Applied Optimization for Wireless, Machine Learning, Big data
Prof Aditya K. Jagnnatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 51
Practical Application: Multiple Input Multiple Output (MIMO) Beamformer Design

Hello welcome to another module in this massive open online course, so we are looking at MIMO beam forming; how to design the optimal transmit and receive Beamformers in a multiple input multiple output wireless communication system. So let us continue our discussion.

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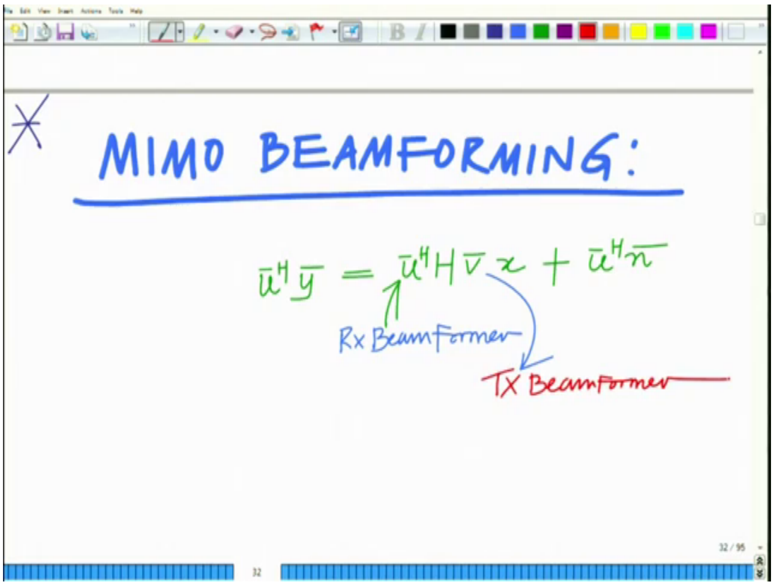

So, what we want to look at is we want to look at MIMO beamforming and we have said after combining with u bar Hermitian u bar is at receive beamformer we have u bar Hermitian H v bar times x plus u bar Hermitian noise this is the noise vector. Now, what we have to do is this is your receive beamformer and u bar and v bar is a transmit beam former and we want to design this to maxi jointly design transmit and receive beamformers to maximization are first what we will do is we will set H v bar.

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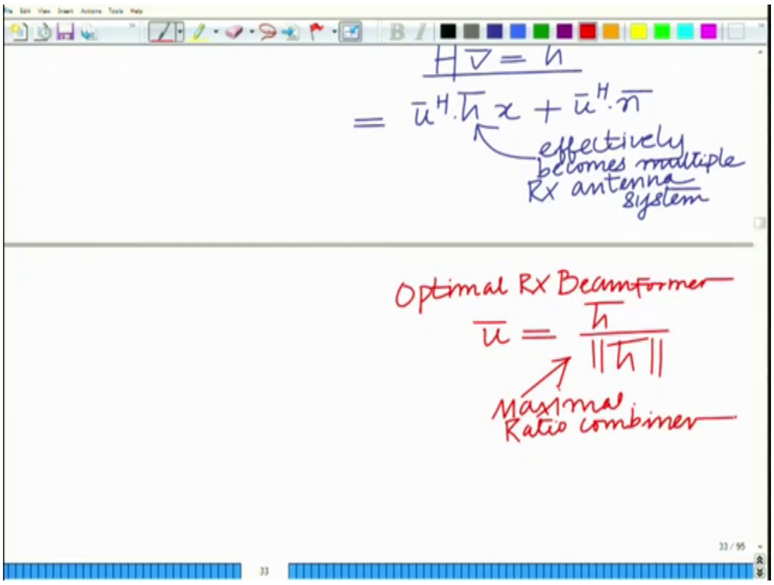

We will set this quantity equal to h bar and now you will observe something interesting this becomes u bar Hermitian h bar into x plus u bar Hermitian times n bar. Now if you see this h bar this effectively becomes a single input multiple output system by setting H v bar equal to h bar this effectively becomes us a SIMO system or a multiple simply multiple receiver antenna system.

