, one can derive the probability of error that is the final expression for the end to, this is also known as the end to end error in this that is probability of E ok. And using that now one can come up with a framework for optimal power distribution between the source and relay that is our ultimate aim. The optimization problem pertains to how to distribute the power optimally between the source and relay in a wireless communication system, which we will deal with in the next module.

Thank you very much.