For which we already know the optimal beamformer effectively becomes your multiple RX antenna system and once it become an RX antenna system what you have is that this is equal to. Now, therefore, now you know what is optimal beam former u bar, the optimal beam former u bar for this multiple antenna system optimal RX beamformer is the maximal ratio combiner; that is u bar equals h bar divided by norm h bar ok. And therefore, we know this is the maximal ratio combiner, this is your optimal beamformer maximum this is a maximal ratio combiner.

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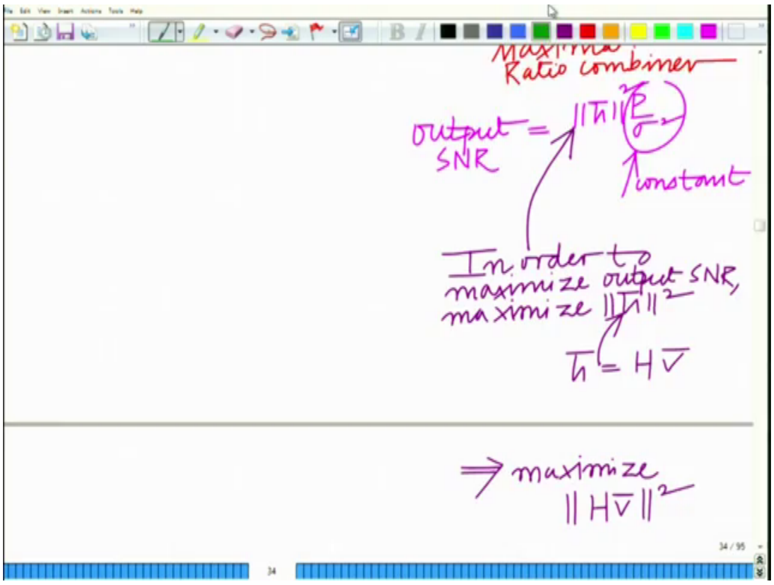

And the output SNR of the maximal ratio combiner is given as norm h bar square P over sigma square remember P is the transmit power sigma is the noise power. So, this quantity is constant P over sigma square because, P is the transmit power sigma square is the noise power implies that we have to maximize norm h bar square. So, to maximize the SNR we have to maximize norm h bar square ok. So, in order to maximize SNR, so that is optimisation problem maximize or let us say maximize output SNR we have to maximize norm h bar square.

But h bar equals we have seen h bar equals h times v bar or capital H times v bar. So, h bar equals H times v bar which basically implies substituting this we have to maximize norm H times v bar square which is basically norm H times v bar square. And what is norm H times v bar square H times v bar square is the vector Hermitian itself for a complex vector which is basically v bar Hermitian H Hermitian H into v bar which is equal to v bar Hermitian G into v bar.

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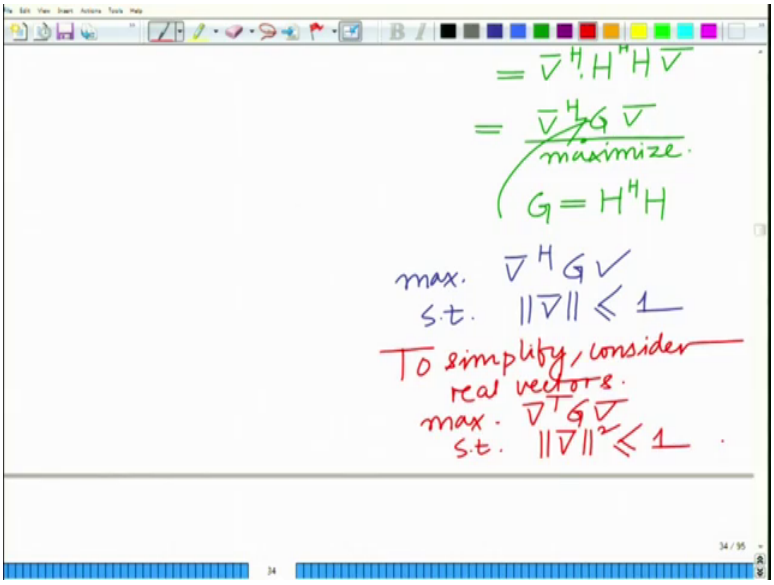

So, we have to maximize this quantity where what is G, G equals H Hermitian H G equals H Hermitian H and we have to maximize. So, optimisation problem, so the net optimisation problem becomes maximize v Hermitian G v subject to the constraint remember the constraint is still there unit norm constraint that is the transit beam former energy of the transmit beamformer or power of the transmit beam former norm v bar is less than or equal to 1. So, this is the resulting problem for optimal beam forming alright we have substituted H capital H bar equals H times v bar and from that you derive this in terms of the optimal this optimisation problem for the optimal beam former.

And a now to just to simplify it what I am going to today is I am going to assume real vectors I am going to place this Hermitian by transpose. So, I am going to say to simplify consider real vectors. So, this becomes v bar transpose G v bar subject to the constraint norm v bar less than one or norm v bar square less than equal to 1 both these constraints are equal equivalent and observe that interestingly this is a non-convex problem.

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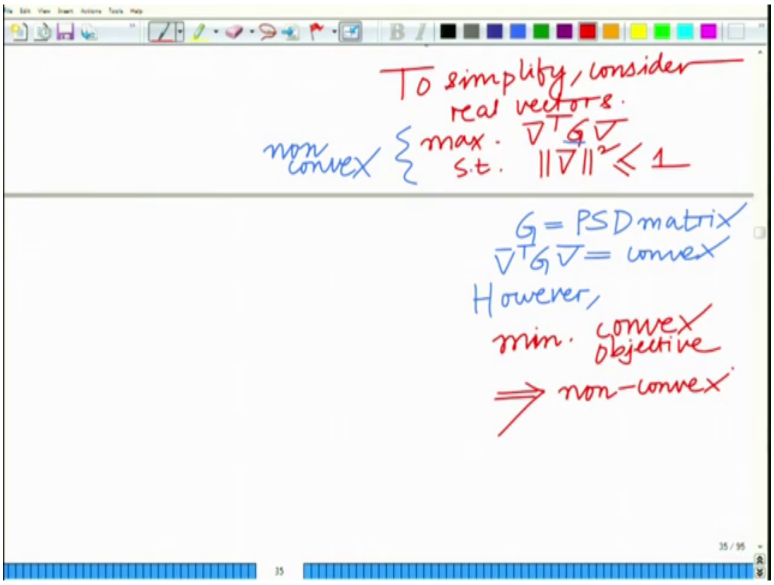

This is one of the few and very interesting non-convex ones because if you look at this v bar transpose G v bar is a convex function correct. However, we are maximizing it we are not minimising remember standard form convex optimisation problem we have a convex objective, but your minimising it here a convex objective your maximizing. So, the problem is a non-convex problem although the objective is convex because your maximizing it is it so non-convex.

In fact, if you minimise it you have to take the minimum and minimiser, but once you take the minimum the convex objective becomes a concave objectives as and therefore, it is a non-convex problem ok. And therefore, is because G is a positive semi definite matrix G equals a PSD matrix and v bar transpose G v bar equals convex however, what your doing is your minimising a convex objective. Your performing minimization over a convex objective function implies that this is non-convex and therefore, now what will do is we will form the Lagrangian the same thing that we have done before.

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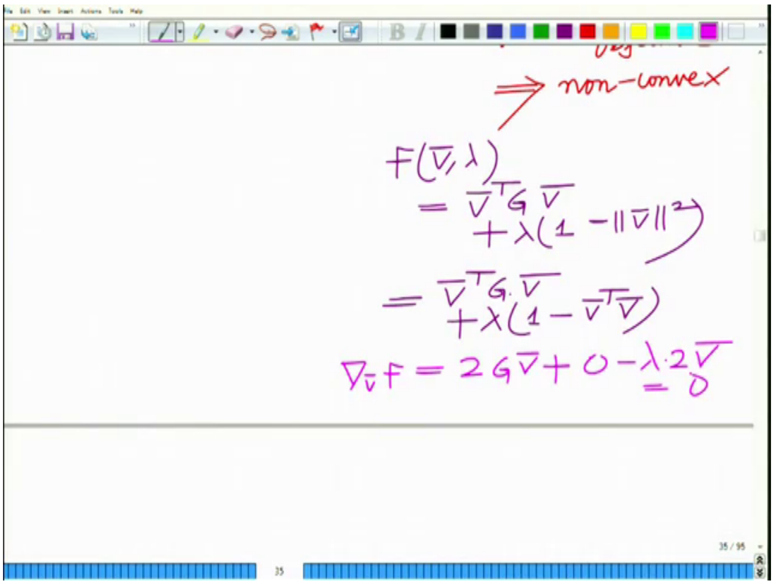

That is F of v bar coma lambda remember you have the objective v bar transpose G v bar plus lambda times 1 minus norm v bar square objective plus eigen plus Lagrange multiplier times constraint. This is equal to v bar transpose G v bar plus lambda times 1 minus v bar transpose v bar now, differentiate it take its gradient with respect to v bar this v bar transpose G v bar twice G v bar plus gradient of lambda with respect to v bar is 0 minus lambda derivative of v bar transpose v bar is twice v bar minus twice v bar or it write minus twice lambda v bar this is equal to 0.

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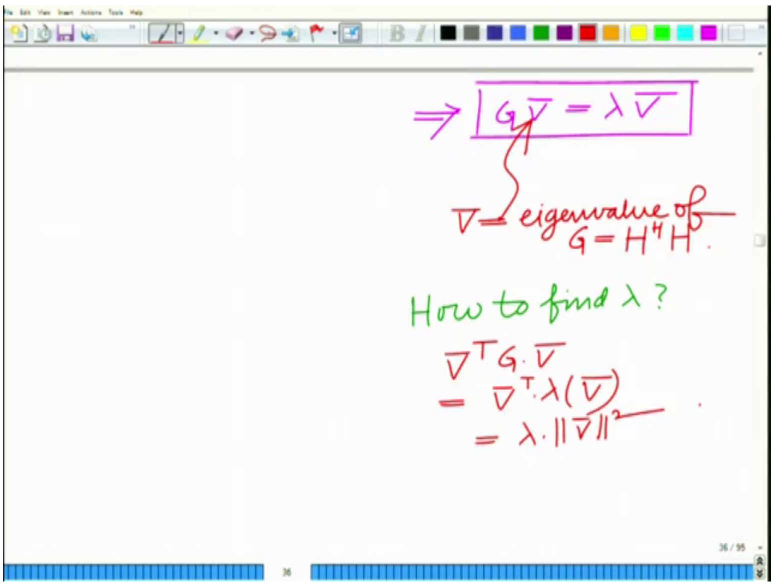

Now, if you solve this the two is cancel and therefore this implies that G v bar equals lambda times v bar and then you observe something very interesting what you observe is the G v bar equals lambda v bar. And recall you have already seen this kind of equation this equation before this is the nothing, but with the definition of the eigenvector of G that is any vector v bar which satisfy this property that is G times v bar vector equal simply a scaling factor lambda times v bar v bar is the eigenvector and lambda is the eigenvalue. So, that is the interesting property so what this shows is that the optimal transmit vector v bar equals eigenvalue of G equals now you can write H Hermitian H that was just for simplicity.

So, there is a eigenvector of H Hermitian H there is an eigenvector of H Hermitian H right, but correct right. So, this has many eigenvalues now which eigenvalue now how to find which eigenvalue or you can say how to find the Lagrange multiplier lambda ah. So, we can say that this will be v transpose G v bar now G v bar equals lambda v bar so this because v transpose lambda into V bar which is lambda times v transpose v bar which is lambda times norm v bar square, but norm v bar square equal to 1 so this becomes lambda.

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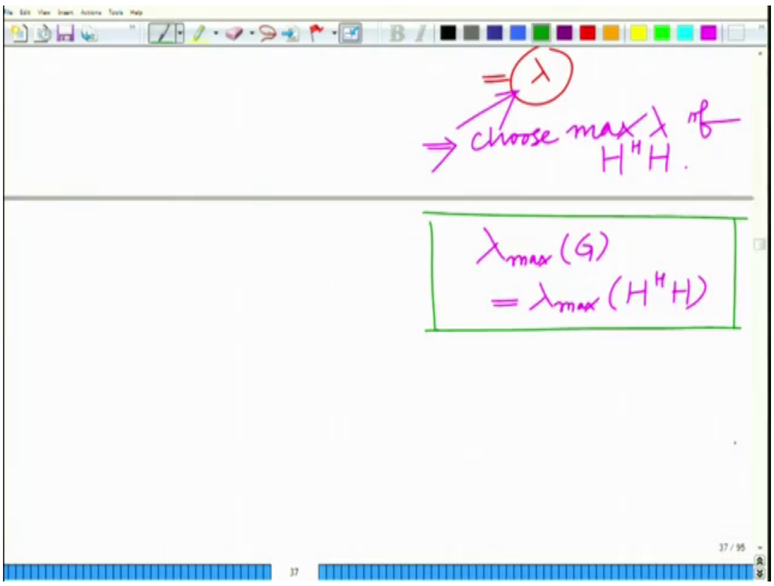

So, this is lambda and you want to maximize this want to maximize v bar transpose G v bar implies choose the maximum lambda or choose eigenvector corresponding to maximum lambda eigenvalue of H Hermitian H. That is let us say we can denote this by lambda max of G which is lambda max of H Hermitian H G is a H Hermitian H. And that is something that is extremely interesting what you have what it says is the transmit beamformer H Hermitian H corresponds to the max eigenvalue corresponding to the maximum eigenvector of H Hermitian H that is the transmit beamformer v bar unit norm transmit beamformer v bar which maximizes the SNR at the receiver.

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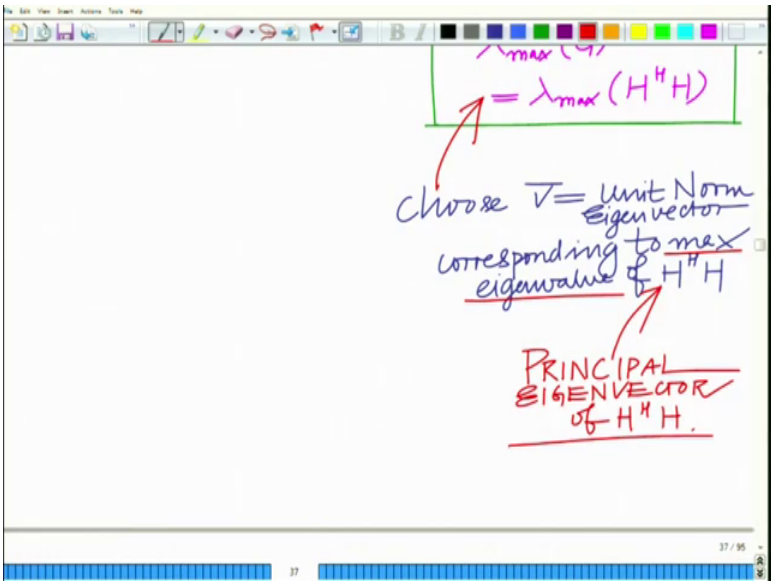

So, choose v bar unit norm choose v bar equals unit norm eigenvector corresponding to maximum eigenvalue of corresponding to the maximum eigenvalue of H Hermitian H that is the interesting aspect. This is also termed this Eigen vector corresponding to the largest eigenvalue this also termed as the principal eigenvector, transmit beam former v Hermitian transmit beamformer v bar is the principal eigenvector corresponding to H Hermitian H.

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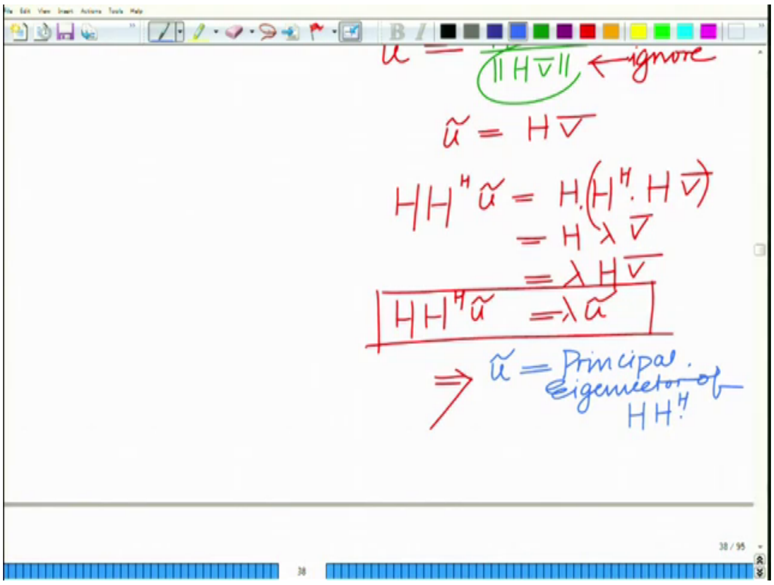

Now whatever u bar the receive beam former remember u bar receive Beamformer we still have to find that that is H v bar divided by norm H v bar. Now, this norm is simply normalization so for the time being ignore this u tilde equals H v bar now look at this now perform H H Hermitian u tilde equals H H Hermitian into H v bar, but v bar is eigenvector of H Hermitian H correct.

So, this will become H now look at this H Hermitian H v bar is equal to lambda V bar so this is lambda times H v bar, but H v bar is u tilde. So, what we have here is we have shown something very interesting H H Hermitian u tilde equals lambda times u tilde implies u tilde is the Eigen that is the receive beam former u tilde is the eigenvector of the matrix H H Hermitian and that is something that is interesting u tilde. Again you can see u tilde or now you can say u tilde equals principal eigenvector of H H Hermitian that is eigenvector corresponding to largest eigenvalue of H H Hermitian.

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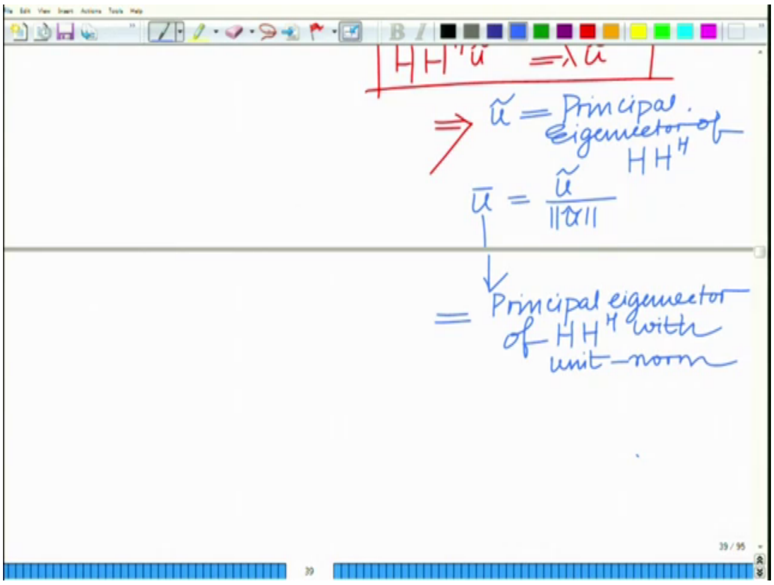

And now u bar is simply u tilde divided by norm of u tilde which is principal eigenvector with unit norm remember you can simply scale the eigenvector by any quantity it will still be an eigenvector. So, this is the you can say principal eigenvector of H H Hermitian with this is a principal eigenvector of H H Hermitian with unit norm great.

And therefore, that basically gives us both the transmit and receive beamformers and we have very interesting expressions for them the transmit beamformer u bar optimal transmit beamformer.

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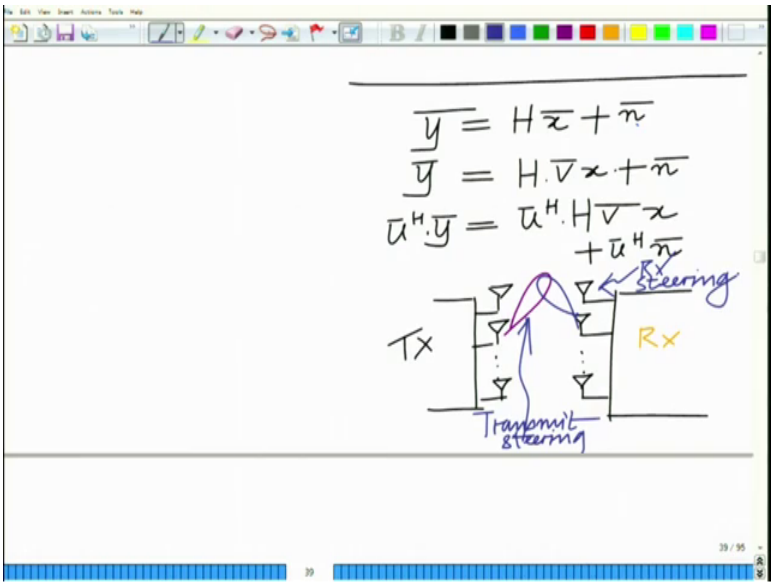

So, to summarise you have the MIMO Beamforming problem y bar equals H x bar plus n bar x bar equals. So, this is v bar times x plus n bar that is your y bar and at the receiver you perform u bar Hermitian v bar y bar which is u bar Hermitian H v bar x plus u bar Hermitian n bar.

And remember what you are doing here is as I already told you have to perform beamforming at both the ends in the MIMO system. So, you have the transmitter you have the receiver your transmitting from the transmitter in a particular direction at the receiver your also collecting or your also processing the signal your steering the receiver antenna array in a particular direction. So, this is basically you are transmit steering remember these all electronic steering so you do not need to physically steer and this is your receive steering.

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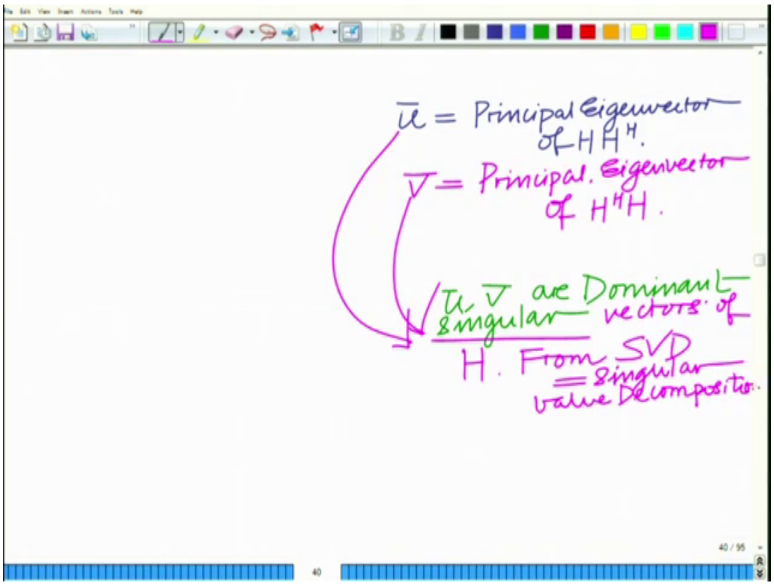

And u bar u bar equals eigenvector of H H Hermitian corresponding to larger eigenvalue or principal H H Hermitian and v bar equals principal eigenvector of H Hermitian H. Now later what we will see is we will see what is known as the singular value decomposition of the channel matrix and it will turn out that u bar and v bar are in fact, the dominant left singular and right u bar is the dominant left singular vector which means singular vector corresponding to the larger singular value.

And similarly v bar is the dominant right singular vector singular vector corresponding to larger singular value. In fact, we will see later that is u bar comma v bar are the dominant singular and this is a key phrase not eigenvectors, but singular vectors of H from the SVD what we call not the EVD from the SVD which is called the singular value decomposition. From the singular value decomposition and further from the singular value decomposition and remember dominant singular values means corresponding to largest singular value.

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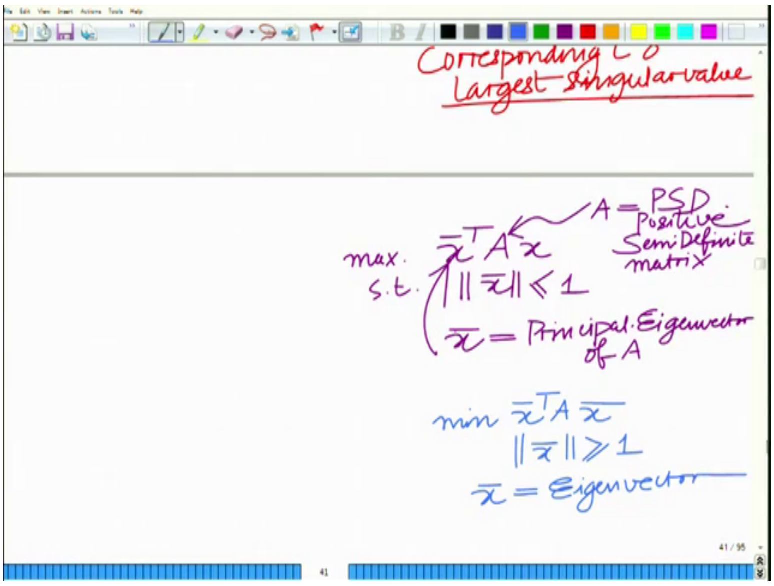

Dominant singular vector means corresponding to the largest singular value and in this context also we have seen a very interesting optimization problem that is if you take a positive semi definite matrix x bar transpose A x maximise subject to the constraint norm x bar equal to 1 or norm x bar less than or equal to 1. Then x bar equals principal eigenvector provided A is positive semi definite remember x bar transpose x provided A is PSD positive semi definite matrix provided x bar is a positive semi definite matrix this is a principal eigenvector of A that is a maximum A bar.

Now similarly if you minimise, now this is another interesting analogue. Now, this problem is convex minimise x bar transpose x bar such that norm x bar of course, this is not convex again in the sense that the constraint is not convex norm x bar greater than or equal to 1, then x bar equals eigenvector corresponding to smallest eigenvalue.

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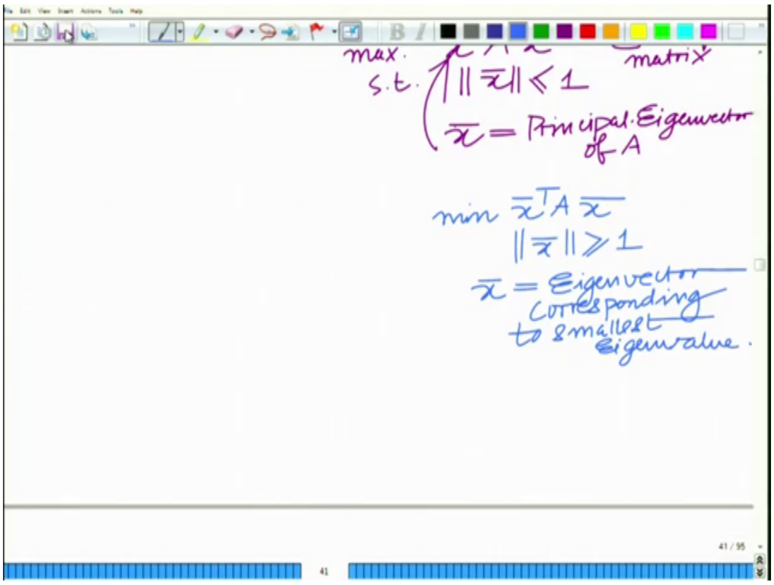

So, there are there is a analog this problem eigenvector corresponding to x bar equals eigenvector corresponding to the smallest eigenvalue. And therefore, this is a very interesting application as we have seen here that is basically with respect to beam forming in MIMO system to determine the top table transmit and receive the unit norm transmit and receive beamformers which are given as v bar optimal transmit beamformer is the principal eigenvector of H Hermitian H or you can also say the dominant left singular vector of H.

And u bar is the principal eigenvector of H H Hermitian that is the dominant left singular vector of H, the channel matrix H is the channel matrix of the MIMO wireless system alright. So, with this interesting observation or this completing after completing this interesting example we will stop here and we will continue in the subsequent modules.

Thank you very much.